A model of oscillator with variable mass

H. Rodrigues*, N. Panza, and D. Portes Jr
Departamento de Física, Centro Federal de Educação Tecnológica Celso Suckow da Fonseca,
Av. Maracanã, 229, 20271-110, Rio de Janeiro, RJ, Brazil,
**e-mail: harg.astrophys@gmail.com*

A. Soares
Departamento de Matemática, Centro Federal de Educação Tecnológica Celso Suckow da Fonseca,
Av. Maracanã, 229, 20271-110, Rio de Janeiro, RJ, Brazil.

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We discuss the general form of Newton's second law for variable mass systems. We then derive the equation of motion of one-dimensional oscillator with time-varying mass. The obtained equation of motion is then analytically solved and the solutions are represented by means of Hypergeometric functions. The work is addressed to physics class at undergraduate level.

Keywords: Newton's second law; variable mass systems; oscillators; hypergeometric functions.

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1. Introduction

In mechanics, variable-mass systems are systems which have mass that does not remain constant with respect to time. In such systems, Newton's second law of motion cannot directly be applied because it is valid for constant mass systems only [1]. Instead, a body whose mass m varies with time can be described by rearranging Newton's second law and adding a term to account for the momentum carried by mass entering or leaving the system [1,2,7].

Due to the intrinsic difficulty of the involved concepts and the necessary analytical methods, usually this topic is not addressed thoroughly in basic physics courses. Thus, it may be interesting to propose new approaches to the topic for students of science and engineering at the undergraduate/graduate level.

In this work, we derive the correct form of Newton's second law applied to variable-mass systems. Then, we discuss a simple model for describing the dynamics of an oscillator with a time-varying mass, where the mass is given by an explicit function of the square of time. Thus, the obtained equation of motion is solved in closed form in terms of Hypergeometric functions.

The work is addressed to undergraduate students and teachers. The study of this topic requires acquaintance with basic concepts of calculus and physics at intermediate level.

2. Variable-mass systems and Newton's second law

Consider a particle of mass m which is moving with velocity v at time t. Under the action of the force F between the time instants t and $t + dt$, its velocity changes from v to $v + dv$. According to Newton's second law, the change of the linear momentum, dp, is given by

$$dp = F dt, \tag{1}$$

where $p = mv$. For constant mass, Eq. (1) entails

$$m\frac{dv}{dt} = F. \tag{2}$$

Equation (2) represents Newton's second law as it is often presented in textbooks. This form is particularly useful in obtaining the equation of motion of a particle of constant mass. Equation (1) leads to an alternative form of Newton's second law, however:

$$F = \frac{dp}{dt}, \tag{3}$$

and further

$$F = \frac{d}{dt}(mv). \tag{4}$$

When applied to describe the dynamics of a constant mass particle, Eqs. (2) and (4) provide equivalent expressions of Newton's second law. Furthermore, those equations are invariant under the Galilean transformations, defined by

$$x' = x - ut; \qquad v' = v - u, \tag{5}$$

where u is the velocity of the primed frame of reference relative to the unprimed frame. Applying the transformations (5) to Eqs. (2) and (4), we get $F' = F$.

The description of a variable-mass particle is a bit more difficult. In order to explore this point, let us apply the Galilean transformations (5) to the equation of motion (4). The invariance of Newton's second law enforces that the equation of motion in the primed frame of reference must retain the same form of Eq. (4):

$$F' = \frac{d}{dt}(mv'). \tag{6}$$

The derivative yields

$$F' = m\frac{dv}{dt} + \frac{dm}{dt}\left(v - u\right), \tag{7}$$

and so,

$$F' = \frac{d}{dt}(mv) - \frac{dm}{dt}u \neq F. \tag{8}$$

So, the equation of motion (4) is not Galilean invariant when the mass depends on the time. In order to correctly obtain the equation of motion, we have to apply the principle of conservation of linear momentum for the entire system, which is the basic principle behind the Newton's second law. Thus, consider a single-degree of freedom system of initial mass m, as illustrated in Fig. 1. The center of mass of the system (body 1 in the figure) moves with velocity v at the instant t. The particle of mass Δm (body 2) and mean velocity w is imparted to the system during a time interval Δt. Assuming that the mass of the entire system is conserved during the process, the new mass of the system and the velocity of its center of mass increase to $m + \Delta m$ and to $v + \Delta v$, respectively. The linear momentum of the entire system at the time t is thus given by

$$p(t) = mv + (\Delta m)\, w, \tag{9}$$

while the new linear momentum of the whole system at the time $t + \Delta t$ reads

$$p(t + \Delta t) = (m + \Delta m)(v + \Delta v). \tag{10}$$

Hence, the change in the total linear momentum of the system is

$$\Delta p = m\Delta v + \Delta m \Delta v - \Delta m\,(w - v)\,. \tag{11}$$

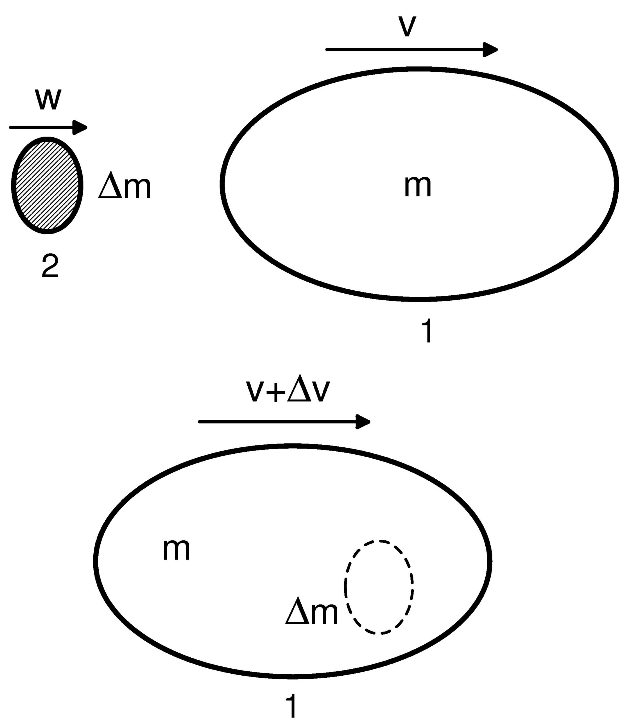

FIGURE 1. (Color online) - The particle of mass Δm and velocity w collides with a particle of mass m and gets stuck in it. After the process, the new particle of mass $m + \Delta m$ moves with velocity $v + \Delta v$.

It follows from Eq. (11) that

$$\frac{\Delta p}{\Delta t} = m\frac{\Delta v}{\Delta t} + \frac{\Delta m}{\Delta t}\Delta v - \frac{\Delta m}{\Delta t}\,(w - v)\,. \tag{12}$$

Taking the limits $\Delta t \to 0$, $\Delta m \to 0$, and $\Delta v \to 0$ in the Eq. (12), and in addition recalling the Newton's second law, $F = \frac{dp}{dt}$, where F is the external force acting on the system, one obtains

$$F = m\frac{dv}{dt} - \frac{dm}{dt}q, \tag{13}$$

where $q = w - v$ is the relative velocity of incident (or escaping) mass with respect to the center of mass of the body. Equation (13) can also be placed in the form [6]

$$m\frac{dv}{dt} = F + \frac{dm}{dt}q. \tag{14}$$

Analogously, for $\frac{dm}{dt} < 0$ (system losing mass) we would obtain

$$m\frac{dv}{dt} = F - \frac{dm}{dt}q. \tag{15}$$

Of course, in the case of isotropic mass variation, that is, isotropic in a system that moves with the body, the net contribution from the $\frac{dm}{dt}$ term is zero. Thus, in this case the correct equation is the common Newton's second law

$$m\frac{dv}{dt} = F. \tag{16}$$

In Eq. (14) the time rate $\frac{dm}{dt}$ should be taken as a positive quantity, since it represents the rate at which the mass of the system increases. On the other hand, if the system is losing mass, then the time rate $\frac{dm}{dt}$ present in Eq. (15) must be taken as a negative quantity. Furthermore, the Eq. (19) is actually invariant under Galilean transformations.

Equations (14) and (15) describe the motion of a time-varying mass particle, and represent the proper extension of Newton's second law. The term $\frac{dm}{dt}(w - v)$ in the right-hand side should be interpreted as a real force acting on the particle, apart from the external force F.

Interestingly, for the particular case $F = 0$ (absence of any external forces) the Eq. (15) leads to the simplified equation of motion

$$m\frac{dv}{dt} = -\frac{dm}{dt}q. \tag{17}$$

Equation (17) is known as the rocket equation, used to describe the motion of a rocket drifting in the free space. The relative velocity q represents the velocity of the gases escaping from the rocket, and is often called the *exhaust velocity* [5, 8].

Also note that Eq. (13) may be put in the form

$$F = \frac{d}{dt}(mv) - \frac{dm}{dt}w, \tag{18}$$

which means that Eq. (18) recovers Eq. (4) in the particular case $w = 0$.

3. One-dimensional oscillator with variable mass

In order to model a variable-mass oscillator, consider a leaking bucket of water which is attached to a spring, as illustrated in Fig. 3. The water exits out through a small hole at the bottom of the bucket. In this situation, and ignoring friction, the system is subjected to the action of three different forces, namely, the elastic force exerted by the spring, the weight of the oscillator, and the force exerted by the flowing water. Assuming that the movement is along the z-axis, and in accordance with Eq. (15), this model can be described by the following equation of motion [7]:

$$m\frac{d^2z}{dt^2} = -\frac{dm}{dt}q - kz - mg, \tag{19}$$

where $z(t)$ is the position of the center of mass measured from the rest position, $q = w - \frac{dz}{dt}$ is the relative velocity of escaping water with respect to the center of mass of the body, k is the stiffness coefficient of the linear restoring force, and g is the constant acceleration of gravity.

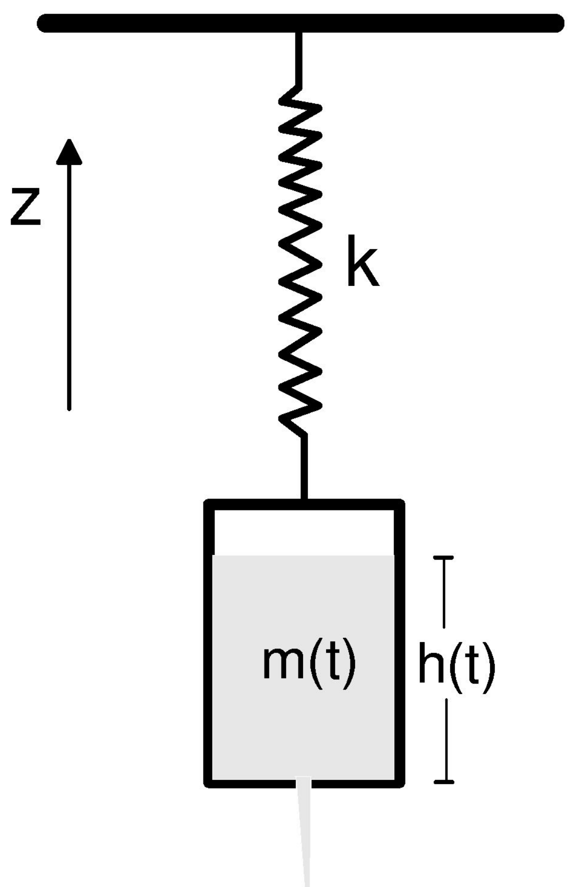

FIGURE 2. (Color online) - Oscillator with a variable mass. A bucket filled with water is attached to a spring. The water flows out through a small hole in the bottom of the bucket.

For $q = 0$, $\frac{dm}{dt} = 0$, and $\frac{d^2z}{dt^2} = 0$ Eq. (19) gives the equilibrium position

$$z_0 = -\frac{m_0 g}{k}, \tag{20}$$

where m_0 is the initial mass of the oscillator. If m is constant, the system oscillates around the position z_0. So, by means of the coordinate transformation

$$z \to z + z_0, \tag{21}$$

the equation of motion (19) turns into

$$m\frac{d^2z}{dt^2} = -\frac{dm}{dt}q - kz + (m_0 - m)\,g. \tag{22}$$

So, at every time t the "instantaneous" equilibrium position is given by

$$z_0(t) = \frac{m_0 - m(t)}{k}g. \tag{23}$$

The mass of water has a quadratic dependence on the time (see Appendix A for details of calculation) which is given by

$$m_w(t) = m_w(0)\left(1 - ft\sqrt{\frac{g}{2h_0}}\right)^2, \tag{24}$$

where $m_w(0)$ is the initial mass of water, $f = \frac{a}{A}$ is the ratio between the cross-sectional area a of the hole, and the cross-sectional area A of the column of water, and h_0 is the initial height of the column of water. The mass of the oscillator is given by the summation of the mass of the bucket m_b, and the time-varying mass of water $m_w(t)$.

Assuming the leaking of water occurs at a very low rate, one can neglect the effect of the first term on the right side of Eq. (22) on the dynamics of the oscillator. In this approach, the equation of motion of the oscillator reads

$$[m_b + m_w(t)]\,\frac{d^2z}{dt^2} = -kz + [m_w(0) - m_w(t)]\,g. \tag{25}$$

4. Solution of the equation of motion

In this section, we present the analytical solution of the equation of motion (25). For this purpose, we have to analyze the problem in two different scenarios. The first one corresponds to the existence of the water within the bucket. According to Eq. (24), the bucket is completely empty after the elapsed time given by

$$\tau = \frac{1}{f}\sqrt{\frac{2h_0}{g}}. \tag{26}$$

In this case, by using Eqs. (24) the equation of motion (25), which is valid for the time interval $0 \le t \le \tau$, can be put in the form

$$\left[m_b + m_w(0)\left(1 - \frac{t}{\tau}\right)^2\right]\frac{d^2z}{dt^2} =$$

$$-kz - m_w(0)g\left[\left(1 - \frac{t}{\tau}\right)^2 - 1\right]. \tag{27}$$

After the elapsed time τ, the oscillations are governed by the differential equation

$$m_b \frac{d^2\widehat{z}}{dt^2} + k\widehat{z} = m_w(0)g, \quad t \geq \tau. \tag{28}$$

Now, we shall consider the solution of the Eq. (27). Defining the new variable

$$x = -\frac{m_w(0)}{m_b}\left(1 - \frac{t}{\tau}\right)^2, \tag{29}$$

and following rather simple calculations, Eq. (27) transforms into

$$x(1-x)\frac{d^2z}{dx^2} + \frac{1}{2}(1-x)\frac{dz}{dx} - \frac{\tau^2 k}{4m_w(0)^2}z = -\frac{\tau^2 g}{4m_w(0)}(m_w(0) + m_b x). \tag{30}$$

Now we assume that the general solution of Eq. (30) can be written in the form

$$z(x) = z_h(x) + z_p(x), \tag{31}$$

where $z_h(x)$ stands for the solution of the homogeneous differential equation associated to Eq. (30), and $z_p(x)$ is a particular solution of the same equation. The former corresponds to the well-known Hypergeometric differential equation [9, 10]:

$$x(1-x)\frac{d^2z}{dx^2} + [c - (a+b+1)x]\frac{dz}{dx} - abz = 0. \tag{32}$$

Equation (32) is invariant by permutation $a \longleftrightarrow b$ with singularities at $x = 0$, 1, and ∞ (all regular). Thus, the solution of equation (32) reads

$$z(x) = c_1\, {}_2F_1(a,b,c,x) + c_2 x^{1-c}\, {}_2F_1(a+1-c, b+1-c, 2-c, x), \tag{33}$$

where $c \notin Z$, c_1 and c_2 are arbitrary constants, and ${}_2F_1(a,b,c,x)$ is the Hypergeometric function, defined by

$${}_2F_1(a,b,c,x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \times \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{x^n}{n!}, \tag{34}$$

with Γ representing the gamma function [9,10]. The radius of convergence of this series is $|x| \leq 1$ for *Re* $(c-a-b) > 0$.

Direct comparison between Eqs. (32) and (33) leads to the system

$$c = \frac{1}{2}, \qquad a + b = -\frac{1}{2}, \qquad ab = \frac{\tau^2 k}{4m_0}, \tag{35}$$

which provides

$$a = -\frac{1}{4} \pm \frac{1}{4}\sqrt{1 - \frac{4\tau^2 k}{m_0}}, \qquad b = -\frac{1}{4} \mp \frac{1}{4}\sqrt{1 - \frac{4\tau^2 k}{m_0}}. \tag{36}$$

Then, the homogeneous solution corresponding to the Eq. (32) can be expressed in the following form $\Big($choosing *e.g.* $a = -\frac{1}{4} + \frac{1}{4}\sqrt{1 - \frac{4\tau^2 k}{m_0}}$ and $b = -\frac{1}{4} - \frac{1}{4}\sqrt{1 - \frac{4\tau^2 k}{m_0}}\ \Big)$:

$$z_h(t) = c_1\, {}_2F_1\left(-\frac{1}{4} + \frac{1}{4}\sqrt{1 - \frac{4\tau^2 k}{m_0}}, -\frac{1}{4} - \frac{1}{4}\sqrt{1 - \frac{4\tau^2 k}{m_0}}, \frac{1}{2}, -\frac{m_0}{m_t}\left(1 - \frac{t}{\tau}\right)^2\right) + c_2\sqrt{\frac{m_0}{m_t}}\left(1 - \frac{t}{\tau}\right) \times {}_2F_1\left(-\frac{1}{4} + \frac{1}{4}\sqrt{1 - \frac{4\tau^2 k}{m_0}}, -\frac{1}{4} - \frac{1}{4}\sqrt{1 - \frac{4\tau^2 k}{m_0}}, \frac{3}{2}, -\frac{m_0}{m_t}\left(1 - \frac{t}{\tau}\right)^2\right). \tag{37}$$

In order to get an oscillatory solution, it is enough to insure that

$$\frac{kh_0}{m_0 g f^2} > \frac{1}{8}. \tag{38}$$

On the other hand, a glance at the right-hand side of Eq. (30) suggests to propose the following particular solution:

$$z_p(t) = \gamma t^2 + \xi t + \eta, \tag{39}$$

where γ, ξ and η are arbitrary constants. Inserting Eq. (39) into Eq. (30) yields

$$\gamma = -\frac{m_0 g}{\tau^2 k + 2m_0}, \qquad \xi = \frac{2m_0 \tau g}{\tau^2 k + 2m_0}, \qquad \eta = \frac{2m_0 g\,(m_0 + m_t)}{k\,(\tau^2 k + 2m_0)}. \tag{40}$$

Finally, inserting the obtained expressions for γ, ξ and η into Eq. (39), the general solution of the equation of motion (30) is obtained by substituting Eqs. (37) and (39) into Eq. (31).

Let us look now for the Eq. (28). Clearly, the solution is given by the simple harmonic form

$$\widehat{z}(t) = c_3 \sin\left(\sqrt{\frac{k}{m_t}} + \phi\right) - z_0, \quad t \geq \tau, \tag{41}$$

the constant z_0 being defined by Eq. (20).

The four constants c_1, c_2, c_3 and ϕ can be determined by using the initial condition $z(0) = 0$ and $v(0) = 0$, together with the condition of continuity of the solutions $z(t)$ and $\widehat{z}(t)$ as well as $v(t)$ and $\widehat{v}(t)$ calculated at the time $t = \tau$. The explicit formulas for these constants are given in the Appendix.

5. Results

The scheme for solving the problem encompasses the following steps. First, assign initial values to all variables: the elapsed time $t = 0$; the initial position $z = z(0)$; the initial velocity $v(0)$; the initial mass of water $m_w(0)$; the initial height of the water column $h(0)$; the value of the ratio f between the cross-sectional areas a and A. Assign values to constants g, k, and the mass of the bucket m_t. Then, evaluate the position $z(t)$ at the time t, given by Eq. (31), for $0 \leq t \leq \tau$, and by Eq. (41), for $t \geq \tau$.

We trigger the time evolution of the oscillator with the initial condition $z(0) = 0$ m and $v(0) = 0$ m s^{-1}. This means that the change in the dynamic state of the system is purely caused by the change in mass of the oscillator with time. Figure 3 depicts the behavior of the position of the oscillator in the z-axis as a function of time for the adopted values of the model parameters as outlined in the caption of the figure. The mass of the oscillator at every time is $m(t) = m_t + m_w(t)$, with $m_w(t)$ given by Eq. (24).

As shown in Fig. 3, the "instantaneous" equilibrium position of the oscillator moves upward while the water in the bucket flows out. The oscillations are obviously caused by the action of the restoring force, as the mass of the oscillator decreases, and so its weight. The system shows a typical oscillatory behavior with "amplitude" and "frequency" which vary as the water leaves the bucket. At the end, there remains only the bucket that oscillates like a one-dimensional harmonic oscillator with constant amplitude and frequency. The final equilibrium position, around which the bucket oscillates after $t > \tau$, can be computed by using Eq. (23), which in the present case has the value 0.98 m.

We also compute the elastic potential energy, U_k, and the gravitational potential energy, W, which are given respectively by

$$U_k = \frac{1}{2}k\left(z - \frac{m_0}{k}g\right)^2, \tag{42}$$

and

$$W = mgz. \tag{43}$$

The mechanical energy of the system is set by the sum of the elastic potential energy, the gravitational potential energy and the kinetic energy, namely, $E = T + U_k + W$.

Figure 4 depicts the behavior of the energy of the oscillator as a function of time for the same set of values of the parameters used in Fig. 3. As discussed in Sec. 3, we can see that the total energy of the oscillator is not conserved due to the mass loss of the system.

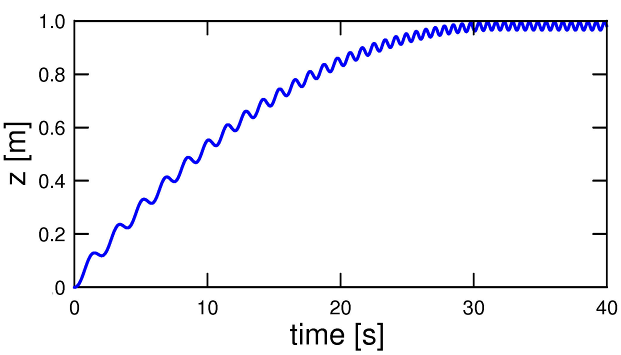

FIGURE 3. (Color online) - Position as a function of time for $f = 0.01$. The used values of the other parameters are $g = 9.8\,\mathrm{m\,s^{-2}}$, $k = 10^2\,\mathrm{kg\,s^{-1}}$ (stiffness coefficient of the spring), $m_b = 1.0$ kg (mass of the bucket), $m_w(0) = 10.0$ kg (initial mass of water), and $h(0) = 0.5$ m (initial height of the water column).

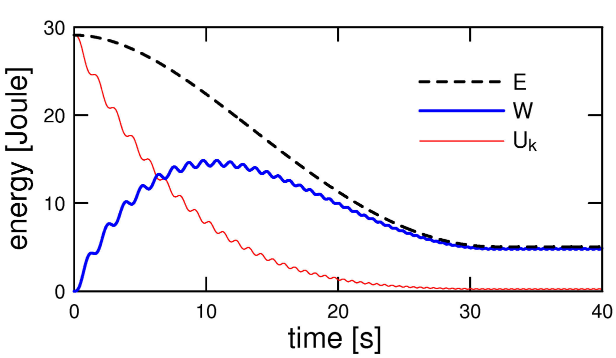

FIGURE 4. (Color online) - Energy as a function of time for $f = 0.01$. The values of the other parameters are the same used in the previous figure.

6. Concluding remarks

In this work we present a set of equations which are used to model the dynamics of a one-dimensional oscillator with a time-varying mass. The time change of the mass is taken into account, using a simple modeling (the bucket of water) where the mass of the oscillator has a quadratic dependence on time.

The resulting equation of motion is analitycally solved in terms of the Hypergeometric functions, and some results for some chosen values of the model parameters have been presented and discussed in the text.

At this point, we point out that the quadratic dependence of mass on time is only a motivator for the leaking oscillator problem, treated here as a purely theoretical problem. Therefore, this result should be considered within its appropriate limitations. Probably, when the bucket is moving, going up and down with the oscillations, the flow rate through the hole could be seen to change as well, deviating slightly from the results obtained here. In other words, we are ignoring the fact that the bucket, as well as the water within it, are accelerating frames. However, we can admit that the quadratic dependence of mass must work as a reasonable approximation in the case the loss of water occurs at a very low rate, and the oscillating bucket experiences smooth motions as investigated in this article. As a suggestion, the model could be investigated experimentally using a motor and a leaking bucket of water, for example, in order to validate or not the assumptions made for the present model for the quadratic dependence of mass.

The present study is intended to be used as a useful approach to the physics of time-varying mass systems at undergraduate e level.

A: Leaking bucket of water

Consider a bucket of water with cross-sectional area A and the column of water of height h. At the bottom of the bucket there is a small hole with cross-sectional area a, with $a \ll A$.

Disregarding losses, we can apply the Bernoulli equation:

$$p_2 + \frac{1}{2}\rho Q^2 + \rho g\left(z_1 + h\right) = p_1 + \frac{1}{2}\rho q^2 + \rho g z_1, \quad \text{(A.1)}$$

where p_1 and p_2 are the pressures at the bottom of the bucket and at the free liquid surface, respectively. The upper part of the bucket is open to the atmosphere, and so the water leaks the bucket freely through the hole. We thus have $p_1 = p_2 = p_0$, where p_0 is the local atmospheric pressure. Q is the velocity at the free liquid surface and q is the exit velocity of the water; h is the height of the free liquid surface relative to the bottom; ρ is the density of the liquid; and z_1 is the position of the bottom of the bucket in the z-axis.

Because $A \gg a$, the velocity Q can be set equal to zero. Thus, we can make these substitutions into the Bernoulli equation to obtain

$$q = \sqrt{2gh}. \quad \text{(A.2)}$$

Notice that Eq. (A.2) is valid even when the surface level is decreasing due to water leakage, provided that the time rate of change of h and Q is sufficiently small.

From the equation of continuity the rate of loss of mass is related to the mass flow trough the equation

$$\frac{dm}{dt} = -\rho q a. \quad \text{(A.3)}$$

On the other hand, the mass of water stored in the bucket at the time t is given by

$$m(t) = \rho A h. \quad \text{(A.4)}$$

Inserting (A.2) into (A.3) yields

$$\frac{dm}{dt} = -\rho A\sqrt{2gh}. \quad \text{(A.5)}$$

From (A.4), we can put (A.5) in the form

$$\frac{dm}{dt} = -f\sqrt{\rho A}\sqrt{2gm}, \quad \text{(A.6)}$$

where $f = \frac{a}{A}$. Thus, Eq. (A.6) leads to

$$\int_{m_w(0)}^{m_w} \frac{dm}{\sqrt{m}} = -\int_0^t f\sqrt{\rho A}\sqrt{2g}dt, \quad \text{(A.7)}$$

where $m_w(0) = \rho A h_0$. By carrying out both integrals in the Eq. (A.7), one obtains Eq. (24).

B

In this appendix we write down the explicit expressions of the constants c_1, c_2, c_3 and ϕ appearing in the general solutions (33) and (41). The constant c_1 is given by

$$c_1 = -\frac{2m_0 g}{k\left(\tau^2 k + 2m_0\right)}\frac{S_1}{S_2}, \quad \text{(B.1)}$$

where

$$\begin{aligned} S_1 = {} & 3m_t\left[k + \tau^{-2}(m_0 + m_t)\right] \times {}_2F_1\left(\frac{1}{4}+\frac{1}{4}\sqrt{1-\frac{4\tau^2 k}{m_0}}, \frac{1}{4}-\frac{1}{4}\sqrt{1-\frac{4\tau^2 k}{m_0}}, \frac{3}{2}, -\frac{m_0}{m_t}\right) \\ & -k(m_0+m_t){}_2F_1\left(\frac{5}{4}-\frac{1}{4}\sqrt{1-\frac{4\tau^2 k}{m_0}}, \frac{5}{4}+\frac{1}{4}\sqrt{1-\frac{4\tau^2 k}{m_0}}, \frac{5}{2}, -\frac{m_0}{m_t}\right), \end{aligned} \quad \text{(B.2)}$$

and

$$\begin{aligned} S_2 = {} & -k\, {}_2F_1\left(\frac{5}{4}-\frac{1}{4}\sqrt{1-\frac{4\tau^2 k}{m_0}}, \frac{5}{4}+\frac{1}{4}\sqrt{1-\frac{4\tau^2 k}{m_0}}, \frac{5}{2}, -\frac{m_0}{m_t}\right) \\ & \times {}_2F_1\left(-\frac{1}{4}+\frac{1}{4}\sqrt{1-\frac{4\tau^2 k}{m_0}}, -\frac{1}{4}-\frac{1}{4}\sqrt{1-\frac{4\tau^2 k}{m_0}}, \frac{1}{2}, -\frac{m_0}{m_t}\right) \\ & + 3k\, {}_2F_1\left(\frac{3}{4}-\frac{1}{4}\sqrt{1-\frac{4\tau^2 k}{m_0}}, \frac{3}{4}+\frac{1}{4}\sqrt{1-\frac{4\tau^2 k}{m_0}}, \frac{3}{2}, -\frac{m_0}{m_t}\right) \\ & \times {}_2F_1\left(\frac{1}{4}+\frac{1}{4}\sqrt{1-\frac{4\tau^2 k}{m_0}}, \frac{1}{4}-\frac{1}{4}\sqrt{1-\frac{4\tau^2 k}{m_0}}, \frac{3}{2}, -\frac{m_0}{m_t}\right) \end{aligned} \quad \text{(B.3)}$$

$$+\,3m_t\tau^{-2}\,{}_2F_1\left(\frac{1}{4}+\frac{1}{4}\sqrt{1-\frac{4\tau^2k}{m_0}},\frac{1}{4}-\frac{1}{4}\sqrt{1-\frac{4\tau^2k}{m_0}},\frac{3}{2},-\frac{m_0}{m_t}\right)$$
$$\times_2F_1\left(-\frac{1}{4}+\frac{1}{4}\sqrt{1-\frac{4\tau^2k}{m_0}},-\frac{1}{4}-\frac{1}{4}\sqrt{1-\frac{4\tau^2k}{m_0}},\frac{1}{2},-\frac{m_0}{m_t}\right). \tag{B.4}$$

Similarly, the constant c_2 can be expressed as

$$c_2=-\frac{6m_0g}{\tau^2k+2m_0}\sqrt{\frac{m_t}{m_0}}\frac{S_3}{S_4}, \tag{B.5}$$

where

$$S_3=-m_t\,{}_2F_1\left(-\frac{1}{4}+\frac{1}{4}\sqrt{1-\frac{4\tau^2k}{m_0}},-\frac{1}{4}-\frac{1}{4}\sqrt{1-\frac{4\tau^2k}{m_0}},\frac{1}{2},-\frac{m_0}{m_t}\right)$$
$$+\,(m_0+m_t)\,{}_2F_1\left(\frac{3}{4}-\frac{1}{4}\sqrt{1-\frac{4\tau^2k}{m_0}},\frac{3}{4}+\frac{1}{4}\sqrt{1-\frac{4\tau^2k}{m_0}},\frac{3}{2},-\frac{m_0}{m_t}\right), \tag{B.6}$$

and

$$S_4=3k\,{}_2F_1\left(\frac{3}{4}-\frac{1}{4}\sqrt{1-\frac{4\tau^2k}{m_0}},\frac{3}{4}+\frac{1}{4}\sqrt{1-\frac{4\tau^2k}{m_0}},\frac{3}{2},-\frac{m_0}{m_t}\right)$$
$$\times_2F_1\left(\frac{1}{4}+\frac{1}{4}\sqrt{1-\frac{4\tau^2k}{m_0}},\frac{1}{4}-\frac{1}{4}\sqrt{1-\frac{4\tau^2k}{m_0}},\frac{3}{2},-\frac{m_0}{m_t}\right)$$
$$+\,3m_t\tau^{-2}\,{}_2F_1\left(\frac{1}{4}+\frac{1}{4}\sqrt{1-\frac{4\tau^2k}{m_0}},\frac{1}{4}-\frac{1}{4}\sqrt{1-\frac{4\tau^2k}{m_0}},\frac{3}{2},-\frac{m_0}{m_t}\right)$$
$$\times\,{}_2F_1\left(-\frac{1}{4}+\frac{1}{4}\sqrt{1-\frac{4\tau^2k}{m_0}},-\frac{1}{4}-\frac{1}{4}\sqrt{1-\frac{4\tau^2k}{m_0}},\frac{1}{2},-\frac{m_0}{m_t}\right)$$
$$-\,k\,{}_2F_1\left(\frac{5}{4}-\frac{1}{4}\sqrt{1-\frac{4\tau^2k}{m_0}},\frac{5}{4}+\frac{1}{4}\sqrt{1-\frac{4\tau^2k}{m_0}},\frac{5}{2},-\frac{m_0}{m_t}\right)$$
$$\times\,{}_2F_1\left(-\frac{1}{4}+\frac{1}{4}\sqrt{1-\frac{4\tau^2k}{m_0}},-\frac{1}{4}-\frac{1}{4}\sqrt{1-\frac{4\tau^2k}{m_0}},\frac{1}{2},-\frac{m_0}{m_t}\right). \tag{B.7}$$

Finally, the remaining constants c_3 and ϕ are obtained from the constants c_1 and c_2 by means of the following formulas:

$$c_3=\sqrt{\left(c_1+\frac{2m_0m_tg}{k\left(\tau^2k+2m_0\right)}\right)^2+\frac{m_0c_2^2}{\tau^2k}}, \tag{B.8}$$

and

$$\phi=\arcsin\left[\frac{1}{c_3}\left(c_1+\frac{2m_0m_tg}{k\left(\tau^2k+2m_0\right)}\right)\right]-\tau\sqrt{\frac{k}{m_t}}. \tag{B.9}$$

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