

## Harmonic oscillator position eigenstates via application of an operator on the vacuum

Francisco Soto-Eguibar and Héctor Manuel Moya-Cessa

*Instituto Nacional de Astrofísica, Óptica y Electrónica,*

*Luis Enrique Erro 1, Santa María Tonantzintla, San Andrés Cholula, Puebla, 72840 México.*

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Harmonic oscillator squeezed states are states of minimum uncertainty, but unlike coherent states, in which the uncertainty in position and momentum are equal, squeezed states have the uncertainty reduced, either in position or in momentum, while still minimizing the uncertainty principle. It seems that this property of squeezed states would allow to obtain the position eigenstates as a limiting case, by doing null the uncertainty in position and infinite in momentum. However, there are two *equivalent* ways to define squeezed states, that lead to different expressions for the limiting states. In this work, we analyze both definitions and show the advantages and disadvantages of using them in order to find position eigenstates. With this in mind, but leaving aside the definitions of squeezed states, we find an operator that applied to the vacuum gives position eigenstates. We also analyze some properties of the squeezed states, based on the new expressions obtained for the eigenstates of the position.

*Keywords:* Position eigenstates; harmonic oscillator; squeezed states; minimum uncertainty states; squeeze operator.

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### 1. Introduction

Quantum states of the harmonic oscillator are very important in the study of the quantum theory of the electromagnetic field because of the fact that they may be used to describe a quantized field [1–3], each harmonic oscillator representing a mode of such electromagnetic field. In fact, this was the concept that Dirac used to build the first quantum theory of the electromagnetic field [2]. The easiest to understand and to manipulate, and the most natural states of the quantum harmonic oscillator are number states  $|n\rangle$ . Number states are eigenstates of the harmonic oscillator Hamiltonian and, of course, are also eigenstates of the number operator  $\hat{n} = \hat{a}^\dagger \hat{a}$ , where  $\hat{a}^\dagger$  and  $\hat{a}$  are the well known creation and annihilation operators, respectively. However, for any  $n$ , no matter how big, the mean field is zero; i.e.,  $\langle n | \hat{E}_x | n \rangle = 0$ , and we know that a classical field changes sinusoidally in time in each point of space; thus, these states can not be associated with classical fields [3, 4].

In the first years of the sixties of the past century, Glauber [5] and Sudarshan [6] introduced the coherent states, and it has been shown that these states are the most classical ones. Coherent states are denoted as  $|\alpha\rangle$ , and one way to define them is as eigenstates of the annihilation operator; that is,  $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ . An equivalent definition is obtained applying the Glauber displacement operator  $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$  to the vacuum:  $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$ ; we see then coherent states as vacuum displaced states. Coherent states also have the very important property that they minimize the uncertainty relation for the two orthogonal field quadratures with equal uncertainties in each quadrature [3, 4].

Since then, other states have been introduced. In particular, squeezed states [3, 4, 7–9] have attracted a great deal of attention over the years because their properties allow to re-

duce the uncertainties either of the position or momentum, while still keeping the uncertainty principle to its minimum. Because of this, they belong to a special class of states named minimum uncertainty states. Once produced, for instance as electromagnetic fields in cavities, they may be monitored via two level atoms in order to check, or measure, that such states have been indeed generated [10, 11].

Based on the above properties, we can think about eigenstates of position as limiting cases of squeezed states. As squeezed states are minimum uncertainty states, we can reduce to zero the uncertainty in the position, while the uncertainty in the momentum goes to infinity, so that we keep the uncertainty principle to its minimum. Of course, there is also the option to reduce to zero the uncertainty in the momentum, while the position gets completely undefined, obtaining that way the possibility to define momentum eigenstates. In Secs. 1 and 2, we analyze the possibility of define the position eigenstates as the limit of extreme squeezing of the squeezed states. In what follows, we will use a unit system such that  $\hbar = m = \omega = 1$ .

There are two equivalent forms to define the squeezed states. In the first one, introduced by Yuen [12], squeezed states are obtained from the vacuum as

$$|\alpha; r\rangle = \hat{S}(r)\hat{D}(\alpha)|0\rangle = \hat{S}(r)|\alpha\rangle, \quad (1)$$

where

$$\hat{S}(r) = \exp\left[\left(\hat{a}^2 - \hat{a}^{\dagger 2}\right)r/2\right] \quad (2)$$

is the so-called squeeze operator. In this view, squeezed states are created displacing the vacuum, and after, squeezing it. Note that when the squeeze parameter  $r$  is zero, the squeezed states reduce to the coherent states. In this work, we will consider only real squeeze parameters, as that is enough for our intentions.

In the definition introduced by Caves [13], the vacuum is squeezed and the resulting state is then displaced; that means, that in this approach squeezed states are given as

$$|\alpha'; r'\rangle = \hat{D}(\alpha') \hat{S}(r') |0\rangle. \tag{3}$$

Both definitions of the squeezed states agree when the squeeze factor is the same,  $r' = r$ , and when the modified amplitude  $\alpha'$  of the Caves approach is given by

$$\alpha' = \mu\alpha - \nu\alpha^*, \tag{4}$$

being

$$\mu = \cosh r \tag{5}$$

and

$$\nu = \sinh r. \tag{6}$$

To analyze the uncertainties in the position and in the momentum of the squeezed states, we introduce, following Loudon and Knight [7], the quadrature operators

$$\hat{X} = \frac{\hat{a} + \hat{a}^\dagger}{2} = \frac{\hat{x}}{\sqrt{2}} \tag{7}$$

and

$$\hat{Y} = \frac{\hat{a} - \hat{a}^\dagger}{2i} = \frac{\hat{p}}{\sqrt{2}} \tag{8}$$

where  $\hat{x}$  is the position operator and  $\hat{p}$  the momentum operator. Note that the quadrature operators are essentially the position and momentum operators; this definition just provides us with two operators that have the same dimensions.

In order to show that really the squeezed states are minimum uncertainty states, we need to calculate the expected values in the squeezed state (1) of the quadrature operators (7) and (8), and its squares. Using (7) and (1), we get

$$\langle \alpha; r | \hat{X} | \alpha; r \rangle = \frac{1}{2} \langle \alpha | \hat{S}^\dagger(r) \frac{\hat{a} + \hat{a}^\dagger}{2} \hat{S}(r) | \alpha \rangle. \tag{9}$$

The action of the squeeze operator on the creation and annihilation operators is obtained using the Hadamard's lemma [14, 15],

$$\hat{S}^\dagger(r) \hat{a} \hat{S}(r) = \mu \hat{a} - \nu \hat{a}^\dagger, \quad \hat{S}^\dagger(r) \hat{a}^\dagger \hat{S}(r) = \mu \hat{a}^\dagger - \nu \hat{a}, \tag{10}$$

such that

$$\hat{S}^\dagger(r) \frac{\hat{a} + \hat{a}^\dagger}{2} \hat{S}(r) = e^{-r} \hat{X}. \tag{11}$$

Therefore, as  $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$  and  $\langle \alpha | \hat{a}^\dagger = \langle \alpha | \alpha^*$ , it is easy to see that

$$\langle \alpha; r | \hat{X} | \alpha; r \rangle = e^{-r} \frac{\alpha + \alpha^*}{2}, \tag{12}$$

and that

$$\langle \alpha; r | \hat{X}^2 | \alpha; r \rangle = e^{-2r} \frac{1 + 2|\alpha|^2 + \alpha^2 + \alpha^{*2}}{4}. \tag{13}$$

So, we obtain for the uncertainty in the quadrature operator  $\hat{X}$ ,

$$\Delta X \equiv \sqrt{\langle \alpha; r | \hat{X}^2 | \alpha; r \rangle - \langle \alpha; r | \hat{X} | \alpha; r \rangle^2} = \frac{e^{-r}}{2}. \tag{14}$$

Proceeding in exactly the same way for the quadrature operator  $\hat{Y}$ , we obtain

$$\Delta Y \equiv \sqrt{\langle \alpha; r | \hat{Y}^2 | \alpha; r \rangle - \langle \alpha; r | \hat{Y} | \alpha; r \rangle^2} = \frac{e^r}{2}. \tag{15}$$

As we already said, we can then think in the position eigenstates and in the momentum eigenstates as limit cases of squeezed states. Indeed, when the squeeze parameter  $r$  goes to infinity, the uncertainty in the position goes to zero, and the momentum is completely undetermined. Of course, when the squeeze parameter goes to minus infinity, we have the inverse situation, and we can think in define that way the momentum eigenstates. In the two following sections, we use the Yuen and the Caves definitions of the squeezed states to test this hypothesis.

## 2. A first attempt à la Yuen

From Eq. (14) above, we can see that in the limit  $r \rightarrow \infty$  the uncertainty for position vanishes and so a position eigenstate should be obtained (from now on, we consider  $\alpha$  real),

$$\lim_{r \rightarrow \infty} \left| \frac{x}{\sqrt{2}}; r \right\rangle \rightarrow |x\rangle_p. \tag{16}$$

We have written a sub index  $p$  in the position eigenstate in order to emphasize that fact. Following the Yuen definition  $|\alpha; r\rangle = \hat{S}(r) \hat{D}(\alpha) |0\rangle = \hat{S}(r) |\alpha\rangle$ , so

$$\left| \frac{x}{\sqrt{2}}; r \right\rangle = \hat{S}(r) \hat{D} \left( \frac{x}{\sqrt{2}} \right) |0\rangle = \hat{S}(r) \left| \frac{x}{\sqrt{2}} \right\rangle. \tag{17}$$

We now write the squeeze operator as [16]

$$\hat{S}(r) = \frac{1}{\sqrt{\mu}} e^{-\frac{\nu}{2\mu} \hat{a}^{\dagger 2}} \frac{1}{\mu^{\hat{a}^\dagger \hat{a}}} e^{\frac{\nu}{2\mu} \hat{a}^2}, \tag{18}$$

where, as we already said,  $\mu = \cosh r$  and  $\nu = \sinh r$ . So,

$$\left| \frac{x}{\sqrt{2}}; r \right\rangle = \frac{1}{\sqrt{\mu}} e^{-\frac{\nu}{2\mu} \hat{a}^{\dagger 2}} \frac{1}{\mu^{\hat{a}^\dagger \hat{a}}} e^{\frac{\nu}{2\mu} \hat{a}^2} \left| \frac{x}{\sqrt{2}} \right\rangle. \tag{19}$$

Now, we develop the first operator (from right to left) in power series, we use the definition of the coherent states,  $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ , and the action of the number operator over the number states ( $\hat{a}^\dagger \hat{a} |n\rangle = \hat{n} |n\rangle = n |n\rangle$ ), to obtain

$$\begin{aligned} \left| \frac{x}{\sqrt{2}}; r \right\rangle &= \frac{1}{\sqrt{\mu}} e^{-\frac{\nu}{2\mu} \hat{a}^{\dagger 2}} \left( \frac{1}{\mu} \right)^{\hat{a}^\dagger \hat{a}} \sum_{n=0}^{\infty} \left( \frac{x}{\sqrt{2}} \right)^n \frac{1}{\sqrt{n!}} |n\rangle \\ &= \frac{1}{\sqrt{\mu}} e^{-\frac{\nu}{2\mu} \hat{a}^{\dagger 2}} \sum_{n=0}^{\infty} \left( \frac{x}{\sqrt{2}} \right)^n \frac{1}{\sqrt{n!}} \left( \frac{1}{\mu} \right)^n |n\rangle. \end{aligned} \tag{20}$$

As  $r \rightarrow \infty$ ,  $\frac{1}{\mu} = \frac{1}{\cosh r} \rightarrow 0$ , which means that the only term that survives from the sum is  $n = 0$ , and then

$$|x\rangle_p \propto e^{-\frac{\nu}{2\mu} \hat{a}^{\dagger 2}} |0\rangle \tag{21}$$

that would give an approximation for how to obtain a position eigenstate from the vacuum. However, note that the above expression does not depend on  $x$  and therefore can not be correct. From all this analysis, we must conclude that the Yuen definition of the squeezed states, whatever representation used, does not give the correct asymptotic states.

### 3. A second attempt à la Caves

We now squeeze the vacuum and after we displace it. Thus, in this case,

$$|x\rangle_p = \lim_{r \rightarrow \infty} \left| \frac{x}{\sqrt{2}}; r \right\rangle = \lim_{r \rightarrow \infty} \hat{D} \left( \frac{x}{\sqrt{2}} \right) \hat{S}(r) |0\rangle. \quad (22)$$

We use again expression  $\left( \hat{S}(r) = \exp \left( -\frac{\nu}{2\mu} \hat{a}^{\dagger 2} \right) \left( \frac{1}{\mu} \right)^{\hat{n} + \frac{1}{2}} \exp \left( \frac{\nu}{2\mu} \hat{a}^2 \right) \right)$  for the squeeze operator [12], where  $\mu$  and  $\nu$  are defined in (5) and (6), and we write the displacement operator as  $\hat{D}(\alpha) = \exp \left( \frac{|\alpha|^2}{2} \right) \exp(-\alpha^* \hat{a}) \exp(\alpha \hat{a}^\dagger)$  [16], to obtain

$$\begin{aligned} \left| \frac{x}{\sqrt{2}}; r \right\rangle &= \exp \left( \frac{x^2}{4} \right) \exp \left( -\frac{x}{\sqrt{2}} \hat{a} \right) \exp \left( \frac{x}{\sqrt{2}} \hat{a}^\dagger \right) \\ &\times \exp \left( -\frac{\nu}{2\mu} \hat{a}^{\dagger 2} \right) \left( \frac{1}{\mu} \right)^{\hat{n} + \frac{1}{2}} \exp \left( \frac{\nu}{2\mu} \hat{a}^2 \right) |0\rangle. \end{aligned} \quad (23)$$

As  $\hat{a}|0\rangle = 0$  and  $\hat{a}^\dagger \hat{a}|0\rangle = \hat{n}|0\rangle = 0$ , we cast the previous formula as

$$\begin{aligned} \left| \frac{x}{\sqrt{2}}; r \right\rangle &= \frac{1}{\sqrt{\mu}} \exp \left( \frac{x^2}{4} \right) \exp \left( -\frac{x}{\sqrt{2}} \hat{a} \right) \\ &\times \exp \left( \frac{x}{\sqrt{2}} \hat{a}^\dagger \right) \exp \left( -\frac{\nu}{2\mu} \hat{a}^{\dagger 2} \right) |0\rangle. \end{aligned} \quad (24)$$

Inserting twice the identity operator, written as  $\hat{I} = \exp \left( \frac{x}{\sqrt{2}} \hat{a} \right) \exp \left( -\frac{x}{\sqrt{2}} \hat{a} \right)$ , we obtain

$$\begin{aligned} \left| \frac{x}{\sqrt{2}}; r \right\rangle &= \frac{1}{\sqrt{\mu}} e^{\frac{x^2}{4}} e^{-\frac{x}{\sqrt{2}} \hat{a}} e^{\frac{x}{\sqrt{2}} \hat{a}^\dagger} \\ &\times e^{\frac{x}{\sqrt{2}} \hat{a}} e^{-\frac{x}{\sqrt{2}} \hat{a}} e^{-\frac{\nu}{2\mu} \hat{a}^{\dagger 2}} e^{\frac{x}{\sqrt{2}} \hat{a}} e^{-\frac{x}{\sqrt{2}} \hat{a}} |0\rangle. \end{aligned} \quad (25)$$

It is clear that  $\exp \left( -\frac{x}{\sqrt{2}} \hat{a} \right) |0\rangle = |0\rangle$ , and using the Hadamard’s lemma [14], it is easy to prove that

$$\exp(-\gamma \hat{a}) \eta(\hat{a}^\dagger) \exp(\gamma \hat{a}) = \eta(\hat{a}^\dagger - \gamma), \quad (26)$$

for any well behaved function  $\eta(\hat{a}^\dagger)$ ; thus

$$\begin{aligned} \left| \frac{x}{\sqrt{2}}; r \right\rangle &= \frac{1}{\sqrt{\mu}} \exp \left( \frac{x^2}{4} \right) \exp \left[ \frac{x}{\sqrt{2}} \left( \hat{a}^\dagger - \frac{x}{\sqrt{2}} \right) \right] \\ &\times \exp \left[ -\frac{\nu}{2\mu} \left( \hat{a}^\dagger - \frac{x}{\sqrt{2}} \right)^2 \right] |0\rangle. \end{aligned} \quad (27)$$

After some algebra,

$$\begin{aligned} \left| \frac{x}{\sqrt{2}}; r \right\rangle &= \frac{1}{\sqrt{\mu}} \exp \left[ -\frac{x^2}{4} \left( 1 + \frac{\nu}{\mu} \right) \right] \\ &\times \exp \left[ -\frac{\nu}{2\mu} \hat{a}^{\dagger 2} + \frac{x}{\sqrt{2}} \left( 1 + \frac{\nu}{\mu} \right) \hat{a}^\dagger \right] |0\rangle. \end{aligned} \quad (28)$$

We take now the limit when  $r \rightarrow \infty$ , or  $\frac{\nu}{\mu} \rightarrow 1$ , so

$$|x\rangle_p \propto \exp \left( -\frac{x^2}{2} \right) \exp \left( -\frac{\hat{a}^{\dagger 2}}{2} + \sqrt{2}x\hat{a}^\dagger \right) |0\rangle. \quad (29)$$

We get an expression that gives us the position eigenstates as an operator applied to the vacuum. Unlike the Yuen case, expression (21), now we have an  $x$  dependence and it looks like a better candidate to be the position eigenstate. In fact, in the next Section, we will show that this really is an eigenstate of the position.

### 4. Leaving squeezed states aside

We will try now an alternative approach to the eigenstates of the position. We can write a position eigenstate, simply by multiplying it by a proper unit operator

$$|x\rangle_p = \sum_{n=0}^{\infty} |n\rangle \langle n|x\rangle_p \quad (30)$$

Therefore the position eigenstate  $|x\rangle_p$  may be written as [17]

$$|x\rangle_p = \sum_{n=0}^{\infty} \psi_n(x) |n\rangle \quad (31)$$

with  $\psi_n(x) = \frac{1}{\sqrt{2^n \sqrt{\pi} n!}} e^{-x^2/2} H_n(x)$ ; such that  $|x\rangle_p$  may be re-written as

$$|x\rangle_p = \frac{e^{-x^2/2}}{\pi^{1/4}} \sum_{n=0}^{\infty} \frac{1}{2^{n/2} n!} H_n(x) \hat{a}^{\dagger n} |0\rangle, \quad (32)$$

that may be added via using the generating function for Hermite polynomials [18]

$$e^{-t^2 + 2tx} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}, \quad (33)$$

to give

$$|x\rangle_p = \frac{e^{-x^2/2}}{\pi^{1/4}} e^{-\frac{\hat{a}^{\dagger 2}}{2} + \sqrt{2}x\hat{a}^\dagger} |0\rangle. \quad (34)$$

The above expression allows us to write the position eigenstate as an operator applied to the vacuum. Note that this expression is the same as the one obtained using the Caves definition for the squeezed states, formula (28). We prove now that indeed (32) is an eigenvector of the position operator; for that, we write the position operator as  $\hat{x} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}$ , thus

$$\hat{x}|x\rangle_p = \frac{e^{-x^2/2}}{\pi^{1/4} \sqrt{2}} (\hat{a} + \hat{a}^\dagger) e^{-\frac{\hat{a}^{\dagger 2}}{2} + \sqrt{2}x\hat{a}^\dagger} |0\rangle. \quad (35)$$

Inserting the identity operator in the above expression as  $\hat{I} = e^{-\frac{\hat{a}^{\dagger 2}}{2}} e^{\sqrt{2}x\hat{a}^\dagger} e^{-\sqrt{2}x\hat{a}^\dagger} e^{\frac{\hat{a}^{\dagger 2}}{2}}$ , we get

$$\begin{aligned} \hat{x}|x\rangle_p &= \frac{e^{-x^2/2}}{\pi^{1/4} \sqrt{2}} e^{-\frac{\hat{a}^{\dagger 2}}{2}} e^{\sqrt{2}x\hat{a}^\dagger} e^{-\sqrt{2}x\hat{a}^\dagger} \\ &\times e^{\frac{\hat{a}^{\dagger 2}}{2}} (\hat{a} + \hat{a}^\dagger) e^{-\frac{\hat{a}^{\dagger 2}}{2}} e^{\sqrt{2}x\hat{a}^\dagger} |0\rangle; \end{aligned} \quad (36)$$

as  $e^{\frac{\hat{a}^\dagger^2}{2}} (\hat{a} + \hat{a}^\dagger) e^{-\frac{\hat{a}^\dagger^2}{2}} = \hat{a} - \hat{a}^\dagger + \hat{a}^\dagger = \hat{a}$ ,  
 $e^{\frac{\hat{a}^\dagger^2}{2}} (\hat{a} + \hat{a}^\dagger) e^{-\frac{\hat{a}^\dagger^2}{2}} = \hat{a} - \hat{a}^\dagger + \hat{a}^\dagger = \hat{a}$ , and  $\hat{a}|0\rangle = 0$ ,  
 we obtain

$$\hat{x}|x\rangle_p = x \frac{e^{-x^2/2}}{\pi^{1/4}} e^{-\frac{\hat{a}^\dagger^2}{2}} e^{\sqrt{2}x\hat{a}^\dagger} |0\rangle = x|x\rangle_p, \quad (37)$$

as we wanted to show.

We can write (32) in terms of coherent states. We have

$$\begin{aligned} e^{\sqrt{2}x\hat{a}^\dagger} |0\rangle &= \sum_{k=0}^{\infty} \frac{1}{k!} (\sqrt{2}x)^k \hat{a}^{\dagger k} |0\rangle \\ &= \sum_{k=0}^{\infty} \frac{(\sqrt{2}x)^k}{\sqrt{k!}} |k\rangle = e^{x^2} |\sqrt{2}x\rangle, \end{aligned} \quad (38)$$

thus

$$|x\rangle_p = \frac{e^{x^2/2}}{\pi^{1/4}} e^{-\frac{\hat{a}^\dagger^2}{2}} |\sqrt{2}x\rangle. \quad (39)$$

With the expressions obtained, it is easy to show that the squeezed states have the form of a Gaussian wave packet. To confirm this, we use the above expression to state that

$$\begin{aligned} \langle\alpha; r|x\rangle_p &= \langle\alpha|\hat{S}^\dagger(r)|x\rangle_p \\ &= \frac{e^{x^2/2}}{\pi^{1/4}} \langle\alpha|\hat{S}^\dagger(r)e^{-\frac{\hat{a}^\dagger^2}{2}}|\sqrt{2}x\rangle. \end{aligned} \quad (40)$$

We write  $\hat{S}^\dagger(r)e^{-\frac{\hat{a}^\dagger^2}{2}}$  as  $e^{-\frac{\hat{a}^\dagger^2}{2}} e^{\frac{\hat{a}^\dagger^2}{2}} \hat{S}^\dagger(r) e^{-\frac{\hat{a}^\dagger^2}{2}}$ , where we have just inserted the identity operator  $\hat{I} = e^{-\frac{\hat{a}^\dagger^2}{2}} e^{\frac{\hat{a}^\dagger^2}{2}}$ , and we use that  $e^{\frac{\hat{a}^\dagger}{2}} \eta(\hat{a}) e^{-\frac{\hat{a}^\dagger}{2}} = \eta(\hat{a} - \hat{a}^\dagger)$ , for any well behaved function  $\eta$ , to obtain

$$\langle\alpha; r|x\rangle_p = \frac{\exp\left[\frac{1}{2}(x^2-r)\right]}{\pi^{1/4}} \langle\alpha|e^{-\frac{\hat{a}^\dagger^2}{2}} e^{\frac{r}{2}\hat{a}^2 - r\hat{a}^\dagger\hat{a}}|\sqrt{2}x\rangle. \quad (41)$$

As the coherent states  $|\alpha\rangle$  are eigenfunctions of the annihilation operator  $\hat{a}$ , it is very easy to show that  $\langle\alpha|e^{-\frac{\hat{a}^\dagger^2}{2}} = \langle\alpha|e^{-\frac{\alpha^{*2}}{2}}$ , so

$$\begin{aligned} \langle\alpha; r|x\rangle_p &= \frac{\exp\left[\frac{1}{2}(x^2 - \alpha^{*2} - r)\right]}{\pi^{1/4}} \\ &\times \langle\alpha|e^{\frac{r}{2}\hat{a}^2 - r\hat{a}^\dagger\hat{a}}|\sqrt{2}x\rangle. \end{aligned} \quad (42)$$

In the Appendix, we disentangle the operator  $e^{\frac{r}{2}\hat{a}^2 - r\hat{a}^\dagger\hat{a}}$  as  $e^{-r\hat{a}^\dagger\hat{a}} e^{\frac{1-e^{2r}}{4}\hat{a}^2}$ , and we get

$$\begin{aligned} \langle\alpha; r|x\rangle_p &= \frac{\exp\left[\frac{1}{2}(x^2 - \alpha^{*2} - r)\right]}{\pi^{1/4}} \\ &\times \langle\alpha|e^{-r\hat{a}^\dagger\hat{a}} e^{\frac{1-e^{2r}}{4}\hat{a}^2}|\sqrt{2}x\rangle. \end{aligned} \quad (43)$$

It is very easy to see that  $e^{\frac{1-e^{2r}}{4}\hat{a}^2}|\sqrt{2}x\rangle = e^{\frac{1-e^{2r}}{2}x^2}|\sqrt{2}x\rangle$ , and that  $e^{i\gamma\hat{n}}|\alpha\rangle = |e^{i\gamma}\alpha\rangle$ , thus

$$\begin{aligned} \langle\alpha; r|x\rangle_p &= \frac{1}{\pi^{1/4}} \exp\left\{\frac{1}{2}\left[(2-e^{2r})x^2 - \alpha^{*2} - r\right]\right\} \\ &\times \langle\alpha|\sqrt{2}e^{-r}x\rangle. \end{aligned} \quad (44)$$

Finally, as  $\langle\delta|\epsilon\rangle = e^{-\frac{1}{2}(|\delta|^2+|\epsilon|^2-2\delta^*\epsilon)}$ , we have

$$\begin{aligned} \langle\alpha; r|x\rangle_p &= \frac{1}{\pi^{1/4}} \exp\left\{\frac{1}{2}\left[(2-e^{2r}-2e^{-2r})x^2\right.\right. \\ &\left.\left.+ 2\sqrt{2}\alpha^*e^{-r}x - \alpha^{*2} - |\alpha|^2 - r\right]\right\}, \end{aligned} \quad (45)$$

as we wanted to show.

### 5. The Husimi Q-function

We can now find the wave function of a coherent state as a function of the position [19]. We use equation (32), that express the eigenstates of the position as an operator acting on the vacuum, and get that

$$\begin{aligned} \langle\beta|x\rangle_p &= \frac{e^{-x^2/2}}{\pi^{1/4}} \langle\beta|e^{-\frac{\hat{a}^\dagger^2}{2} + \sqrt{2}x\hat{a}^\dagger}|0\rangle \\ &= \frac{e^{-\frac{x^2}{2}}}{\sqrt[4]{\pi}} e^{-\frac{\beta^{*2}}{2} + \sqrt{2}\beta^*x} \langle\beta|0\rangle \\ &= \frac{e^{-\frac{x^2}{2}}}{\pi^{1/4}} e^{-\frac{\beta^{*2}}{2} - \frac{|\beta|^2}{2} + \sqrt{2}\beta^*x}, \end{aligned} \quad (46)$$

as  $\langle\beta|\hat{a}^\dagger = \beta^*\langle\beta|$  and  $\langle\beta|n\rangle = e^{-\frac{|\beta|^2}{2}} \frac{\beta^{*n}}{\sqrt{n!}}$ .

The Husimi Q-function [20] can be calculated from (45) simply as

$$Q(\beta) = \frac{1}{\pi} \left| \langle\beta|x\rangle_p \right|^2 = \frac{e^{-x^2} e^{-|\beta|^2}}{\pi^{3/2}} \left| e^{-\frac{\beta^{*2}}{2} + \sqrt{2}\beta^*x} \right|^2, \quad (47)$$

that after some algebra, can be re-written as

$$\begin{aligned} Q(\beta) &= \frac{1}{\pi^{3/2}} \\ &\times \exp\left[-x^2 - |\beta|^2 - \text{Re}(\beta^{*2}) + 2\sqrt{2}\text{Re}(\beta)x\right]. \end{aligned} \quad (48)$$

In the figures, we plot the Husimi Q-function for different values of  $x$ .

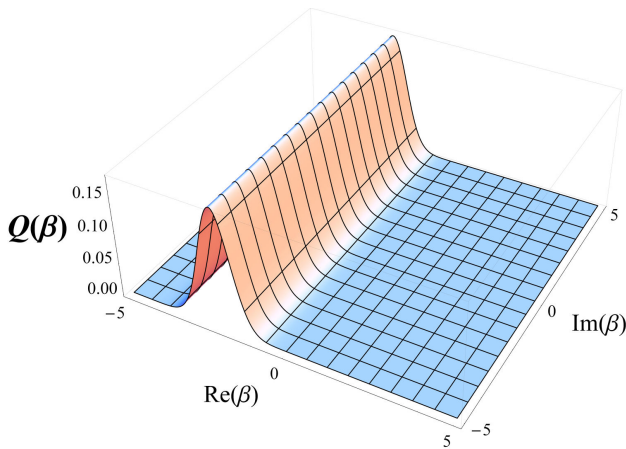


FIGURE 1. The Husimi  $Q$ -function for  $x = -3$ .

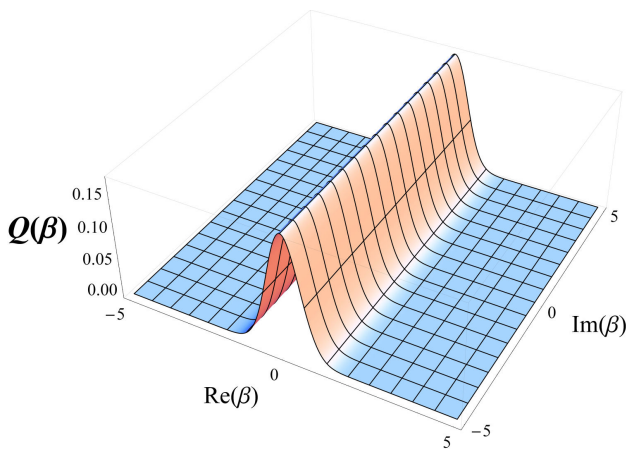


FIGURE 2. The Husimi  $Q$ -function for  $x = 0$ .

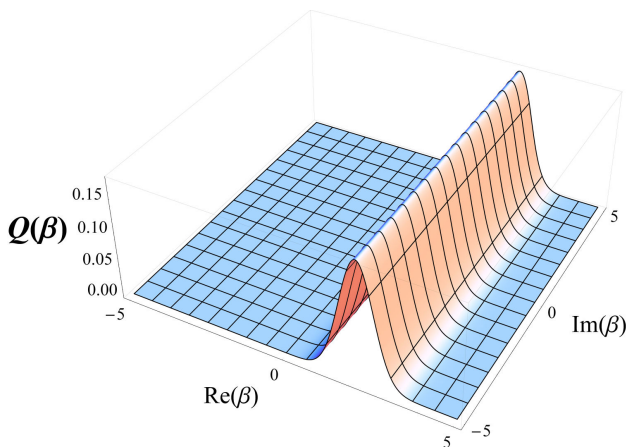


FIGURE 3. The Husimi  $Q$ -function for  $x = 3$ .

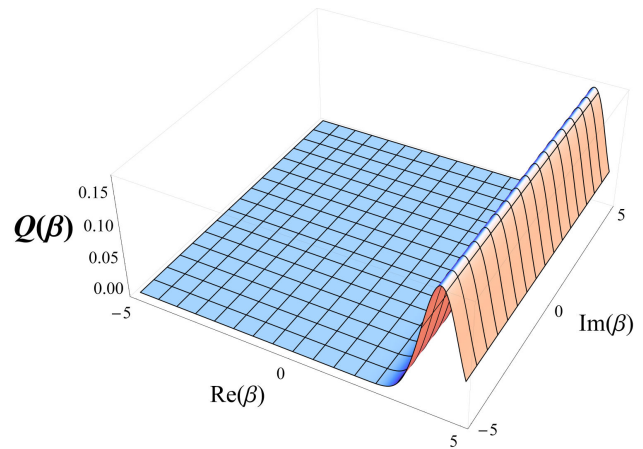


FIGURE 4. The Husimi  $Q$ -function for  $x = 6$ .

### 6. Conclusions

We have found an operator that applied to the vacuum gives us the eigenstates of the position. We did that by two different ways; first, using the Caves definition of the squeezed states, we took the limit of extreme squeezing in the position side, to get the position eigenstate. Second, we used the expansion of an arbitrary wave function in the base of the harmonic oscillator; *i.e.*, we wrote an arbitrary wave function in terms of Hermite polynomials. The expressions obtained allows us to show certain properties of squeezed states, and also allow us to write in a very easy way the Husimi  $Q$ -function of the position eigenstates. The same procedure can be followed to find the eigenstates of the momentum, but taken the limit when the squeeze parameters goes to  $-\infty$ .

We can also conclude that from the point of view of this work, the Caves approach to define squeezed states is more adequate, because it gives the correct eigenstates of the position; while the Yuen definition, formula (1), gives an expression that is incorrect. So, we must first squeeze the vacuum, and after, displace it.

### A Appendix

In this appendix, we show how to disentangle the operator  $e^{-\frac{r}{2}\hat{a}^2+r\hat{a}^\dagger\hat{a}}$ . We define

$$\hat{F}(r) \equiv e^{-\frac{r}{2}\hat{a}^2+r\hat{a}^\dagger\hat{a}}, \tag{49}$$

and we suppose that (48) can be rewritten as

$$\hat{F}(r) = \exp [f(r)\hat{a}^\dagger\hat{a}] \exp [g(r)\hat{a}^2], \tag{50}$$

where  $f(r)$  and  $g(r)$  are two unknown well behaved functions; as  $\hat{F}(0) = \hat{I}$ , being  $\hat{I}$  the identity operator, these functions must satisfy the conditions  $f(0) = g(0) = 0$ . At first sight, one can think that in the proposal (45) should be a term of the form  $\exp [h(r)\hat{a}^{\dagger 2}]$ ; however, this is not the case

because  $[\hat{a}^2, \hat{a}^\dagger \hat{a}] = 2\hat{a}^2$ . We differentiate with respect to  $r$ , so to find

$$\begin{aligned} \frac{d\hat{F}}{dr} &= \frac{df}{dr} \hat{a}^\dagger \hat{a} \exp [f\hat{a}^\dagger \hat{a}] \exp [g\hat{a}^2] \\ &+ \frac{dg}{dr} \exp [f\hat{a}^\dagger \hat{a}] \hat{a}^2 \exp [g\hat{a}^2], \end{aligned} \quad (51)$$

where for simplicity in the notation, we have dropped all  $r$ -dependency; we write  $\hat{I} = \exp [-f\hat{a}^\dagger \hat{a}] \exp [f\hat{a}^\dagger \hat{a}]$  for the identity operator in the second term, to obtain

$$\begin{aligned} \frac{d\hat{F}}{dr} &= \frac{df}{dr} \hat{a}^\dagger \hat{a} \exp [f\hat{a}^\dagger \hat{a}] \exp [g\hat{a}^2] + \frac{dg}{dr} \exp [f\hat{a}^\dagger \hat{a}] \hat{a}^2 \\ &\times \exp [-f\hat{a}^\dagger \hat{a}] \exp [f\hat{a}^\dagger \hat{a}] \exp [g\hat{a}^2]. \end{aligned} \quad (52)$$

Using the Hadamard's lemma [14, 15], it is very easy to prove that

$$\exp [f\hat{a}^\dagger \hat{a}] \hat{a}^2 \exp [-f\hat{a}^\dagger \hat{a}] = e^{-2f} \hat{a}^2, \quad (53)$$

$$\frac{d\hat{F}}{dr} = \left( \frac{df}{dr} \hat{a}^\dagger \hat{a} + \frac{dg}{dr} e^{-2f} \hat{a}^2 \right) \hat{F}. \quad (54)$$

Equating this equation to the one obtained differentiating the original formula for  $\hat{F}(r)$ , equation (44), we get the following system of first order ordinary differential equations

$$\frac{df}{dr} = 1, \quad \frac{dg}{dr} e^{-2f} = -\frac{1}{2}. \quad (55)$$

The solution of the first equation, that satisfies the initial condition  $f(0) = 0$ , is the function  $f(r) = r$ . Substituting this solution in the second equation and solving it with the initial condition  $g(0) = 0$ , we obtain  $g(r) = \frac{1-e^{2r}}{4}$ . Thus, finally we write

$$e^{-\frac{r}{2}\hat{a}^2 + r\hat{a}^\dagger \hat{a}} = e^{r\hat{a}^\dagger \hat{a}} e^{\frac{1-e^{2r}}{4}\hat{a}^2}. \quad (56)$$

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