Variational symmetries of Lagrangians

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We present an elementary derivation of the equation for the infinitesimal generators of variational symmetries of a Lagrangian for a system with a finite number of degrees of freedom. We also give a simple proof of the existence of an infinite number of Lagrangians for a given second-order ordinary differential equation.

Keywords: Lagrangians; symmetries; constants of motion; ordinary differential equations.

Presentamos una derivación elemental de la ecuación para los generadores infinitesimales de simetrías variacionales de una lagrangiana para un sistema con un número finito de grados de libertad. Damos también una prueba simple de la existencia de un número infinito de lagrangianas para una ecuación diferencial ordinaria de segundo orden dada.

Descriptores: Lagrangianas; simetrías; constantes de movimiento; ecuaciones diferenciales ordinarias.

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1. Introduction

In classical mechanics, the Lagrangian of a given mechanical system leads to its equations of motion, and the identification of the continuous symmetries of the Lagrangian allows one to find constants of motion (or first integrals). However, the identification of such symmetries is often based on the existence of ignorable coordinates, which depends on the coordinates chosen, and the symmetries commonly considered are restricted to rotations and translations (see, *e.g.*, Refs. [1–3]).

The Lagrangian formalism is also useful in many other areas. Any second-order ordinary differential equation (ODE) can be seen as the Euler–Lagrange equation for some Lagrangian (in fact, for an infinite number of Lagrangians) and many systems of second-order ODEs can be derived from a Lagrangian.

It turns out that a Lagrangian may possess many non-trivial continuous symmetries and, what is more relevant, in many cases, some of them can be readily found by solving an equation applicable in any coordinate system. The aim of this paper is to present an elementary derivation of the equation that determines the so-called variational symmetries of a Lagrangian, and of the expression for the constants of motion associated with these symmetries. The results presented here are applicable to any system of second-order ODEs derivable from a Lagrangian, and to any second-order ODE, not necessarily related to classical mechanics.

In Sec. 2 we consider systems with one degree of freedom (or a single second-order ODE), deriving the basic equations that determine the variational symmetries of a Lagrangian and the corresponding first integrals. In order to apply these results to any second-order ODE, we show how to find a Lagrangian for a given second-order ODE. It may be remarked that even though, at least since the nineteenth century, it is known that any second-order ODE possesses an infinite number of Lagrangians, this result is not presented in the standard textbooks on classical mechanics (see, however, Ref. [4]). In Sec. 3 the formulas applicable to the case with an arbitrary number of degrees of freedom are given. Throughout this paper various examples are given, illustrating the concepts and methods introduced here.

2. Systems with one degree of freedom

In order to present the ideas in a simple way, it is convenient to consider firstly the case where there is only one degree of freedom, or we have a single second-order ODE.

2.1. Variational symmetries of a Lagrangian

We shall consider one-parameter families of transformations

$$x' = x'(x,t,s), \qquad t' = t'(x,t,s), \tag{1}$$

where s is a parameter that takes values in some neighborhood of zero, and we shall assume that the transformation (1) reduces to the identity for $s = 0$; that is $x'(x,t,0) = x$ and $t'(x,t,0) = t$. For a fixed value of s, Eqs. (1) give a transformation from the plane (x,t) into the plane (x',t'). Such transformations are called *point transformations* (see, *e.g.*, Refs. [5–8]; more general transformations are also useful, see, *e.g.*, Refs. [5, 6, 9]). Some examples of one-parameter families of point transformations are

$$x' = x\mathrm{e}^{s} - \frac{1}{2}gt^2(\mathrm{e}^{3s} - \mathrm{e}^{s}), \qquad t' = t\mathrm{e}^{3s/2}, \tag{2}$$

where g is a constant,

$$x' = \frac{x}{1-ts} - \frac{gt^3 s}{2(1-ts)^2}, \qquad t' = \frac{t}{1-ts}, \tag{3}$$

and

$$x' = x\mathrm{e}^{-s}, \qquad t' = t\mathrm{e}^{2s}. \tag{4}$$

It may be noticed that the transformations (2) and (4) are defined for all $s \in \mathbb{R}$, but (3) is defined only for $s \neq 1/t$.

Actually, Eqs. (2)–(4) are examples of local one-parameter *groups* of transformations, which means that, if we define $\varphi_s(x,t) \equiv (x',t')$, then

$$\varphi_s(\varphi_u(x,t)) = \varphi_{s+u}(x,t), \tag{5}$$

for all values of s and u such that both sides of the equation are defined. For instance, in the case of the transformations (4), $\varphi_s(x,t) = (x\mathrm{e}^{-s}, t\mathrm{e}^{2s})$; hence, $\varphi_s(\varphi_u(x,t)) = \varphi_s(x\mathrm{e}^{-u}, t\mathrm{e}^{2u}) = ((x\mathrm{e}^{-u})\,\mathrm{e}^{-s}, (t\mathrm{e}^{2u})\,\mathrm{e}^{2s}) = (x\mathrm{e}^{-s-u}, t\mathrm{e}^{2s+2u}) = \varphi_{s+u}(x,t)$. These one-parameter groups of transformations arise in a natural way in the solution of systems of first-order ODEs (see the examples below).

We shall say that the one-parameter family of transformations (1) is a *variational symmetry* of a given Lagrangian $L(x,\dot{x},t)$ if

$$L\left(x', \frac{\mathrm{d}x'}{\mathrm{d}t'}, t'\right)\frac{\mathrm{d}t'}{\mathrm{d}t} = L\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}, t\right) + \frac{\mathrm{d}}{\mathrm{d}t}F(x,t,s), \tag{6}$$

for all values of s for which the transformation is defined, where $F(x,t,s)$ is some function. Some authors reserve the name variational symmetry for the point transformations satisfying Eq. (6) without the last term of the right-hand side (*e.g.*, Refs. [5–8]), and the point transformations satisfying (6) with $\mathrm{d}F/\mathrm{d}t \neq 0$ are sometimes called Noether symmetries [5] or divergence symmetries [6]. As we shall show below, each one-parameter family of point transformations satisfying Eq. (6) yields a constant of motion for the ODE given by the Lagrangian L.

More precisely, a transformation satisfying Eq. (6) maps any solution of the Euler–Lagrange equation corresponding to L, into another solution, which follows from the fact that the Euler–Lagrange equation determines the local extrema of the integral

$$\int_{t_0}^{t_1} L\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}, t\right)\,\mathrm{d}t,$$

with fixed endpoints (x_0,t_0), (x_1,t_1) [1–3, 8]. Condition (6) amounts to

$$\int_{t'_0}^{t'_1} L\left(x', \frac{\mathrm{d}x'}{\mathrm{d}t'}, t'\right)\,\mathrm{d}t' = \int_{t_0}^{t_1} L\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}, t\right)\,\mathrm{d}t + \int_{t_0}^{t_1} \frac{\mathrm{d}}{\mathrm{d}t}F(x,t,s)\,\mathrm{d}t, \tag{7}$$

where t'_0, t'_1 are the values of t' corresponding to the points (x_0,t_0), (x_1,t_1), respectively, according to the transformation (1). The last term on the right-hand side of Eq. (7) is equal to the difference of the values of F at the endpoints, and it is therefore a constant when one considers curves with the same endpoints. Hence, a curve that minimizes (or maximizes) the first term on the right-hand side of Eq. (7) is mapped into a curve that minimizes (or maximizes) the integral on the left-hand side.

For instance, the family of transformations (2) is a variational symmetry of the Lagrangian

$$L(x,\dot{x},t) = \frac{1}{6}\dot{x}^3 + \frac{1}{2}g\dot{x}^2 t - g^2 xt, \tag{8}$$

where g is a constant [the constant appearing in Eqs. (2)]. In fact, treating the derivative as a quotient of differentials (or, equivalently, using the chain rule), from Eqs. (2) we have

$$\frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\mathrm{e}^s\mathrm{d}x - (\mathrm{e}^{3s}-\mathrm{e}^s)gt\mathrm{d}t}{\mathrm{e}^{3s/2}\mathrm{d}t} = \mathrm{e}^{-s/2}\dot{x} - (\mathrm{e}^{3s/2} - \mathrm{e}^{-s/2})gt,$$

hence,

$$\begin{aligned} L\left(x', \frac{\mathrm{d}x'}{\mathrm{d}t'}, t'\right)\frac{\mathrm{d}t'}{\mathrm{d}t} &= \left\{\frac{1}{6}\left[\mathrm{e}^{-s/2}\dot{x} - \left(\mathrm{e}^{3s/2}-\mathrm{e}^{-s/2}\right)gt\right]^3 \right. \\ &\quad + \frac{1}{2}g\left[\mathrm{e}^{-s/2}\dot{x} - \left(\mathrm{e}^{3s/2}-\mathrm{e}^{-s/2}\right)gt\right]^2 t\mathrm{e}^{3s/2} \\ &\quad \left. - g^2\left[x\mathrm{e}^s - \frac{1}{2}gt^2\left(\mathrm{e}^{3s}-\mathrm{e}^s\right)\right]t\mathrm{e}^{3s/2}\right\}\mathrm{e}^{3s/2} \\ &= L\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}, t\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}g^2t^2x\left(1-\mathrm{e}^{4s}\right)\right. \\ &\quad \left. + \frac{1}{24}g^3t^4\left(1 - 6\mathrm{e}^{4s} + 5\mathrm{e}^{6s}\right)\right]. \end{aligned}$$

(Note that s is a parameter, that does not depend on t.)

On the other hand, the transformations (4) are variational symmetries for the Lagrangian

$$L(x,\dot{x},t) = \frac{t^2}{2}\left(\dot{x}^2 - \frac{x^6}{3}\right). \tag{9}$$

Indeed, from Eqs. (4) we have

$$\frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\mathrm{e}^{-s}\mathrm{d}x}{\mathrm{e}^{2s}\mathrm{d}t} = \mathrm{e}^{-3s}\dot{x},$$

and, therefore,

$$\begin{aligned} L\left(x', \frac{\mathrm{d}x'}{\mathrm{d}t'}, t'\right)\frac{\mathrm{d}t'}{\mathrm{d}t} &= \frac{(t\mathrm{e}^{2s})^2}{2}\left[(\mathrm{e}^{-3s}\dot{x})^2 - \frac{(x\mathrm{e}^{-s})^6}{3}\right]\mathrm{e}^{2s} \\ &= \frac{t^2}{2}\left(\dot{x}^2 - \frac{x^6}{3}\right), \end{aligned}$$

showing that Eq. (6) holds with $F = 0$.

Finding the variational symmetries of a given Lagrangian, making use of the definition (6), is not an easy task. However, as we shall see, this problem is simplified if we start looking for the *infinitesimal generators* of such symmetries. Indeed, differentiating both sides of Eq. (6) with respect to s, at $s = 0$, making use of the chain rule and the definitions

$$\eta(x,t) \equiv \left.\frac{\partial x'(x,t,s)}{\partial s}\right|_{s=0}, \qquad \xi(x,t) \equiv \left.\frac{\partial t'(x,t,s)}{\partial s}\right|_{s=0}, \tag{10}$$

we obtain

$$\frac{\partial L}{\partial x}\eta + \frac{\partial L}{\partial \dot{x}}\left(\frac{\partial}{\partial s}\frac{\mathrm{d}x'}{\mathrm{d}t'}\right)_{s=0} + \frac{\partial L}{\partial t}\xi + L(x,\dot{x},t)\left(\frac{\partial}{\partial s}\frac{\mathrm{d}t'}{\mathrm{d}t}\right)_{s=0} = \left(\frac{\partial}{\partial s}\frac{\mathrm{d}F}{\mathrm{d}t}\right)_{s=0}. \tag{11}$$

Treating the derivative $\mathrm{d}x'/\mathrm{d}t'$ as a quotient of differentials, using the elementary rules of differentiation and the definitions (10), we see that

$$\left(\frac{\partial}{\partial s}\frac{\mathrm{d}x'}{\mathrm{d}t'}\right)_{s=0} = \left.\frac{(\mathrm{d}t')\frac{\partial}{\partial s}\mathrm{d}x' - (\mathrm{d}x')\frac{\partial}{\partial s}\mathrm{d}t'}{(\mathrm{d}t')^2}\right|_{s=0} = \left.\frac{(\mathrm{d}t')\,\mathrm{d}\left(\frac{\partial x'}{\partial s}\right) - (\mathrm{d}x')\,\mathrm{d}\left(\frac{\partial t'}{\partial s}\right)}{(\mathrm{d}t')^2}\right|_{s=0} = \frac{\mathrm{d}\eta}{\mathrm{d}t} - \dot{x}\frac{\mathrm{d}\xi}{\mathrm{d}t}.$$

On the other hand,

$$\left(\frac{\partial}{\partial s}\frac{\mathrm{d}t'}{\mathrm{d}t}\right)_{s=0} = \left.\frac{\partial}{\partial s}\left(\frac{\partial t'}{\partial t} + \dot{x}\frac{\partial t'}{\partial x}\right)\right|_{s=0} = \left.\left(\frac{\partial}{\partial s}\frac{\partial t'}{\partial t} + \dot{x}\frac{\partial}{\partial s}\frac{\partial t'}{\partial x}\right)\right|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left.\frac{\partial t'}{\partial s}\right|_{s=0} = \frac{\mathrm{d}\xi}{\mathrm{d}t}$$

and, similarly,

$$\left(\frac{\partial}{\partial s}\frac{\mathrm{d}F}{\mathrm{d}t}\right)_{s=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left.\frac{\partial F}{\partial s}\right|_{s=0}.$$

Thus, Eq. (11) amounts to

$$\frac{\partial L}{\partial x}\eta + \frac{\partial L}{\partial \dot{x}}\left(\frac{\mathrm{d}\eta}{\mathrm{d}t} - \dot{x}\frac{\mathrm{d}\xi}{\mathrm{d}t}\right) + \frac{\partial L}{\partial t}\xi + L\frac{\mathrm{d}\xi}{\mathrm{d}t} = \frac{\mathrm{d}G}{\mathrm{d}t}, \tag{12}$$

where

$$G \equiv \left.\frac{\partial F}{\partial s}\right|_{s=0}$$

is some function of (x,t). (When $G = \mathrm{const.}$, condition (12) reduces to Eq. (4.27) of Ref. [7] and Eq. (9.38) of Ref. [8]; *cf.* also Sec. 13.7 of Ref. [3]. As we shall see below, the condition $G = \mathrm{const.}$ is a very strong restriction on the symmetries of a given Lagrangian.)

For a given Lagrangian, $L(x,\dot{x},t)$, Eq. (12) determines the infinitesimal generators (represented by the functions ξ and η) of variational symmetries of L. For a given function $L(x,\dot{x},t)$, Eq. (12) is a partial differential equation for the two functions ξ and η, which depend on (x,t) (recall that, *e.g.*, $\mathrm{d}\eta/\mathrm{d}t = \partial\eta/\partial t + \dot{x}\,\partial\eta/\partial x$). The left-hand side of Eq. (12) is a linear operator acting on the functions ξ and η and, therefore, any linear combination, with constant coefficients, of solutions of Eq. (12) is also a solution of this equation (see, *e.g.*, Eqs. (22), below).

As pointed out above, the interest in the variational symmetries of L comes from the fact that, each pair of functions ξ, η, that satisfies Eq. (12) gives rise to a *constant of motion*; that is, to a function with a total derivative with respect to the time equal to zero, if the Euler–Lagrange equations hold. In fact, from the Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}, \tag{13}$$

and the chain rule, we have

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial \dot{x}}\ddot{x} + \frac{\partial L}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\dot{x}\right) + \frac{\partial L}{\partial t}$$

therefore, Eq. (12) can be written as

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}}\right)\eta + \frac{\partial L}{\partial \dot{x}}\left(\frac{\mathrm{d}\eta}{\mathrm{d}t} - \dot{x}\frac{\mathrm{d}\xi}{\mathrm{d}t}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(L - \frac{\partial L}{\partial \dot{x}}\dot{x}\right)\xi + L\frac{\mathrm{d}\xi}{\mathrm{d}t} = \frac{\mathrm{d}G}{\mathrm{d}t},$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{x}}\eta + \xi\left(L - \frac{\partial L}{\partial \dot{x}}\dot{x}\right) - G\right] = 0, \tag{14}$$

thus showing that the expression inside the brackets is a constant of motion (*cf.* Eq. (10.31) of Ref. [5] and Eq. (10.31) of Ref. [10]). When $G = \mathrm{const.}$, Eq. (14) reduces to Eq. (13.158) of Ref. [3] and Eq. (9.25) of Ref. [8].

2.1.1. Example

As a first example, we shall consider the standard Lagrangian for a particle of mass m in a uniform gravitational field,

$$L(x,\dot{x},t) = \frac{1}{2}m\dot{x}^2 - mgx. \tag{15}$$

Substituting this last expression into Eq. (12) we obtain

$$-mg\eta + m\dot{x}\left(\frac{\partial\eta}{\partial t} + \dot{x}\frac{\partial\eta}{\partial x} - \dot{x}\frac{\partial\xi}{\partial t} - \dot{x}^2\frac{\partial\xi}{\partial x}\right) + \left(\frac{1}{2}m\dot{x}^2 - mgx\right)\left(\frac{\partial\xi}{\partial t} + \dot{x}\frac{\partial\xi}{\partial x}\right) = \frac{\partial G}{\partial t} + \dot{x}\frac{\partial G}{\partial x}.$$

Since η, ξ, and G depend on (x,t) only, the only way in which this last equation can be identically satisfied is that the coefficient of each power of $\dot{x}$ on each side of the equation coincides (recall that we are not using the equations of motion; here x, $\dot{x}$ and t are independent variables). Thus, by equating the coefficients of $\dot{x}^3$, $\dot{x}^2$, $\dot{x}$, and $\dot{x}^0$ on both sides of the equation we obtain the four equations

$$\frac{\partial\xi}{\partial x} = 0, \tag{16}$$

$$\frac{\partial\eta}{\partial x} - \frac{1}{2}\frac{\partial\xi}{\partial t} = 0, \tag{17}$$

$$m\frac{\partial\eta}{\partial t} - mgx\frac{\partial\xi}{\partial x} = \frac{\partial G}{\partial x}, \tag{18}$$

$$-mg\eta - mgx\frac{\partial\xi}{\partial t} = \frac{\partial G}{\partial t}. \tag{19}$$

Equation (16) implies that ξ is some function of t only,

$$\xi = A(t) \tag{20}$$

and from Eq. (17) it follows that

$$\eta = \frac{1}{2}x\frac{\mathrm{d}A}{\mathrm{d}t} + B(t), \tag{21}$$

where $B(t)$ is another function of t only. Using now Eqs. (18) and (19), the equality of the partial derivatives $\partial^2 G/\partial t\partial x$ and $\partial^2 G/\partial x\partial t$ gives

$$m\frac{\partial^2\eta}{\partial t^2} = -mg\frac{\partial\eta}{\partial x} - mg\frac{\partial\xi}{\partial t},$$

that is,

$$\frac{1}{2}x\frac{\mathrm{d}^3A}{\mathrm{d}t^3} + \frac{\mathrm{d}^2B}{\mathrm{d}t^2} = -\frac{3}{2}g\frac{\mathrm{d}A}{\mathrm{d}t}.$$

Since A and B are functions of t only, we have

$$\frac{\mathrm{d}^3A}{\mathrm{d}t^3} = 0, \quad \text{and} \quad \frac{\mathrm{d}^2B}{\mathrm{d}t^2} = -\frac{3}{2}g\frac{\mathrm{d}A}{\mathrm{d}t}.$$

These equations imply that

$$A(t) = c_1t^2 + c_2t + c_3,$$

where c_1, c_2, c_3 are arbitrary constants, and

$$B(t) = -\frac{1}{2}c_1gt^3 - \frac{3}{4}c_2gt^2 + c_4t + c_5,$$

where c_4 and c_5 are two additional arbitrary constants. Substituting these expressions into the previous results we obtain

$$\begin{aligned}\xi &= c_1t^2 + c_2t + c_3,\\ \eta &= c_1\left(xt - \frac{1}{2}gt^3\right) + c_2\left(\frac{1}{2}x - \frac{3}{4}gt^2\right) + c_4t + c_5\end{aligned} \tag{22}$$

showing that in this case the solution of Eq. (12) contains five arbitrary constants. Making use Eqs. (18) and (19) we find that, up to an irrelevant constant term,

$$\begin{aligned}G = {}& c_1\left(\frac{1}{2}mx^2 - \frac{3}{2}mgt^2x + \frac{1}{8}mg^2t^4\right)\\ &+ c_2\left(-\frac{3}{2}mgtx + \frac{1}{4}mg^2t^3\right)\\ &+ c_4\left(mx - \frac{1}{2}mgt^2\right) + c_5(-mgt).\end{aligned} \tag{23}$$

It may be remarked that, if one assumes that G is equal to zero, or a trivial constant, then $c_1 = c_2 = c_4 = c_5 = 0$, and, instead of the five-parameter family of symmetries obtained above, one is left with just a one-parameter group of variational symmetries of the Lagrangian (15), which is the obvious one ($x' = x$, $t' = t + s$), which is related to the fact that L does not depend explicitly on the time.

The constant of motion associated with the infinitesimal generator (22) is [see Eq. (14)]

$$\begin{aligned}&c_1m\left(x\dot{x}t - \frac{1}{2}g\dot{x}t^3 - \frac{1}{2}\dot{x}^2t^2 - \frac{1}{2}x^2 + \frac{1}{2}gxt^2 - \frac{1}{8}g^2t^4\right)\\ &+ c_2m\left(\frac{1}{2}x\dot{x} - \frac{3}{4}g\dot{x}t^2 - \frac{1}{2}\dot{x}^2t + \frac{1}{2}gxt - \frac{1}{4}g^2t^3\right)\\ &+ c_3m\left(-\frac{1}{2}\dot{x}^2 - gx\right) + c_4m\left(\dot{x}t - x + \frac{1}{2}gt^2\right)\\ &+ c_5m(\dot{x} + gt).\end{aligned} \tag{24}$$

Since the constants $c_1, \ldots, c_5$ are arbitrary, each of the functions inside the parentheses is a constant of motion, though they cannot be functionally independent; for a second-order ODE, there are only two functionally independent first integrals. In this case, any constant of motion must be some function of, *e.g.*,

$$\varphi_1 \equiv \dot{x} + gt, \qquad \varphi_2 \equiv x - \dot{x}t - \frac{1}{2}gt^2. \tag{25}$$

The values of φ_1 and φ_2 are the values of $\dot{x}$ and x at $t = 0$, respectively. It may be noticed that the constant of motion multiplying c_3 is minus the total energy.

The functions ξ and η can be conveniently combined in the linear partial differential operator

$$\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}. \tag{26}$$

This combination is invariant under coordinate transformations and constitutes a *vector field*, in the terminology of the theory of differentiable manifolds (see, *e.g.*, Refs. [6, 11]). Substituting Eqs. (22) into Eq. (26) we obtain the vector field

$$\begin{aligned}\mathbf{X} = {}& c_1\left[t^2\frac{\partial}{\partial t} + \left(xt - \frac{1}{2}gt^3\right)\frac{\partial}{\partial x}\right]\\ &+ c_2\left[t\frac{\partial}{\partial t} + \left(\frac{1}{2}x - \frac{3}{4}gt^2\right)\frac{\partial}{\partial x}\right]\\ &+ c_3\frac{\partial}{\partial t} + c_4t\frac{\partial}{\partial x} + c_5\frac{\partial}{\partial x},\end{aligned}$$

which is a linear combination of the five vector fields

$$\begin{aligned}&\mathbf{X}_1 \equiv t^2\frac{\partial}{\partial t} + \left(xt - \frac{1}{2}gt^3\right)\frac{\partial}{\partial x},\\ &\mathbf{X}_2 \equiv t\frac{\partial}{\partial t} + \left(\frac{1}{2}x - \frac{3}{4}gt^2\right)\frac{\partial}{\partial x},\\ &\mathbf{X}_3 \equiv \frac{\partial}{\partial t}, \qquad \mathbf{X}_4 \equiv t\frac{\partial}{\partial x}, \qquad \mathbf{X}_5 \equiv \frac{\partial}{\partial x}.\end{aligned} \tag{27}$$

These vector fields form a basis for the infinitesimal generators of the variational symmetries of the Lagrangian (15) and also form a basis of a Lie algebra, with the bracket given by the commutator, but this fact will not be elaborated here.

Once we have obtained the infinitesimal generator of a variational symmetry of a Lagrangian, we can use it directly to find a first integral [by means of Eq. (14)], without the need to know a one-parameter family of transformations corresponding to that infinitesimal generator. However, it is always possible, in principle, to find a unique local one-parameter *group* of transformations whose infinitesimal generator is defined by a given pair of functions $\eta(x,t)$ and $\xi(x,t)$. All we have to do is to find the solution of the system of first-order ODEs

$$\frac{\mathrm{d}x'}{\mathrm{d}s} = \eta(x',t'), \qquad \frac{\mathrm{d}t'}{\mathrm{d}s} = \xi(x',t'), \tag{28}$$

with the initial condition (x,t). Equation (28) follows from the definition (10) written as

$$\eta(x'(x,t,s),t'(x,t,s))\Big|_{s=0} = \left.\frac{\partial x'(x,t,s)}{\partial s}\right|_{s=0},$$
$$\xi(x'(x,t,s),t'(x,t,s))\Big|_{s=0} = \left.\frac{\partial t'(x,t,s)}{\partial s}\right|_{s=0},$$

demanding that the equalities hold for all values of s (not only for $s=0$), treating x and t as parameters (that specify the initial conditions). For example, in the case of the vector field $\mathbf{X}_1$ [see Eq. (27)], the system of equations (28) takes the form

$$\frac{\mathrm{d}x'}{\mathrm{d}s} = x't' - \frac{1}{2}gt'^3, \qquad \frac{\mathrm{d}t'}{\mathrm{d}s} = t'^2. \tag{29}$$

From the second of these last equations (separating variables) we obtain

$$-\frac{1}{t'} = s + \text{const}.$$

and the integration constant is determined by the condition that $t' = t$ at $s = 0$; hence, $-1/t' = s - 1/t$, that is

$$t' = \frac{t}{1-ts}.$$

Inserting this expression into the first equation in (29) we obtain the linear equation

$$\frac{\mathrm{d}x'}{\mathrm{d}s} - \frac{t}{1-ts}x' = -\frac{1}{2}g\left(\frac{t}{1-ts}\right)^3$$

(recall that here t is a parameter determining the initial condition). Thus

$$(1-ts)x' = -\frac{gt^2}{2(1-ts)} + \text{const}.$$

The integration constant has to be chosen in such a way that $x' = x$ for $s = 0$. Hence,

$$x' = \frac{x}{1-ts} - \frac{gt^3 s}{2(1-ts)^2},$$

which is the local group of point transformations given in Eqs. (3).

In a similar way, one finds that the one-parameter group of transformations generated by $\mathbf{X}_4$ is that of the Galilean transformations $t' = t$, $x' = x + st$; $\mathbf{X}_3$ generates the time displacements $t' = t + s$, $x' = x$; $\mathbf{X}_5$ generates the translations $x' = x + s$, $t' = t$; and $\mathbf{X}_2$ is the infinitesimal generator of the group of point transformations

$$x' = x\mathrm{e}^{s/2} + \frac{1}{2}gt^2(\mathrm{e}^{s/2} - \mathrm{e}^{2s}), \qquad t' = t\mathrm{e}^{s}.$$

2.2. The existence of an infinite number of Lagrangians

Given a second-order ODE

$$\ddot{x} = f(x,\dot{x},t), \tag{30}$$

which may correspond to a mechanical system or may have some other origin, we want to find some function, $L(x,\dot{x},t)$, such that the Euler–Lagrange equation (13) be equivalent to Eq. (30). To this end, we note that Eq. (13) amounts to

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}} + \dot{x}\frac{\partial}{\partial x}\frac{\partial L}{\partial \dot{x}} + \ddot{x}\frac{\partial}{\partial \dot{x}}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

and therefore we are looking for a function $L(x,\dot{x},t)$ such that

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}} + \dot{x}\frac{\partial}{\partial x}\frac{\partial L}{\partial \dot{x}} + f(x,\dot{x},t)\frac{\partial}{\partial \dot{x}}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \tag{31}$$

holds for all values of x, $\dot{x}$, and t. Taking the partial derivative with respect to $\dot{x}$ on both sides of Eq. (31), assuming that the partial derivatives of L commute and letting

$$M \equiv \frac{\partial^2 L}{\partial \dot{x}^2}, \tag{32}$$

one obtains

$$\frac{\partial M}{\partial t} + \dot{x}\frac{\partial M}{\partial x} + M\frac{\partial f}{\partial \dot{x}} + f(x,\dot{x},t)\frac{\partial M}{\partial \dot{x}} = 0, \tag{33}$$

which is a first-order linear partial differential equation for M. Making use of Eq. (30), Eq. (33) can also be written as

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -M\frac{\partial f}{\partial \dot{x}}. \tag{34}$$

This last equation shows that M is defined up to a multiplicative constant of motion; that is, if M_1 and M_2 are two solutions of Eq. (34), then $\mathrm{d}(M_1/M_2)/\mathrm{d}t = 0$, which means that there is an infinite number of Lagrangians for Eq. (30) since M_1/M_2 can be a trivial constant (*i.e.*, a real number) or a function of x, $\dot{x}$, ant t with a total derivative with respect to the time equal to zero as a consequence of Eq. (30) (see the example in Sec. 2.2.3, below).

Once we have a solution of Eq. (34), from Eq. (32) we can find an expression for L, containing two indeterminate functions of x and t. Substituting the expression for L thus obtained into Eq. (31), the Lagrangian is determined up to the total derivative with respect to the time of an arbitrary function of (x,t).

As pointed out in Ref. [4], Eq. (33) is the equation for the *Jacobi last multiplier* of the system of equations

$$\mathrm{d}t = \frac{\mathrm{d}x}{\dot{x}} = \frac{\mathrm{d}\dot{x}}{f(x,\dot{x},t)}.$$

Some recent applications of this relationship can be found in, *e.g.*, Refs. [12–14], and the references cited therein.

2.2.1. *Example. The Emden–Fowler equation*

In the case of the Emden–Fowler equation

$$\ddot{x} + \frac{2}{t}\dot{x} + x^5 = 0, \tag{35}$$

Eq. (34) takes the form

$$\frac{\mathrm{d}M}{\mathrm{d}t} = M\frac{2}{t},$$

hence, we can choose $M = t^2$, *i.e.*, $\partial^2 L/\partial\dot{x}^2 = t^2$, and

$$L = \frac{1}{2}t^2\dot{x}^2 + g(x,t)\dot{x} + h(x,t), \tag{36}$$

where g and h are some functions of two variables. Substituting this expression for L and $f(x,\dot{x},t) = -2\dot{x}/t - x^5$ into Eq. (31) we obtain

$$\frac{\partial g}{\partial t} - x^5 t^2 = \frac{\partial h}{\partial x},$$

which can be written as

$$\frac{\partial g}{\partial t} = \frac{\partial}{\partial x}\left(h + \frac{x^6 t^2}{6}\right).$$

Thus,

$$g = \frac{\partial\Phi(x,t)}{\partial x}, \qquad h + \frac{x^6t^2}{6} = \frac{\partial\Phi(x,t)}{\partial t},$$

where $\Phi(x,t)$ is an *arbitrary* function of two variables and, substituting into Eq. (36), we find the Lagrangian

$$L = \frac{t^2}{2}\left(\dot{x}^2 - \frac{x^6}{3}\right) + \frac{\mathrm{d}\Phi(x,t)}{\mathrm{d}t},$$

which reduces to Eq. (9) if $\Phi = \mathrm{const}$.

Following the steps presented in Sec. 2.1, we find that all the variational symmetries of the Lagrangian (9) are given by

$$\xi = 2ct, \qquad \eta = -cx, \tag{37}$$

where c is an arbitrary constant, and $G = \mathrm{const.}$, therefore,

$$-\frac{1}{6}c(3t^2x\dot{x} + t^3x^6 + 3t^3\dot{x}^2) = \mathrm{const.} \tag{38}$$

[see Eq. (14)]. Thus, in place of the second-order ODE (35), we have the first-order ODE (38), which amounts to

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{-t^2x \pm \sqrt{t^4x^2 - \frac{4}{3}t^6x^6 + 4kt^3}}{2t^3},$$

where k is a constant, or, equivalently,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \pm\frac{1}{t}\sqrt{u^2 - \frac{4}{3}u^4 + 4ku},$$

where $u \equiv x^2t$. (This new variable arises in a natural manner from $(\xi\,\partial/\partial t + \eta\,\partial/\partial x)u = 0$.) (*Cf.* also Ref. [15].) The one-parameter group of transformations generated by (37), with $c = 1$ is the one given by Eqs. (4).

2.2.2. *Example. A damped harmonic oscillator*

Another illustrative example is given by the equation

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = 0, \tag{39}$$

which corresponds to a damped harmonic oscillator (here γ and ω are constants). Equation (34) takes the form $\mathrm{d}M/\mathrm{d}t = M\gamma$ and we can choose $M = \mathrm{e}^{\gamma t}$. Hence

$$L = \frac{1}{2}\mathrm{e}^{\gamma t}\dot{x}^2 + g(x,t)\dot{x} + h(x,t), \tag{40}$$

where $g(x,t)$ and $h(x,t)$ are some functions of two variables. The Lagrangian (40) reproduces the ODE (39) if and only if

$$-\mathrm{e}^{\gamma t}\omega^2 x + \frac{\partial g}{\partial t} = \frac{\partial h}{\partial x}.$$

Choosing $g = 0$ and $h = -\mathrm{e}^{\gamma t}\omega^2x^2/2$, we obtain the well-known Lagrangian

$$L = \frac{1}{2}\mathrm{e}^{\gamma t}(\dot{x}^2 - \omega^2x^2). \tag{41}$$

One finds that the infinitesimal generators of the variational symmetries of this Lagrangian [*i.e.*, the solutions of Eq. (12)] form a five-dimensional vector space. One of these is given by $\xi = 1$, $\eta = -\gamma x/2$, with $G = \mathrm{const}$. In fact, the vector field

$$\mathbf{X} = \frac{\partial}{\partial t} - \gamma\frac{x}{2}\frac{\partial}{\partial x}$$

generates the one-parameter group of point transformations

$$t' = t + s, \qquad x' = x\mathrm{e}^{-\gamma s/2},$$

which leaves the Lagrangian (41) invariant.

2.2.3. *Example. A nonstandard Lagrangian*

As a final example, we find a nonstandard Lagrangian for a particle in a uniform gravitational field. Starting from the equation of motion $\ddot{x} = -g$, from Eq. (34) we have $\mathrm{d}M/\mathrm{d}t = 0$. Therefore, M must be a trivial constant [as in the case of the standard Lagrangian (15)], or an *arbitrary* function of the constants of motion (25). Choosing $M = \dot{x} + gt$ we find

$$L = \frac{\dot{x}^3}{6} + \frac{g\dot{x}^2t}{2} + \dot{x}g(x,t) + h(x,t),$$

where g and h are two functions to be determined. Substituting L into the Euler–Lagrange equation we see that, in order to reproduce the equation of motion, we can choose $g = 0$ and $h = -g^2xt$; in this way we obtain the Lagrangian (8).

3. Systems with an arbitrary number of degrees of freedom

Since the derivations are almost identical to those presented in Sec. 2, in this section we only give the definitions and main results applicable to the case of a mechanical system with an arbitrary number, n, of degrees of freedom or, more generally, we have a system of n second-order ODEs derivable from a Lagrangian $L(q_i, \dot{q}_i, t)$.

The one-parameter family of point transformations

$$q'_i = q'_i(q_1, \ldots, q_n, t, s), \qquad t' = t'(q_1, \ldots, q_n, t, s), \tag{42}$$

$i = 1, 2, \ldots, n$, is a variational symmetry of the Lagrangian $L(q_i, \dot{q}_i, t)$ if

$$L\left(q'_i, \frac{\mathrm{d}q'_i}{\mathrm{d}t'}, t'\right) \frac{\mathrm{d}t'}{\mathrm{d}t} = L\left(q_i, \frac{\mathrm{d}q_i}{\mathrm{d}t}, t\right) + \frac{\mathrm{d}}{\mathrm{d}t}F(q_i, t, s), \qquad \text{for all } s, \tag{43}$$

where F is some function. Assuming that $q'_i(q_1, \ldots, q_n, t, 0) = q_i$ and $t'(q_1, \ldots, q_n, t, 0) = t$, with the aid of the definitions

$$\eta_i(q_j, t) \equiv \left.\frac{\partial q'_i(q_j, t, s)}{\partial s}\right|_{s=0}, \qquad \xi(q_i, t) \equiv \left.\frac{\partial t'(q_i, t, s)}{\partial s}\right|_{s=0}, \tag{44}$$

from Eq. (43) one finds that the functions (44) correspond to the infinitesimal generator of a variational symmetry of L if

$$\sum_{i=1}^{n}\left[\frac{\partial L}{\partial q_i}\eta_i + \frac{\partial L}{\partial \dot{q}_i}\left(\frac{\mathrm{d}\eta_i}{\mathrm{d}t} - \dot{q}_i\frac{\mathrm{d}\xi}{\mathrm{d}t}\right)\right] + \frac{\partial L}{\partial t}\xi + L\frac{\mathrm{d}\xi}{\mathrm{d}t} = \frac{\mathrm{d}G}{\mathrm{d}t}, \tag{45}$$

for some function $G(q_i, t)$.

Making use of the Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0,$$

from Eq. (45) it follows that

$$\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\eta_i + \xi\left(L - \sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) - G \tag{46}$$

is a constant of motion.

Equation (45) is a partial differential equation for $n+1$ functions of $n+1$ variables, whose solution yields a vector field

$$\xi\frac{\partial}{\partial t} + \sum_{i=1}^{n}\eta_i\frac{\partial}{\partial q_i},$$

which is the infinitesimal generator of a local group of variational symmetries of L. This group is determined by the system of first-order ODEs

$$\frac{\mathrm{d}q'_i}{\mathrm{d}s} = \eta_i(q'_j, t'), \qquad \frac{\mathrm{d}t'}{\mathrm{d}s} = \xi(q'_j, t'),$$

with the initial condition $q'_i(0) = q_i$, $t'(0) = t$.

By contrast with the case of a single second-order ODE, considered in Sec. 2.2, not every system of two or more second-order ODEs can be derived from a Lagrangian (see, *e.g.*, Ref. [8] and the references cited therein).

3.1. Example

A simple example is given by the Lagrangian

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgy, \tag{47}$$

which is the standard Lagrangian for a particle of mass m in a uniform gravitational field. Substituting Eq. (47) into Eq. (45), equating the coefficients of $\dot{x}^3$, $\dot{x}^2\dot{y}$, $\dot{x}\dot{y}^2$, $\dot{y}^3$, $\dot{x}^2$, $\dot{x}\dot{y}$, $\dot{y}^2$, $\dot{x}$, $\dot{y}$, and the terms that do not contain $\dot{x}$ or $\dot{y}$, on both sides of the equation, one finds that the infinitesimal generator of any variational symmetry of (47) must be a linear combination of the eight vector fields

$$\mathbf{X}_1 \equiv \frac{\partial}{\partial x}, \qquad \mathbf{X}_2 \equiv \frac{\partial}{\partial y}, \qquad \mathbf{X}_3 \equiv \frac{\partial}{\partial t}, \qquad \mathbf{X}_4 \equiv t\frac{\partial}{\partial x}, \qquad \mathbf{X}_5 \equiv t\frac{\partial}{\partial y}, \tag{48}$$

and

$$\mathbf{X}_6 \equiv \left(\frac{1}{2}gt^2 + y\right)\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$
$$\mathbf{X}_7 \equiv t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \left(\frac{y}{2} - \frac{3}{4}gt^2\right)\frac{\partial}{\partial y},$$
$$\mathbf{X}_8 \equiv t^2\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x} + \left(yt - \frac{1}{2}gt^3\right)\frac{\partial}{\partial y}. \tag{49}$$

The vector fields $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$ generate translations along the x, y, and t axes, respectively; $\mathbf{X}_4$ and $\mathbf{X}_5$ generate Galilean transformations; while the other three vector fields in (49) correspond to symmetries that are not obvious. $\mathbf{X}_6$ is especially interesting because in the limit $g = 0$ it generates rotations about the origin in the xy plane; the constant of motion associated with this symmetry is $m(y\dot{x} - x\dot{y}) - mg(tx - t^2\dot{x}/2)$, which reduces to a component of the angular momentum when $g = 0$.

4. Final remarks

If one looks for all the point transformations [Eqs. (1), or (42)] that map any solution of an ODE, or of a system of ODEs, into another solution, one finds that not all of them are variational symmetries of the Lagrangian leading to that ODE or system of ODEs. Moreover, different Lagrangians corresponding to the same ODE or system of ODEs may have different variational symmetries. For instance, the ODE

$\ddot{x} = -g$, where g is a constant, possesses an eight-parameter group of point symmetries [7], while the Lagrangians (15) and (8), which lead to this equation, admit five-parameter and three-parameter groups of variational symmetries, respectively.

In spite of the fact that, for a given ODE or system of ODEs, the variational symmetries may not be the more general point symmetries of the equation or system of equations, the variational symmetries are very useful because there exists a first integral associated with each of them, which can be readily calculated [Eqs. (14) and (46)], though, as we have seen, the first integrals obtained in this manner need not be functionally independent. On the other hand, not all first integrals are associated with variational symmetries. In Ref. [16], it is shown that it is possible to find first integrals of EDOs or systems of EDOs, without making use of a Lagrangian, with the aid of the so-called adoint symmetries of the system of equations. A similar result is presented in Ref. [17], making use of Lie group analysis.

The review paper [18] presents various generalizations of the basic results given here, making use of the language of differentiable manifolds, vector fields and differential forms.

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