Equation of motion for describing a massive pulsating system with spherical symmetry

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An equation of motion for describing the motion of a massive pulsating system with spherical symmetry has been deduced. Such equation has applications in astrophysics and cosmology as long as the physical and the symmetry conditions are satisfied.

Keywords: Fluid dynamics: mathematical formulations; mechanical properties; compressibility.

Se deduce una ecuación para describir el movimiento de un sistema masivo pulsante con simetría esférica. Esta ecuación puede tener aplicaciones en astrofísica y cosmología, siempre y cuando las condiciones impuestas en la deducción sean satisfechas.

Descriptores: Dinámica de fluidos; formulaciones matemáticas; propiedades mecánicos; compresibilidad.

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1. Introduction

In a previous paper (Paper I) an equation of motion to describe the pulsation of a system with spherical symmetry has been proposed and resolved [5]. This equation of motion contains exactly the kinetic energy and the potential energy where the energy of the gravitational field and the thermal energy are present. Using the knowledge obtained in the courses of theoretical mechanics [9], basic information of hydrodynamics [19] and with educational purposes in mind, one derives in-depth such equation of motion. We begin assuming that the constituent of such system is basically a gas without strong interactions between the particles so that it can be described as a non compressible fluid via the hydrodynamical equations, namely the continuity equation, the Euler momentum equation and the Poisson equation for taking into account the gravitational interaction between the particles. The deduced equation is precisely Eq. (36) and in order to generalize the applications of such equation a normalization has been done, namely $y = (R(t)/R_0)$ (cf. Eq. (37)), where R_0 means the hydrostatic radius. Thus the differential equation to be solved transforms into $\dot{y}^2{=}(2MG/R_0^3)((1/y){-}(1/2)(1/y^2)){+}E$, with $E{=}\text{const.}$ describing the total energy in mass units. After a carefully adjustment the equation deduced here can be applied either to understand or to solve problems in astrophysics and cosmology. In the framework of the astrophysics a direct application of the equation is for studying the pulsation of Cepheids. In this case, the solution depends of a free parameter (a) only which accounts either for the anharmonicity or harmonicity of the pulsation of the star [15]. The pulsation of Cepheids has been broadly studied either in relationship with radial and non-radial pulsations or by considering linear or non-linear perturbation, however the topic is not completely understood and instead of doing computer simulations that require the introduction of *ad hoc* parameters and constants [20,13], the aim of the present paper is to help via Eq. (36) to treat the Cepheid problem by a strictly analytic method which can be step by step well understood and fitted with astronomical observations. Additionally, by considering that the Cepheid is never motionless and with the normalization conditions $\xi(t) = R(t)/R_{\text{obs}}$, $\zeta(t) = R_0/R_{\text{obs}}$, $\chi(t) = R_{\text{obs}}/R_\odot$, where R_{obs} is the observed radius of the star by the astronomers and $R_\odot$ the radius of the sun, the problem described by the differential equation mentioned above transforms into a variational problem which must be solved *a la* Lanczos [14,7,17]. This topic has been exhaustively discussed in Paper I as well. Additionally, with the aid of the parameter a and the physical restrictions imposed by the perturbation analysis into $\xi(t)$, $\zeta(t)$ and $\chi(t)$, some relations to obtain characteristic parameters of the star can be obtained namely the mass of the system, the surface gravity, the amplitude of the elongation, the maximal velocity of the pulsation and the luminosity. For Cepheids, at least four quantities have been observed and they can be matched with the results provided by Eq. (36) [8,1,18,12].

The outline of the paper is as follows: Sec. 2 contains the deduction of the equation of motion, in Sec. 3 the constant of motion is obtained and in Sec. 4 the results are discussed.

2. Our Hydrodynamical model

For the obtention of the equation of motion, one supposes that the system pulsates as a *totum*, *i.e.*, particle and energy interchanges between different internal layers of the system which could originate additional internal disturbances are not present. Moreover, the following assumptions must be considered: the oscillations are adiabatical and the system can loss mass obeying the condition $\delta Q \ll dU - \delta W$. Additionally, there are energy conservation so that

$$\frac{d}{dt}(E_{\text{kin}} + E_{\text{pot}}) = 0. \tag{1}$$

Where E_{kin} means kinetic energy and E_{pot} potential energy [16]. Then one needs expressions for such energies.

2.1. Kinetic energy

We begin turning to the mass conservation of the system:

$$\frac{dM}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi R^3 \rho\right) = \frac{4\pi}{3}\left(3\rho R^2 \dot{R} + R^3 \dot{\rho}\right), \tag{2}$$

where M is the mass, R the radius of the system and ρ its density. The point over the variable means derivation with respect to the time. From Eq. (2) follows that

$$\dot{\rho} + 3\frac{\dot{R}}{R}\rho = 0. \tag{3}$$

On the other hand, the continuity equation for non compressible fluids reads

$$\rho\vec{\nabla} \cdot \vec{v} + \dot{\rho} = 0. \tag{4}$$

Since the system has a spherical symmetry, using Eq. (3) and supposing the velocity has radial dependence only, Eq. (4) takes the form

$$\frac{\rho}{r^2}\frac{d}{dr}\left(r^2 v\right) - 3\frac{\dot{R}}{R}\rho = 0, \tag{5}$$

[16] and a restriction for the radial dependence of the velocity follows immediately:

$$v(r) = \frac{\dot{R}}{R}r. \tag{6}$$

Assuming the system of mass M and radius R is divided into concentric rings of mass dm and thickness dr, the differential of the kinetic energy is

$$dE_{\text{kin}} = \frac{1}{2}dmv^2 = \frac{1}{2}4\pi\rho r^2 dr\frac{\dot{R}^2}{R^2}r^2 = 2\pi\rho\frac{\dot{R}^2}{R^2}r^4 dr, \tag{7}$$

where the expression for the velocity of Eq. (6) has been used. The integration of Eq. (7) from $r = 0$ to $r = R$ gives

$$E_{\text{kin}} = 2\pi\rho\frac{\dot{R}^2}{R^2}\int_0^R r^4 dr = \frac{3}{10}M\dot{R}^2, \tag{8}$$

which is the kinetic energy for the pulsation of the system.

2.2. Potential energy

The energy of the gravitational field E_G and the thermal energy of the gas contributes substantially to the potential energy of the system. For a potential and a density with dependence of the radius only, the Poisson equation reads

$$\nabla^2\phi(r) = 4\pi G\rho(r), \tag{9}$$

where G is the gravitational constant [3] and for a system with spherical symmetry, Eq. (9) reduces to

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = 4\pi G\rho(r)r^2. \tag{10}$$

The first integration of this equation gives

$$\frac{\partial\phi}{\partial r} = \frac{G}{r^2}\int_0^r 4\pi\rho(r')r'^2 dr' + \frac{\text{Constant}}{r^2} \tag{11}$$

The integral of the last equation correspond to the mass contained in a sphere of radius r and when $r = R$, one has $\partial\phi(r = R)/\partial r = MG/R^2$ which in conjunction with Constant $= 0$ can be used as a boundary conditions. Additionally, for avoiding a singularity in the origin (center of the sphere) a mass $M' =$ constant at this point can be assumed, so that the magnitude of the gravitational field ($\|\overline{g}\| = g_r$) due to a mass contained in a sphere of radius r is given by

$$g_r = -\frac{\partial\phi}{\partial r} = -\frac{GM_r}{r^2}, \tag{12}$$

with

$$M_r = 4\pi\int_0^r \rho(r')r'^2 dr'. \tag{13}$$

For constant density the integration of Eq. (13) is immediate and then $g_r = -(4\pi/3)G\rho r$ but also $g_r = -\partial\phi(r)/\partial r$ (cf. Eq. 12) so that the integration for the potential provides the following expression

$$\phi(r) = \frac{2\pi G\rho r^2}{3} + A. \tag{14}$$

To determine the constant A the continuity condition at the surface $\phi_S = -(MG/R)$ must be taken into account, thus one has

$$\frac{2\pi G\rho R^2}{3} + A = \frac{1}{2}\frac{MG}{R} + A = -\frac{MG}{R}, \tag{15}$$

and solving this equation for A one obtains

$$A = -\frac{3MG}{2R}. \tag{16}$$

Consequently, the gravitational potential is given by

$$\phi(r) = -\frac{3MG}{2R}\left(1 - \frac{1}{3}\frac{r^2}{R^2}\right). \tag{17}$$

On the other hand, in order to calculate the gravitational potential energy of an ensemble of particles, must be considered that the gravitational force that undergoes a test particle ν due to a particle μ is

$$\overline{F}_{\mu\nu} = -\frac{Gm_\mu m_\nu}{|\overline{r}_\mu - \overline{r}_\nu|^2}\hat{r}_{\mu\nu}, \tag{18}$$

being $\hat{r}_{\mu\nu}$ an unitary vector pointing from the particle μ to the test particle ν, m_μ the mass of the particle μ , m_ν the mass of the test particle ν and $\overline{r}_\mu$, $\overline{r}_\nu$ the position vectors of the particles μ and ν respectively [11].

Since the gravitational field of the particle μ is conservative then $\overline{g}_\mu = -\overline{\nabla}\phi_\mu$ and consequently the field of the particle μ and the force given by Eq. (17) keeps a similar relation with the potential energy of the test particle ν, namely $\overline{g}_\mu m_\nu = \overline{F}_{\mu\nu} = -\overline{\nabla}\phi_\mu m_\nu = -\overline{\nabla}E_{pot\nu}$ where the potential energy of the test particles ν in the field of the particle μ is defined as $E_{pot\nu} = -\phi_\mu m_\nu = -(Gm_\mu m_\nu/|\overline{r}_\mu - \overline{r}_\nu|)$. Thus for all the μ particles, the gravitational potential energy of the ν test particles is given by

$$E_{pot\mu} = -\sum_\mu \frac{Gm_\mu m_\nu}{|\overline{r}_\mu - \overline{r}_\nu|}. \tag{19}$$

And for a substratum constituted by identical particles, the total gravitative potential energy for the $\mu \neq \nu$ particles reads

$$E_{\text{pot}} = -\frac{1}{2}\sum_\mu\sum_\nu \frac{Gm_\mu m_\nu}{|\overline{r}_\mu - \overline{r}_\nu|} = \frac{1}{2}\sum_\mu m_\nu \phi(\overline{r}_\mu), \tag{20}$$

where $\phi(\overline{r}_\mu)$ is the potential created by each one of all the μ particles. For a continuous distribution of matter, the last equation can be written as

$$E_{\text{pot}} = \frac{1}{2}\int \rho\phi(r)d^3x = \frac{1}{2}\int \rho\phi(r)r^2 dr d\Omega = \frac{1}{2}\int_0^R 4\pi\rho\phi(r)r^2 dr. \tag{21}$$

Where d^3x represents a three dimensional integration and Ω the solid angle. By substituting the expression (17) for $\phi(r)$ and after a straightforward algebra the following expression for the gravitational potential energy is obtained:

$$E_{\text{pot}} = -\frac{3}{5}\frac{M^2G}{R}. \tag{22}$$

2.3. Gravitational pressure

In order to obtain the pressure due to the gravitational field, a hydrostatic equilibrium is assumed so that the following equation is full satisfied

$$\rho\frac{\partial}{\partial t}\overline{v} = -\rho\overline{\nabla}\phi - \overline{\nabla}p_g. \tag{23}$$

For a constant acceleration of the particles and a spherical symmetry we have $\partial p_g/\partial r = \rho g_r$ so that using $g_r = -(4\pi/3)G\rho r$ and carried out the integration for the pressure, one obtains

$$p_g = -\frac{2\pi}{3}\rho^2 G r^2 + B. \tag{24}$$

Where B is an integration constant. Using the boundary condition $p_g(r = R) = 0$ follows immediately that $B=(2\pi/3)\rho^2 GR^2$ and taking into account $\rho=(M/(4\pi/3)R^3)$ the gravitational pressure is then

$$p_g = \frac{3}{8\pi}\frac{M^2G}{R^4}\left(1 - \frac{r^2}{R^2}\right). \tag{25}$$

It must be pointed here that in this case a singularity at $r = 0$ is not present because at this point the pressure is $p_{gc} = (3/8\pi)(M^2G/R^4)$.

The mean value of binding pressure can be obtained via $\langle p_g\rangle = (1/V)\int p_g dx^3$ [2] and with the aid of (25) one obtains

$$\langle p_g\rangle = \frac{3}{20\pi}\frac{M^2G}{R^4}, \tag{26}$$

and at this point it must be noticed that $(p_{gc}/\langle p_g\rangle) = (5/2)$.

On the other hand, the gas pressure is given by

$$p_{\text{gas}} = nkT = \frac{\rho(1+Z)}{m_H A}kT. \tag{27}$$

Here k is the Boltzmann constant, T is the temperature of the system, A the nucleonic number, Z is the electrons number and $n = (\rho/m) = (\rho(1+Z)/m_H A)$ is the particle density where the mass density ρ is normalized to the hydrogen mass m_H and with $((1+Z)/m_H A)$ containing the information about the chemical composition of the system [10].

In order to satisfy the hydrostatic equilibrium requirement, the gravitational pressure must equals the gas pressure $(\langle p_g\rangle = \langle p_{\text{gas}}\rangle)$, so that with the the aid of Eqs. (25) and (27) an expression for the temperature for every interior point r can be obtained:

$$T(r) = \frac{1}{2}\frac{MG}{kR}\frac{m_H A}{(1+Z)}\Big(1 - \frac{r^2}{R^2}\Big). \tag{28}$$

In a similar way as it has be done above for the pressure, the mean temperature can be obtained through

$$\langle T\rangle = \frac{1}{V}\int T(r)d^3x = \frac{1}{V}\int_0^R 4\pi T(r)r^2 dr. \tag{29}$$

Substituting Eq. (28) and carrying out the integration, one obtains

$$\langle T\rangle = \frac{3}{20\pi}\frac{M^2G}{k\rho R^4}\frac{m_H A}{(1+Z)} = \frac{1}{5}\frac{MG}{kR}\frac{m_H A}{(1+Z)}, \tag{30}$$

which is also an expression of the temperature as function of the mass of the system. From Eq. (28) the central temperature can be obtained and follows immediately that $(T_c/\langle T\rangle) = (5/2)$. In addition, substituting Eq. (30) into Eq. (27), one obtains for $\langle p_{\text{gas}}\rangle$ the same value as that obtained for $\langle p_g\rangle$ in Eq. (26) which is in agreement with the hydrodynamical equilibrium condition.

In the potential energy either the thermal energy associated to the equation of state of an ideal gas described by Eq. (27) as the gravitational energy given by Eq. (22) must be present, *i.e.*,

$$E_{\text{pot}} = +\frac{3}{2}kT\frac{1+Z}{m_H A}M - \frac{3}{5}\frac{M^2G}{R}. \qquad (31)$$

We have used the fact that for a system of N particles with tree freedom degrees its thermal energy is $E_{\text{thermal}} = (3/2)NkT$ and for the particle number we have taken $N = (1 + Z/m_H A)M$. Nevertheless, the dependence of the potential energy from the chemical composition can still eliminated. For an adiabatical pulsation [6], one has

$$TR^2 = T_0R_0^2 = \text{constant}, \qquad (32)$$

where T_0 and R_0 are the values of the temperature and the radius of the system by hydrostatic equilibrium. In such state the pressure of the gas equals the gravitational pressure, *i.e.*,

$$\frac{\rho_0(1+Z)}{m_H A}kT_0 = \frac{3}{20\pi}\frac{M^2G}{R_0^4}, \qquad (33)$$

with ρ_0 the mass density by hydrostatic equilibrium as well. Multiplying the last equation by R_0^2, substituting $\rho = (4\pi/3)MR_0^{-3}$ and using Eq. (32), one obtains

$$\frac{kT(1+Z)}{m_H A} = \frac{1}{5}\frac{MGR_0}{R^2}, \qquad (34)$$

whose substitution into Eq. (31) provides the following expression for the potential energy

$$E_{\text{pot}} = -\frac{3}{5}\frac{M^2G}{R} + \frac{3}{10}M^2G\frac{R_0}{R^2} \qquad (35)$$

Finally, taken into account either the kinetic energy as the potential energy, the equation of motion which describes the radial pulsation of the spherical system is

$$\frac{3}{10}M\dot{R}^2 - \frac{3}{5}\frac{M^2G}{R} + \frac{3}{10}M^2G\frac{R_0}{R^2} = \text{contant}, \qquad (36)$$

and using the substitution

$$y = \frac{R}{R_0}, \qquad (37)$$

the equation of motion transforms into

$$\dot{y}^2 = 2\frac{MG}{R_0^3}\left(\frac{1}{y} - \frac{1}{2}\frac{1}{y^2}\right) + E, \qquad (38)$$

where $E < 0$ means the total dimensionless energy and it is negative because we have a binding system.

3. Integral of motion

In the present section we will show that indeed the equation of motion given by Eq. (36) has the integral of motion demanded in Eq. (1) [4]. According with Eq. (8) and Eq. (35) the lagrangian is

$$L = E_{\text{kin}} - E_{\text{pot}} = \frac{3}{10}M\dot{R}^2 + \frac{3}{5}\frac{M^2G}{R} - \frac{3}{10}M^2G\frac{R_0}{R^2}. \qquad (39)$$

So that the associated equation of motion obtained with the condition $(d/dt)(\partial L/\partial\dot{R}) - (\partial L/R) = 0$ is

$$3M\ddot{R} + \frac{3}{5}\frac{M^2G}{R^2} - \frac{3}{5}M^2G\frac{R_0}{R^3} = 0, \qquad (40)$$

and using the substitution proposed in Eq. (37) the last differential equation transforms into

$$\ddot{y} + \frac{MG}{R_0^3}\left(\frac{1}{y^2} - \frac{1}{y^3}\right) = 0. \qquad (41)$$

Multiplying the last equation by $\gamma(t)\dot{y}$ being $\gamma(t)$ an arbitrary function of the time and after a straightforward algebra one arrives into the following equation

$$\frac{d}{dt}\left[\gamma(t)\frac{\dot{y}^2}{2}\right] + \frac{MG}{R_0^3}\left\{\frac{d}{dt}\left[\frac{-\gamma(t)}{y}\right] + \frac{d}{dt}\left[\frac{\gamma(t)}{2y^2}\right]\right\}$$
$$+ \dot{\gamma}(t)\left(-\frac{1}{2}y^2 + \frac{MG}{R_0^3}\left(\frac{1}{y} - \frac{1}{2y^2}\right)\right) = 0. \qquad (42)$$

The coefficient of $\dot{\gamma(t)}$ is the sum of the kinetic energy and the potential energy per unit mass, *i.e.*, the total energy E per unit mass, and once must be pointed out that it is less than zero because we have a binding system since $E_{\text{pot}} > E_{\text{kin}}$. In order to have a constant of motion, it is necessary that $dE\gamma/dt = E\dot{\gamma}$ and introducing this condition into Eq. (42), one arrives to

$$\frac{d}{dt}\left[\gamma(t)\frac{\dot{y}^2}{2}\right] + \frac{MG}{R_0^3}\left\{\frac{d}{dt}\left[\frac{-\gamma(t)}{y}\right]\right.$$
$$\left. + \frac{d}{dt}\left[\frac{\gamma(t)}{2y^2}\right]\right\} + \frac{d}{dt}(E\gamma) = 0 \qquad (43)$$

And the constant of motion is

$$\gamma(t)\left[\frac{\dot{y}^2}{2} + \frac{MG}{R_0^3}\left(-\frac{1}{y} + \frac{1}{2y^2}\right) + E\right] = \text{Constant} \qquad (44)$$

Which is properly the total energy per unit mass.

4. Discussion and Conclusions

In the present work an equation of motion to describe the pulsation of a system with spherical symmetry has been deduced. Such equation contains basically an expression for the kinetic energy and the potential energy where the energy of the gravitational field and the thermal energy are present. The Eq. (36) has applications into astrophysics and cosmology as well as into physical problems for which the physical and the symmetry conditions are satisfied. For astrophysical problems, a direct application of such equation is for studying the pulsation of Cepheids. In this case, the solution depends only of a free parameter (a) which accounts either for the anharmonicity or harmonicity of the pulsation of the star. The pulsation of Cepheids has been broadly studied but the topic is not completely understood and instead of doing computer simulations that require the introduction of *ad hoc* parameters, the Cepheid problem, as presented in Paper I, can be treated by a strictly analytic method which can be step by step well understood and fitted with astronomical observations. By considering that the Cepheid is never motionless and with the normalization conditions $\xi(t) = R(t)/R_{\text{obs}}$, $\zeta(t) = R_0/R_{\text{obs}}$, $\chi(t) = R_{\text{obs}}/R_{\odot}$, where R_{obs} is the observed radius of the star by the astronomers and $R_{\odot}$ the radius of the sun, the problem described by the differential equation presented here transforms into a variational problem which must be solved via the variation principle *a la* Lanczos [14]. This topic has also been exhaustively discussed in Paper I where the application to Cepheids has been reported. Additionally, with the aid of the parameter a and the physical restrictions imposed by the variational analysis into $\xi(t)$, $\zeta(t)$ and $\chi(t)$, expressions to know the mass of the star, the surface gravity, the amplitude of the elongation, the maximal velocity of the pulsation and the luminosity of Cepheids, have been obtained.

Additionally, the constant of motion is obtained and it is properly the total energy per unit mass multiplied by a function of the time acting on the system as a *totum*, here must be pointed out that such function of the time takes relevance when it is associated with the Hubble function, so that the cosmological application of our results may be relevant and it is a topic of another paper.

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