

On thermal waves' velocity: some open questions in thermal waves' physics

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This paper presents some considerations about the truly character of thermal waves. Starting from the comparison of typical characteristic velocities, it is shown why a limiting frequency must exist, above which the parabolic treatment of the heat transfer in presence of time varying periodical heat sources is no longer valid due to the constancy of the speed of light in vacuum. Although it is demonstrated that this frequency is much smaller than that at which the thermal wave velocity can become the speed of light, many questions remain open, such as the behavior of thermal waves at intermediary frequencies. The discussion presented here can be useful to stimulate further discussion on this theme among students, teachers and scientists dealing with heat transport under non-stationary conditions.

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Thermal waves (TW) are temperature oscillations resulting from periodical heating of a material [1]. They are often described as the solutions of the parabolic heat diffusion equation (PHDE) in the presence of a periodical (sinusoidal for a sake of simplicity) time varying heat source modulated in intensity at a given frequency, f [1].

Consider an isotropic homogeneous semi-infinite solid, whose surface is heated uniformly (in such a way that the one dimensional approach used in what follows is valid) by a source (light, for example) of periodically modulated intensity ($I_0/2 \operatorname{Re}[(1+\exp(i\omega t))]$), where I_0 is the intensity of the light source (energy per unit area and unit time), ω is the angular modulation frequency, t is the time, $i=(-1)^{1/2}$ and Re denotes the real part.

The temperature distribution $T(x,t)$ within the solid with thermal diffusivity α can be obtained solving the (parabolic) heat diffusion equation (PHDE) [1,2]

$$\frac{\partial^2 T(x,t)}{\partial x^2} - \frac{1}{\alpha} \frac{\partial T(x,t)}{\partial t} = 0, \quad x > 0, \quad t > 0 \quad (1)$$

with the boundary condition

$$-k \left. \frac{\partial T(x,t)}{\partial x} \right|_{x=0} = \frac{I_0}{2} \operatorname{Re}[(1 + \exp(i\omega t))] \quad (2)$$

where k is the thermal conductivity, related to the thermal diffusivity, α through

$$k = \alpha \rho c. \quad (3)$$

Here ρ is the density and c is the specific heat at constant pressure.

The condition (2) express that the thermal energy generated at the surface of the solid (for example by the absorption

of light) diffuses into its bulk by diffusion. It is supposed here that all the deposited energy is transformed into heat. From now on, the operator $\operatorname{Re}()$ will be omitted, taking into account the convention that the real part of the expressions of the temperature must be taken to obtain physical quantities [3].

The solution of the problem with interest for practical applications [1] is the one related to the time dependent component. If we separate this component from the spatial distribution, the temperature can be expressed as:

$$T(x,t) = \theta(x) \exp(i\omega t). \quad (4)$$

Substituting in Eq. (1) we obtain

$$\frac{d^2 \theta(x)}{dx^2} - q^2 \theta(x) = 0 \quad (5)$$

where

$$q = \sqrt{\frac{i\omega}{\alpha}} = (1+i) \sqrt{\frac{\omega}{2\alpha}} = \frac{(1+i)}{\mu} \quad (6)$$

and

$$\mu = \sqrt{\frac{2\alpha}{\omega}} \quad (7)$$

The general solution of the above problem is then

$$T(x,t) = \frac{I_0}{2\varepsilon\sqrt{\omega}} \exp\left(-\frac{x}{\mu}\right) \cos\left(\frac{x}{\mu} + \omega t + \frac{\pi}{4}\right) \quad (8)$$

where $\varepsilon = k\alpha^{-1/2}$ is the thermal effusivity.

Expression (8) has the meaning of a plane wave. Like other waves it has an oscillatory spatial dependence of the

form $\exp(iqx)$, with a wave vector q given by Eq. (6). Because it has several wave-like features, Eq. (8) represents a thermal or temperature wave (TW). The detection of TWs is the basics of the so-called photothermal techniques that have gained in interest since the early 1970s due to their potential not only for optical spectroscopy, but also for the measurement of thermal properties of materials [1]. Although the main properties of TWs have been discussed in detail by several authors, this work will be focused on one of their main properties, namely the propagation velocity.

From Eq. (8) it is easy to see that TW's wave-length is given by $\lambda = 2\pi\mu$ so that they propagate with phase velocity, V_p , given by:

$$V_p = \lambda f = \omega\mu = \sqrt{2\alpha\omega} \quad (9)$$

As in other wave phenomena, the phase velocity is defined as the velocity of points of constant amplitude in a wave of the form given by the above expression. Since Eq. (5) is a linear ordinary differential equation describing the motion of a thermal wave, then the superposition of solutions will be also a solution of it (we have approximated the temperature distribution by just the first harmonic of that superposition because the higher harmonics damp out more quickly due to the damping coefficient increase with frequency). This superposition represents a group of waves with angular frequencies in the interval $\omega, \omega + d\omega$ travelling in space as "packets" with a group velocity:

$$V_g = \frac{1}{\frac{dqR}{d\omega}} = 2\sqrt{2\alpha\omega} = 2V_p \quad (10)$$

where $qR = Re(q) = 1/\mu$. This velocity is the phase velocity of the envelope, *i.e.* the velocity at which thermal energy propagates. In other words, it is the velocity of points of constant amplitude in a group of waves and is calculated from the dispersion relation (Eq. (6)) as usual.

The group velocity is twice the wave's phase velocity [2]. If TW are truly waves, then this velocity must be smaller than the light speed in vacuum, 3×10^8 m/s, otherwise one of the postulates of the special relativity theory will be violated. Therefore, in order to keep

$$V_g < c \quad (11)$$

the following condition must be achieved

$$f < \frac{c^2}{16\pi\alpha} = f_c \quad (12)$$

which is obtained after substituting Eq. (10) into Eq. (11). The frequency f_c will be called here the critical frequency.

Values of f_c for different materials (with different α values) are given in Table I. For example, for a poor heat conductor such as balsa wood ($\alpha = 0.5 \times 10^{-7}$ m²/s) the maximal available frequency is about 10^{22} Hz. For frequencies values higher than f_c the group velocity becomes larger than c . Thus, it can be concluded that for each material a critical

TABLE I. Values of thermal diffusivity, second sound velocity (Eq. (14)) and the critical frequency (Eq. (12)) for some solids materials. The used value of τ is 10^{-12} s. Thermal diffusivities were taken from: <http://www.fiz-chemie.de/infotherm/servlet/infothermSearch> (downloaded March, 2015)

MATERIAL	α (m ² /s)	f_c (Hz)	u (m/s)
Balsa Wood	5.00×10^{-8}	3.58×10^{22}	223
Glass (non-porous)	4.00×10^{-7}	4.48×10^{21}	632
Steel	3.70×10^{-6}	4.84×10^{20}	1923
Gadolinium	5.47×10^{-6}	3.27×10^{20}	2339
Brass	3.00×10^{-5}	5.97×10^{19}	5477
Tantalum	5.68×10^{-5}	3.15×10^{19}	7536
Silicon	9.38×10^{-5}	1.91×10^{19}	9685

frequency, f_c , exists above which heat cannot be modulated, otherwise relativity theory can be violated. This is a result that contradicts any experience because there is not a technical upper limit for the modulation frequency. In other words, frequencies given by Eq. (12) can be easily exceeded in practice. Therefore, for these frequencies Fourier treatment of heat transport becomes inadequate.

An explanation to this paradoxical result can be given if we look at the hyperbolic heat diffusion equation (HHDE) [3-5]

$$\nabla^2 T - \frac{1}{\alpha} \frac{\partial T}{\partial t} - \frac{1}{u^2} \frac{\partial^2 T}{\partial t^2} = 0 \quad (13)$$

which considers that a build-up time, τ , must exist for the onset of the thermal flux after a temperature gradient is suddenly imposed on the sample [2-7]. This time is also called the relaxation time. Here

$$u = \left(\frac{\alpha}{\tau}\right)^{\frac{1}{2}} \quad (14)$$

For the same case study described above of periodic excitation in the form given by Eq. (2), we obtain from Eq. (13) after a variables separation (Eq. (4))

$$\frac{d^2\theta(x)}{dx^2} - q_c^2\theta(x) = 0 \quad (15)$$

an expression similar to Eq. (5) but with the "new" complex wave number q_c given by

$$q_c = 2\pi f \sqrt{\frac{\tau}{\alpha}} \sqrt{i \frac{f_L}{2\pi f} - 1} \quad (16)$$

and

$$f_L = \frac{1}{\tau} \quad (17)$$

It is well-known [4] that for modulation frequencies such that $f \ll f_L$ the HHDE reduces to the PHDE.

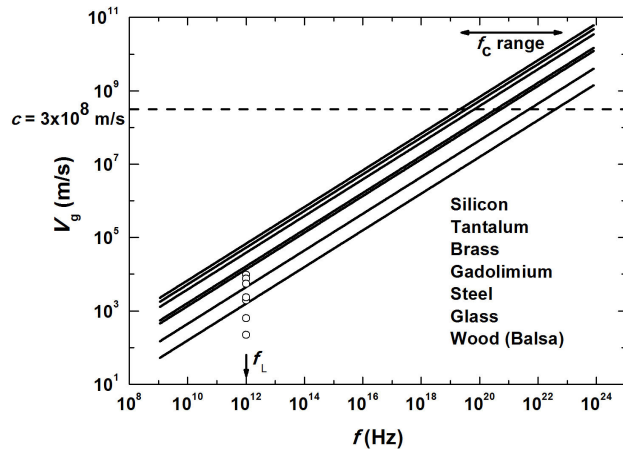


FIGURE 1. Group velocity as a function of frequency for the materials shown in Table I. The critical frequency range is shown with the horizontal arrow. The speed of light in vacuum is shown with the horizontal line. Values of the second sound velocity for each material are shown with open circles in the figure. Note from Eqs. (10) and (14) that they are related to the group velocity value evaluated at the frequency $f = f_L = 1/\tau$, namely $v_g(f_L)$, by $u = v_g(f_L)/(4\pi^{1/2}) \sim v_g(f_L)/7$.

But for frequencies such that $f \gg f_L$ the general solution for the temperature field differs strongly from that of the PHDE [5] being

$$T(x, t) = \frac{I_0 \sqrt{\tau}}{2\varepsilon} \exp\left(-\frac{xu}{2\alpha}\right) \cos\left(\frac{\omega}{u}x - \omega t\right) \quad (18)$$

Accordingly, the thermal waves will propagate at the velocity u , which represents a (finite) speed of propagation of the thermal signal, and diverges only for the unphysical assumption of $\tau = 0$.

With the exception of liquid Helium [7], there are not experimental data reported for the relaxation time [8]. Theoretical predictions give values for this parameter ranging from 10^{-14} s for some metals to some seconds for materials with non-homogeneous inner structure such as tissues and granular materials [9]. As mentioned by several authors [10-12], in many cases the values reported have generated great controversy, in particular for the last mentioned materials. Therefore, experimental measurements of the relaxation time are necessary, which have been remained elusive [3,9,13]. In this work it will be supposed that the relaxation time is of the order of about 10^{-12} s, a good assumption for most materials at room temperature [6-9], so that that $f_L \sim 10^{12}$ Hz and u takes typical values of about of 10^2 - 10^4 m/s (see Table I), well below the c -value, thus becoming independent on the modulation frequency. This value is of the same order of magnitude that the sound's speed in solids. This is one of the reasons why, historically, u is called second sound velocity [7].

Figure 1 shows in a logarithmic plot the group velocity as a function of the modulation frequency for different solid materials (Eq. (10)). Note that the frequency range at which this velocity approaches the light speed in vacuum lies approximately between 10^{19} and 10^{22} Hz, much above the value of the limiting frequency f_L .

Thus, several questions remain open in TWs physics: What is the TWs behavior at frequencies in the vicinity of f_L ? At which frequency does the TWs velocity actually change from V_p to u , thus becoming independent on modulation? Although part of this topic has been discussed before [3-5], the debate is not settled, and new experiments and theoretical analysis are needed to understand it better and to shine light on some open questions in the analysis of heat transfer phenomena, in particular those taking place in the presence of periodical heat sources, and to motivate further analysis and discussion on this topic not only between students and teachers, but also among researchers dealing with heat propagation problems under time varying periodical heating conditions. In the past, hyperbolic non-Fourier conduction has strictly been studied from a mathematical viewpoint with insufficient attention to its practical importance. Non-Fourier effects have long been known to exist in the form of second-sound thermal waves in superfluid helium [7]. More recently, they have been observed in a variety of phenomena involving ultrafast heating such as supernovae explosions [14], ultrafast laser heating [15] and complex fluids (e.g. colloidal suspensions of nanometer sized particles in basic fluids, where heat transfer times can be substantially reduced due to the reduced dimensions of the particles) [16,17]. Therefore, the existence of non-Fourier heat transfer is a topic of importance to be introduced in modern physics courses. Finally, it is worth mentioning that the hyperbolic heat diffusion equation is one of the possible solutions to avoid the inconsistency of the parabolic approach, which violates the special relativity theory at very high modulation frequencies. However, some authors [16] suggest that the hyperbolic approach also violates the second law of thermodynamics in the very high modulation frequency regime, since heat can flow from the colder region to the hotter in the sample, against the temperature gradient. It is sure that this subject is not closed yet!

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