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Two-dimensional harmonic and Green's functions on a spherical surface

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The solutions of the Laplace-Beltrami equation on a spherical surface are constructed by the method of separation of variables, as the products of the Fourier basis functions of the azimuthal angle and the integer powers of tangent or cotangent functions of half the polar angle. The Legendre operator acting on the latter functions yields zero. The construction of the Green's function as the solution of the corresponding Poisson-Beltrami equation with a unit point source on the spherical surface is also constructed using the two-dimensional spherical harmonic basis.

Keywords: Laplace-Beltrami and Poisson-Beltrami operators and equations; two-dimensional spherical harmonics; Green's functions; separability and integrability.

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1. Introduction

The motivation of the work reported in this article is the interest in the study of vortices and vortex sheets in two-dimensions, on planes and spherical surfaces. Bogomolov [1] studied two-dimensional flows of an incompressible fluid on the surface of a sphere assuming that the fluid is contained within an infinitely thin spherical shell. He was able to describe the motion of point vortices in this manner. An extension to more general types of vortex flows was provided by Kimura & Okamoto [2], who used Green's functions in order to do so. An exact solution for distributed vortex equilibria on a sphere has been obtained by Crowdy & Cloke [3] involving a combination of point vortices and vortex patches. Our recent work [4] reported the construction of complete sets of circular, elliptical and bipolar harmonic vortices on a plane. Currently, we have formulated the investigation of harmonic vortices on a spherical surface, recognizing the need to construct the solutions of the Laplace–Beltrami equation as well as the Green's function for its Poisson–Beltrami equation. Since the homogeneous equation is separable and integrable in the familiar spherical coordinates, we think that a didactic presentation of its solution will be of interest and useful for students and teachers, beyond the topic of our personal motivation. The construction of the Green's function for the Poisson–Beltrami equation, with the Laplace–Beltrami operator and a unit point source, is accomplished by using its expansion in the basis of the two-dimensional harmonics on the spherical surface. Notice that the latter are different from the familiar spherical harmonics in three dimensions, which are eigenfunctions of the Laplace–Beltrami operator.

Section 2 identifies the Laplace–Beltrami operator as the angular part of the familiar Laplace operator in three dimensions in spherical coordinates. The restriction to the surface of a sphere of radius $r = a$ eliminates the radial coordinate dependence, and the Laplace equation becomes the Laplace–Beltrami equation. The latter is separable in the polar and azimuthal angles, involving the same operators appearing in the three-dimensional case; the equation in the azimuthal angle and its solutions are also common for two and three dimensions; the difference resides in the fact that the Legendre operator acting on the polar angle function must yield zero. The last equation for the successive integer separation constants $m = 0, 1, 2, \ldots$ is readily integrable. Section 3 introduces the Poisson–Beltrami equation for the Green's function, in which the Laplace–Beltrami operator acting on such a function leads to the unit point source at the position $\theta = \theta'$, $\phi = \phi'$ with their respective Dirac delta-function distribution representations. The construction of the Green's function is accomplished by expanding it in the complete basis of two-dimensional spherical harmonics, and using the Fourier expansion of the Dirac delta-function in the azimuthal angle as well as the linear independence of such a basis, followed by the integration of the respective ordinary differential equations in the polar angle. Section 4 includes a discussion of the results and their applications.

2. Laplace–Beltrami equation: separation and integration

The Laplace equation in three dimensions and spherical coordinates is [5]:

$$\left\{\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\right\}\Psi(r,\theta,\varphi) = 0. \tag{1}$$

Its angular part inside the square brackets is called the Laplace–Beltrami operator. The restriction to the two dimensional spherical surface of radius $r = a$ is accomplished by choosing such a value in the function Ψ, so that its radial derivatives vanish, and the Laplace equation (1) becomes the Laplace–Beltrami equation:

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] \times \Psi(r = a, \theta, \varphi) = \Delta_B\Psi(r = a, \theta, \varphi) = 0, \quad (2)$$

with Δ_B the Laplace–Beltrami operator.

The familiar structure of the operator in the equation allows the factorization of the solutions:

$$\Psi(\theta, \varphi) = \Theta(\theta)\Phi(\varphi), \quad (3)$$

and the separation into the ordinary differential equations

$$\frac{d^2\Phi}{d\varphi^2} = -m^2\Phi, \quad (4)$$

$$\left(\sin\theta\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} - m^2\right)\Theta = 0. \quad (5)$$

Equation (4) is recognized to be common with the one in the three-dimensional spherical harmonics. Its eigenfunctions are $\sin(m\phi)$ and $\cos(m\phi)$ with integer eigenvalues $m = 0, 1, 2, 3, \ldots$, insuring the periodicity and uniqueness of the solutions. Equation (5) is different from the eigenvalue equation of the Legendre operator leading to the familiar three-dimensional spherical harmonics. Here, we proceed to construct its solutions explicitly.

It is convenient to rewrite (5) as an eigenvalue equation in the form:

$$\sin\theta\frac{d}{d\theta}\sin\theta\frac{d\Theta}{d\theta} = m^2\Theta, \quad (6)$$

because it suggests dealing only with the first-order differential equations:

$$\sin\theta\frac{d\Theta}{d\theta} = \pm m\Theta. \quad (7)$$

We proceed to construct the solutions for the successive values of m. For $m = 0$ the simplest solution is a constant, say a_0, and the second solution is such that

$$\sin\theta\frac{d\Theta_0}{d\theta} = b_0, \quad (8)$$

with b_0 a constant. Then, integrating the above equation:

$$\Theta_0 = b_0 \ln(\cot(\theta/2)). \quad (9)$$

Equation (6) has singularities at $\theta = 0, \pi$, which are removable in the northern and southern hemispheres by using the tangent and cotangent solutions, respectively; the cotangent is divergent at $\theta/2 = 0$ and the tangent is divergent at $\theta/2 = \pi/2$. Now let us consider $m > 0$. In this case, Eq. (7) is also immediately integrable, with solutions:

$$\ln\Theta_m = -m\ln(\cot(\theta/2)), \quad \text{or}$$
$$\Theta_m = \tan^m(\theta/2). \quad (10)$$

The second solutions for $m < 0$ become:

$$\Theta_{-m} = \cot^m(\theta/2). \quad (11)$$

In short, the most general solutions of the Laplace–Beltrami equation are the superpositions of the two-dimensional spherical harmonics:

$$\Psi(\theta, \varphi) = a_0 + b_0\ln(\cot(\theta/2)) + \sum_{m=1}^{\infty}(a_m\cot^m(\theta/2) + b_m\tan^m(\theta/2)) \times (c_m\cos(m\varphi) + d_m\sin(m\varphi)). \quad (12)$$

3. Green's function for the Poisson–Beltrami equation on the two-dimensional spherical surface

The Poisson–Beltrami equation for the Green's function on the two-dimensional spherical surface is written in terms of the Laplace–Beltrami operator on its left-hand side and the unit source at the point (a, θ', ϕ') with Dirac-delta distribution functions:

$$\Delta_B G(r = a, \theta, \varphi; r' = a, \theta', \varphi') = -4\pi\frac{\delta(\theta - \theta')\,\delta(\varphi - \varphi')}{\sin\theta}. \quad (13)$$

The Green's function and the delta-function in the azimuthal angle can be represented as Fourier series, incorporating their invariance under the exchange of their unprimed and primed variables, or field and source points:

$$G(\theta, \varphi; \theta', \varphi') = g_0(\theta, \theta') + \sum_{m=1}^{\infty} g_m(\theta, \theta')\cos(m(\varphi - \varphi')), \quad (14)$$

$$\delta(\varphi - \varphi') = \frac{1}{2\pi}\left[1 + 2\sum_{m=1}^{\infty}\cos(m(\varphi - \varphi'))\right]. \quad (15)$$

Substitution of the expansions (14) and (15) in Eq. (13), using the linear independence of the Fourier basis leads to the ordinary differential equations for $m = 0$ and the natural numbers, successively:

$$\left(\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}\right)g_0(\theta, \theta') = -2\delta(\theta - \theta'), \quad (16)$$

$$\left(\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} - \frac{m^2}{\sin\theta}\right)g_m(\theta, \theta') = -4\delta(\theta - \theta'). \quad (17)$$

For $\theta \neq \theta'$, the right-hand side of both Eqs. (16) and (17) vanish, with the consequent implication that the g functions must be solutions of the Laplace–Beltrami equation of orders $m = 0$ and $m = 1, 2, 3, \ldots$, respectively, constructed in the previous section for the intervals $\theta < \theta'$ and $\theta' < \theta$. Additionally, the g functions must be symmetric under the exchange of θ and θ', and continuous at $\theta = \theta'$. These conditions are satisfied by the respective choices:

$$g_0(\theta, \theta') = a_0 \ln(\cot(\theta_< /2)), \tag{18}$$

$$g_m(\theta, \theta') = a_m \frac{\cot^m(\theta_> /2)}{\cot^m(\theta_< /2)}, \tag{19}$$

where $\theta_<$ and $\theta_>$ are the smaller and larger of θ and θ', respectively. The coefficients a_0 and a_m are determined by integrating the equations (16) and (17) over the interval $\theta' - \epsilon \leq \theta \leq \theta' + \epsilon$, and then taking the limit as ϵ tends to zero, as is shown explicitly below:

$$\lim_{\epsilon \to 0} \int_{\theta'-\epsilon}^{\theta'+\epsilon} \frac{d}{d\theta} \sin\theta \frac{dg_0}{d\theta} d\theta = \lim_{\epsilon \to 0} \sin\theta \frac{dg_0}{d\theta}\bigg|_{\theta'-\epsilon}^{\theta'+\epsilon}$$

$$= \sin\theta' \left(\frac{dg_0}{d\theta}\bigg|_+ - \frac{dg_0}{d\theta}\bigg|_- \right) = -2, \tag{20}$$

$$\lim_{\epsilon \to 0} \int_{\theta'-\epsilon}^{\theta'+\epsilon} \left(\frac{d}{d\theta} \sin\theta \frac{d}{d\theta} - \frac{m^2}{\sin\theta} \right) g_m d\theta$$

$$= \lim_{\epsilon \to 0} \left[\sin\theta \frac{dg_m}{d\theta}\bigg|_{\theta'-\epsilon}^{\theta'+\epsilon} - \int_{\theta'-\epsilon}^{\theta'+\epsilon} \frac{m^2 g_m}{\sin\theta} d\theta \right]$$

$$= \sin\theta' \left(\frac{dg_m}{d\theta}\bigg|_+ - \frac{dg_m}{d\theta}\bigg|_- \right)$$

$$- \lim_{\epsilon \to 0} \int_{\theta'-\epsilon}^{\theta'+\epsilon} \frac{m^2 g_m}{\sin\theta} d\theta = -4, \tag{21}$$

where the subindices $+$, $-$ indicate that the limit is taken from the right and from the left of θ', respectively. The last integral in (21) vanishes because of the continuity of g_m. Thus, we obtain:

$$\frac{dg_0}{d\theta}\bigg|_+ - \frac{dg_0}{d\theta}\bigg|_- = -\frac{2}{\sin\theta'}, \tag{22}$$

$$\frac{dg_m}{d\theta}\bigg|_+ - \frac{dg_m}{d\theta}\bigg|_- = -\frac{4}{\sin\theta'}. \tag{23}$$

In both equations we have for the first term $\theta_< = \theta'$, $\theta_> = \theta$ and for the second one $\theta_< = \theta$, $\theta_> = \theta'$. Then, replacing g_0 and g_m with the appropriate expressions given in Eqs. (18) and (19) the results $a_0 = -2$, $a_m = 2/m$ are obtained, so that the Green's function is:

$$G(\theta,\varphi;\theta',\varphi') = -2 \Bigg[\ln(\cot(\theta_< /2)) - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\cot^m(\theta_> /2)}{\cot^m(\theta_< /2)} \cos(m(\varphi - \varphi')) \Bigg]. \tag{24}$$

4. Discussion

In Sec. 2, the Laplace–Beltrami equation (2) in spherical coordinates was identified. Its factorizable solutions (3), led to the ordinary differential equations (4) and (5) in the azimuthal and polar angles, respectively. The first one and its eigenfunctions $\sin(m\varphi)$ and $\cos(m\varphi)$ are common with those of the familiar two-dimensional circular and three-dimensional spherical harmonic functions. The second one shares the Legendre operator of the three-dimensional Laplace equation; the difference between the respective equations resides in the vanishing of the right-hand side in (5) versus the eigenvalue term in the Legendre equation. The casting of Eq. (5) in the form of (6), involving the successive application of the same operator two times, simplifies the solution of the second-order differential equation by transforming it into the integrable first-order differential equations (7) with the two possible signs of m. Thus, the two independent solutions in the polar angle for each m are given in (8), (9), (10) and (11). Correspondingly, the most general solution of (6) is constructed as the superposition of the respective Laplace–Beltrami harmonic functions as expressed by Eq. (12).

In Sec. 3, the Poisson–Beltrami equation (13) with a unit point source on the spherical surface introduces the corresponding Green's function. The equation is solved by writing the Dirac-delta distribution in the azimuthal angle as the completeness relationship for the Fourier cosine basis, (15). The Green's function is written also as a Fourier series in the same basis with coefficients depending on the polar angle $g_m(\theta, \theta')$. The Green's function must satisfy the conditions of being invariant under the exchange of the source and field points, harmonic at field points other than the source point, and continuous at the source point. The choices made in Eqs. (18) and (19) guarantee that these conditions are satisfied, distinguishing between points on one side or the other of the circle parallel to the equator at the source point. Substitutions of the respective Fourier series in (14), and the orthogonality of the Fourier basis lead to the ordinary differential equations (16) and (17) for the g Fourier coefficients. Integration of the latter in the vicinity of the polar angle of the source point allows us to identify the discontinuity of the derivatives of the g functions at the corresponding parallel circle, and the determination of the unknown constants in Eqs. (18) and (19). Thus, the construction of the Laplace–Beltrami harmonic function expansion of the Green's function is completed as shown in (24).

It is also pertinent to point out some of the relationships between the results here presented and those presented in a couple of the research articles cited in the introduction. Indeed, the spherical harmonic expansion of the Green's function of our Eq. (12) is one of the key elements in Ref. 2. On the other hand, the two-dimensional harmonic and Green's functions in the plane in circular coordinates - Eqs. (14) and (60) in Ref. 4 - and on the spherical surface - equations (9), (10), (11) and (24) - are connected via stereographic projections between the points on the equatorial plane $(\rho = a\cot(\theta/2), \varphi)$ and the points on the spherical surface $(r = a, \theta = 2\text{arccot}(\rho/a), \varphi)$ [6].

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