Chua's circuit from the linear system perspective

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This paper analyzes the Chua's circuit, considering it linear by parts in its internal representation model. This approach allows graduate students to develop skills in matrix algebra and linear systems, using physics as a fundamental tool to model and simulate dynamic systems. Through this approach, the student will understand the importance of the knowledge of state equations to form the matrices that represent the circuit, where the main matrix dictates its values and eigenvectors, as well as its similar matrix. A physical simulation of the Chua's circuit is carried out, along with the corresponding experiments. The electric circuit is composed by resistors, capacitors, inductance, diodes and an operational amplifier.

Keywords: Nonlinear systems; phase portrait; analog circuits; differential equation; Chua's circuit.

Este trabajo analiza el circuito de Chua tomado lineal por partes en su modelo de representación interna. Este enfoque permite a los estudiantes de ciencias desarrollar sus competencias en algebra matricial y sistemas lineales, con el uso de la física como herramienta fundamental para modelar y simular sistemas dinámicos. El estudiante aprenderá que esta herramienta necesita del conocimeinto de las ecuaciones de estado para formar las matrices que representan dicho circuito. Donde la matriz principal dicta sus valores y vectores propios, así como su matriz similar. Posteriormente se procederá a simular físicamente el circuito de Chua y finalmente se realizarán los experimentos correspondientes. El circuito eléctrico esta constituido por resistencias, capacitores, bobina, diodos y amplificador operacional.

Descriptores: Sistemas lineales; retrato de fase; electrónica analógica; ecuaciones diferenciales ordinarias; circuito de Chua.

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1. Introduction

As students pass through their education at the undergraduate and graduate level, there is a need to develop skills in various disciplines; for instance, physics and mathematics are indispensable tools to address and generate practical solutions to current problems. Therefore, students must be able not only to analyze theoretical and technical solutions, but also to observe their relation to the context and other aspects of professional fields. Within this context, a linear systems course is presented as an excellent opportunity to integrate the acquired skills in the areas of physics and mathematics in order to design, model and simulate linear systems. A topic of high importance is the concept of state space resulting from the use of matrices.

It is important to use software tools that allow us to model and simulate such systems. Also, it is well known that the use of software as an alternative in the search of a particular numerical solution from a mathematical expression, is very convenient. Once we have used the concepts of physics and mathematics, as well as the technological resources through software, it is essential to continue all the way to the experimental realization; which leaves an unforgettable impression on students.

In this paper, we take a chaotic system that has the particularity of being linear in sections. The study of chaotic phenomena using tools from linear systems, gives students great skill in handling the matrix language, and on the other hand gives them the opportunity to deal with nonlinear systems.

In the last two decades, the study of chaos theory has been of great interest in mathematics, physics and engineering areas; the erratic behavior represented by mathematical models and their implementation across physical systems are studied. Generally, these systems are modeled by a set of nonlinear ordinary differential equations. In particular, one of the most widely studied systems is the Chua's circuit [1,2], considered as the transition from practice to theory. In other words, this system was the one that first hooked the attention of many researchers because in terms of theoretical development, it is relatively easy to physically implement different types of oscillators. One example is that different authors have shown that for linear systems attractors parts can be generated with multiscroll, as is the case of Suykens and Vandewalle [3, 4], where they introduced a family of n-scroll attractors. We decided to study the Chua's circuit because it gives us the ease of having a linear piecemeal approach, where each part is a linear system.

To carry out the objectives mentioned above, some definitions and explanations (grounded mathematically) are given, to understand what is happening physically and theoretically. After being in context follows the study in detail of the Chua's circuit from the equations of state, also we will show that it is easier to study this circuit with a similar matrix than with the original matrix.

Once the mathematical system is understood and solved, the next step consists in making the correspondent simulation with the help of a computer package. Later, details of the performed experiments are given, where the behavior of this circuit is documented [5]. This work is structured as follows: in Sec. 2 definitions and theorems are given with their demonstrations. In Sec. 3, the mathematical model of Chua's circuit is developed through Kirchhoff's laws. In Sec. 4, the numerical solutions using the software Mathematica are shown as well as experimental results of the circuit, discussion and conclusions are exposed in Sec. 5.

2. Definitions and Theorems

In this section definitions of terms such as square matrix, eigenvalues, eigenvectors, matrix transformation, similar matrix, and some others are given, as well as theorems involving them. A good part of these definitions are developed numerically in Sec. 4.

Definition 2.1 *We say that a square matrix A $n \times n$ is invertible (not singular) if $\exists$ other $n \times n$ square matrix B such that: $AB = BA = I$. B is called the inverse matrix A, denoted by A^{-1}.*

Theorem 2.1 *If the matrix A is invertible, then its inverse is unique.*

***Proof*:** *Suppose A is invertible. B and C are inverses matrices of A, then $B = IB = (CA)B = C(AB) = CI = C$. As it demonstrated by the uniqueness of the inverse of A.*

Definition 2.2 *be a matrix $A_{n\times m}$. Then $Ker(A) = \{x \in F^m : Ax = 0\}$*

Theorem 2.2 *If $A_{n\times n}$ is invertible, then $Ker(A) = \{0\}$.*

***Proof*:** *Let $x \in Ker(A)$, then $Ax = 0$. Now $A^{-1}Ax = A^{-1}0$*, then $x = 0$. *It follows that in the kernel of A only can find the* 0.

Definition 2.3 *We say that a matrix A is similar to a matrix B if $\exists$ Q an invertible matrix such that: $A = Q^{-1}BQ$ where A, B, $Q \in M_{n\times n}(F)$*

Definition 2.4 *A matrix $A \in M_{n\times n}(F)$ is said diagonalizable if this is similar to a diagonal matrix D.*

Definition 2.5 *be $A \in M_{n\times n}(F)$, an element non zero $x \in F^n$ is called eigenvector of matrix A if $\exists$ an scalar λ such that $Ax = \lambda x$. Scalar λ is called eigenvalue corresponding to the eigenvector x.*

Theorem 2.3 *Let $A \in M_{n\times n}(F)$, then an scalar $\lambda \in F$ is an eigenvalue of A if and only if* $\det(A - \lambda I) = 0$.

***Proof*:** ($\Rightarrow$) *Suppose that λ is an eigenvalue of A, it follows that $\exists$ $x \in F^n$ nonzero such that*:

$Ax = \lambda x$ (*by definition of eigenvalue*), then

$Ax - \lambda x = 0$ (*subtracting both sides $-\lambda x$*)

$(A - \lambda I)x = 0$ (*distributive upside down on the right*)

It follows that $(A - \lambda I) = 0$ (because x is not zero) $\det(A - \lambda I)x = 0$ (*applying determining both sides*).

($\Leftarrow$) *Suppose* $\det(A - \lambda I)x = 0$, *it follows that the matrix $(A - \lambda I)$ is not invertible and thus $\exists$ x (nonzero) $\in Ker(A - \lambda I)$ such that $(A - \lambda I)x = 0$, clearly $Ax = \lambda x$ and then λ is an eigenvalue of A associated with eigenvector x.*

Definition 2.6 *if $A \in M_{n\times n}(F)$, the polynomial $f(t) =$* $\det(A - \lambda I)$ *is called characteristic polynomial of A.*

Theorem 2.4 *Let $A, B \in M_{n\times n}(F)$. If A is similar to B, then A and B have the same characteristic polynomial.*

***Proof*:** *If we show that similar matrices have the same determinant, then the test is reduced to demonstrate that $(A - \lambda I)$ is similar $(B - \lambda I)$.*

First is $A \in M_{n\times n}(F)$ similar to $B \in M_{n\times n}(F)$, then $\exists$ $Q \in M_{n\times n}(F)$ such that $A = Q^{-1}BQ$.

Now the determinant of A is to be $\det(A) = \det(Q^{-1}BQ) = \det(Q^{-1})\det(B)\det(Q) = \det(B)$ (*since* $\det(Q^{-1}) = 1/\det(Q)$).

We conclude that similar matrices have the same determinant. Let $f(t)$ the characteristic polynomial of A and $g(t)$ the characteristic polynomial of the matrix B. Then consider $Q^{-1}(B - \lambda I)Q = Q^{-1}BQ - \lambda I = A - \lambda I$.

It appears that $A - \lambda I$ is similar to the matrix $B - \lambda I$, therefore $\det(A-\lambda I)=\det(B-\lambda I)$ *this proofs that if A is similar to B, then* A *and B have the same characteristic polynomial.*

Theorem 2.5 *Let A be a square matrix and f(t) its characteristic polynomial. So*

(a) *A scalar λ is an eigenvalue of A if and only if λ is a root of f(t).*

(b) *A has as maximum n different eigenvalues.*

***Proof*:**(a) ($\Rightarrow$) *Suppose λ is eigenvalue of A, then $\exists$ $x \in F^n$ nonzero such $Ax = \lambda x$. Then $(A - \lambda I)x = 0$, it follows that $(A - \lambda I)$ is not invertible and that x is non-zero so* $\det(A - \lambda I) = 0 = f(\lambda)$ *and thus λ is the root of f(t).*

($\Leftarrow$) *Suppose it λ is a root of f(t), then $f(\lambda) =$* $\det(A - \lambda I) = 0$. *It follows that $(A - \lambda I)$ is not invertible so $\exists$ $x \in F^n$ nonzero such that $(A-\lambda I)x = 0$. Then $Ax = \lambda x$. That is, λ is an eigenvalue of A.*

(b) *$f(t)$ is a polynomial of degree n, by the* ***Fundamental Theorem of Algebra*** *$f(t)$ has n roots. It follows that A has at most n distinct eigenvalues.*

Theorem 2.6 *be $A_{n\times n}$ and be λ an eigenvalue of A. A vector $\in F^n$ is eigenvector of A corresponding to λ if and only if $x \neq 0$* and $x \in$ *Ker*$(A - \lambda I)$.

***Proof*:** ($\Rightarrow$) *Suppose that x is an eigenvector of A corresponding to the eigenvalue λ, then by definition x is not null and $Ax = \lambda x$. Then $(A - \lambda I)x = 0$, it follows that $x \in Ker(A - \lambda I)$.*

($\Leftarrow$) *Suppose x is not zero and $x \in$* Ker$(A - \lambda I)$, *then $(A - \lambda I)x = 0$, it follows that $Ax = \lambda Ix$ and therefore x is an eigenvector of A corresponding to λ.*

Definition 2.7 *be $A_{n\times n}$ and is λ an eigenvalue of A. $E_\lambda = \{x \in F^n : Ax = \lambda x\} =$* Ker$(A - \lambda I)$. *$E_\lambda$ is called the eigenspace of A corresponding to λ.*

Given a linear transformation T: $V \rightarrow V$, where V is a dimensionally finite vector space over a field K, this linear transformation is a matrix representation, and this matrix representation is obtained by applying the transformation vectors to a fixed base. Suppose that the basis of the vector space is β, and the matrix representing the transformation on that basis denoted by $[T]_\beta$, is a matrix in which the operations are not so simple. The problem lies in finding another basis β_1 of V such that $[T]_{\beta_1}$ it is diagonal or block diagonal.

Any linear operator whose characteristic polynomial factors decompose in grade 1 there is a basis β for V such that

$$[T]_\beta = J_1 \oplus ... \oplus J_k \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_k \end{pmatrix} \tag{1}$$

where J_i is a square matrix even if it is 1×1 in case the matrix is diagonal this matrix is called a block of Jordan, the matrix $[T]_\beta = J_1 \oplus \ldots \oplus J_k$ is called Jordan canonical form [6].

This is introduced because it is easier to work with diagonal matrices, if the matrix is known, it is implied that the linear transformation is also known and can be modified through this transformation matrix as carried out in modeling.

3. Modeling Chua's circuit

Chua's circuit (the simplest electronic circuit where a double scroll appears) is modelled in this section through the state equations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{2}$$

Which form the internal representation, understanding that t is always present in the variables and for simplicity, we remove and rewrite the system as

$$\begin{aligned} \dot{x}b &= Ax + Bu \\ y &= Cx + Du, \end{aligned} \tag{3}$$

Where A, B, C and D are matrices and let x be the vector column. Now it is time to introduce Chua's circuit (shown in Fig. 1). A v_m input was added to the circuit and taking as its output the potential v_2, the system can thus be interpreted through its state equations.

The Chua's diode has the skeleton shown in Fig. 2.

Now applying the laws of Kirchhoff of current and voltage, an internal representation model is obtained:

$$\begin{aligned} \dot{v_1} &= -\frac{(G+G_m)}{c_1} v_1 + \frac{G}{c_1} v_2 + \frac{G_m}{c_1} v_m - \frac{1}{c_1} I_x \\ \dot{v_2} &= \frac{G}{c_2} v_1 - \frac{G}{c_2} v_2 - \frac{1}{c_2} I_L \\ \dot{I_L} &= \frac{1}{L} v_2 \end{aligned} \tag{4}$$

where

$$I_x = \begin{cases} -G_1 v_1 - (G_0 - G_1) v_D, & \forall \quad v_1 > v_D \\ -G_0 v_1, \quad \forall \quad |v_1| \leq v_D \\ -G_1 v_1 + (G_0 - G_1) v_D, & \forall \quad v_1 < v_D \end{cases} \tag{5}$$

The latter equation gives the possibility of interpreting the Chua's circuit as three systems, which are obtained when the Diode 1 is on; when both diodes are not working and when Diode 2 is on, respectively. Then, the A matrix in (2) now is transformed in three A_i, where $i = 1, 2, 3$.

$$A_1 = A_3 = \begin{pmatrix} -\frac{(G+G_m-G_1)}{c_1} & \frac{G}{c_1} & 0 \\ \frac{G}{c_2} & -\frac{G}{c_2} & -\frac{1}{c_2} \\ 0 & \frac{1}{L} & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -\frac{(G+G_m-G_0)}{c_1} & \frac{G}{c_1} & 0 \\ \frac{G}{c_2} & -\frac{G}{c_2} & -\frac{1}{c_2} \\ 0 & \frac{1}{L} & 0 \end{pmatrix} \tag{6}$$

For example, if matrix A_1 is chosen, it's obtained:

$$\begin{aligned} \dot{x} &= \begin{pmatrix} \dot{v_1} \\ \dot{v_2} \\ \dot{I_L} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{(G+G_m-G_1)}{c_1} & \frac{G}{c_1} & 0 \\ \frac{G}{c_2} & -\frac{G}{c_2} & -\frac{1}{c_2} \\ 0 & \frac{1}{L} & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ I_L \end{pmatrix} \\ &+ \begin{pmatrix} \frac{G_m}{c_1} \\ 0 \\ 0 \end{pmatrix} V_m, \\ y &= (0\ 1\ 0)\, x + (0) V_m \end{aligned} \tag{7}$$

Here it becomes clear that the matrices B, C and D are the same in all three cases, the only changing matrix is A.

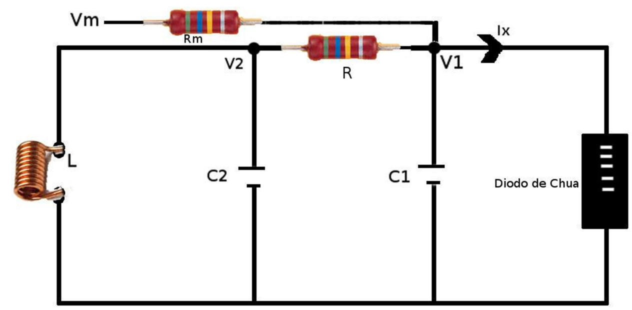

FIGURE 1. Chua's circuit.

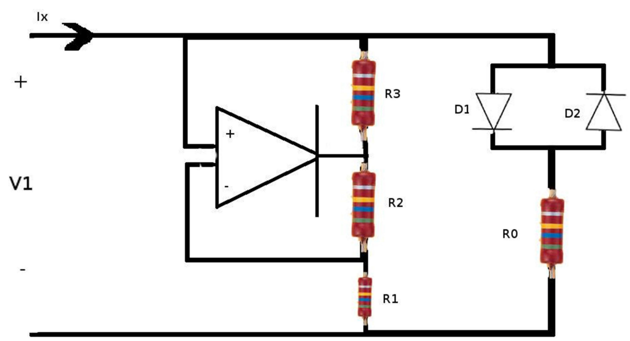

FIGURE 2. Chua's diode.

TABLE I. Values used in the Chua's circuit.

element	value
R_m	10 k Ω
R_0	1.2 k Ω
R_1	750 Ω
R_2	220 Ω
R_3	220 Ω
R	1.8 k Ω
L	18 mH
c_1	10 nF
c_2	100 nF
G_0	$\frac{R_2}{R_1 R_3}$
G_m	$\frac{1}{R_m}$
G_1	$\frac{R_0 R_2 - R_1 R_3}{R_0 R_1 R_3}$
G	$\frac{1}{R}$
V_D	0.7 Volts

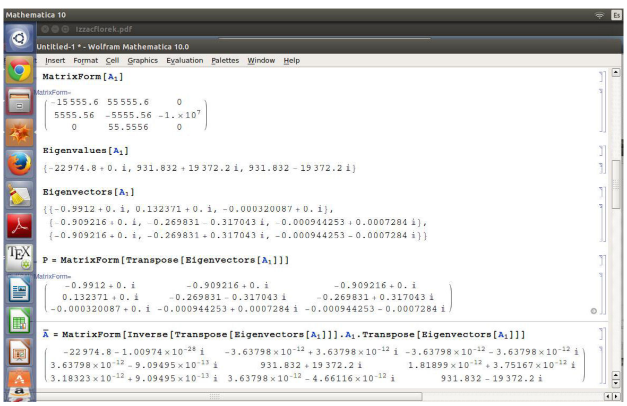

FIGURE 3. Computed values using Mathematics 10.0.

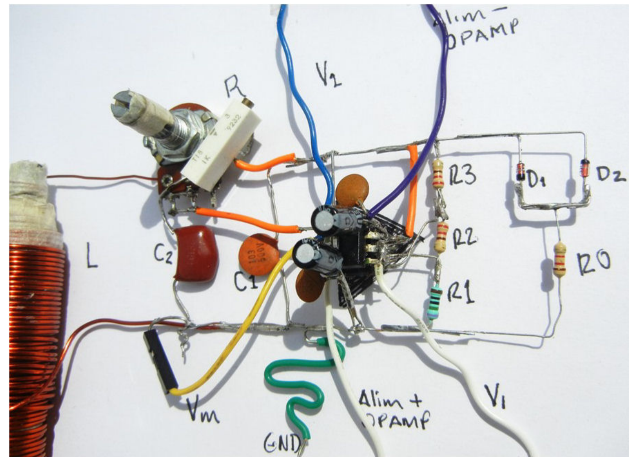

FIGURE 4. Experimental circuit of Chua.

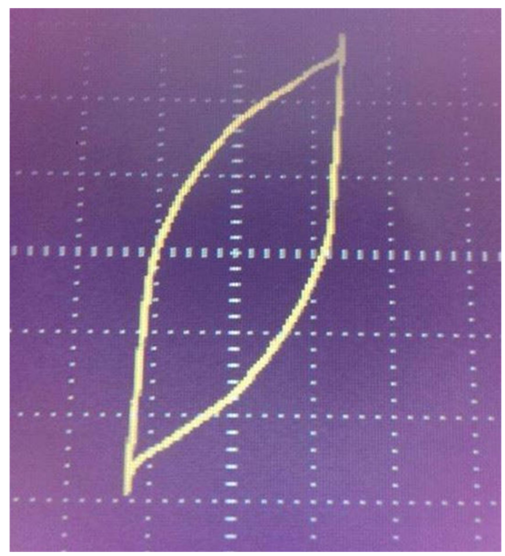

FIGURE 5. Phase portrait of dissipation Ball.

4. Numeric and Experimental Results

The equilibrium points that describe the system are found in this section, as well as the description of the two basins of attraction experimentally observed in Chua's circuit. Before providing the numerical results, some definitions are cited, which will be validated by experimental results.

Definition 4.1 *A pair* (M, f^t) *where* f^t *satisfies the properties of group and M is almost always a metric space is said dynamic system.*

Definition 4.2 *The set* $\cup_{t=-\infty}^{\infty} f^t x_0$ *is called path or orbit for a given point* x_0.

Definition 4.3 *Let* $x \in M$, $F \in C^1$ (*differentiable*) *we assume that* S *is a topological sphere.* D *is called dissipation ball if*:

a) (*Grad* φ,$F(x)$) $< 0 \forall x \in S$, φ *is a differentiable function from* M *to the field* K.

b) $\forall x \in M \ \exists t$: $f^t \ x \in D$.

Theorem 4.1 *Two state equations* $\{A, B, C, D\}$ *and* $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ *are equivalent zero state or have the same transfer function,* $\Leftrightarrow$ D= $\bar{D}$, CA^{m}B=$\bar{C}\bar{A}^m\bar{B}$, $m = 0, 1, 2, \ldots$

See demo [7].

Chua's circuit is constructed with the parameters given in Table I.

It should be specified that the below presented analysis consists on Diode 1 working (that is, only for A_1), since the process is similar to the other two cases. The software Mathematics 10.0 running on the Ubuntu-Linux platform was used to obtain the numerical values of the matrix A_1, its eigenvalues, eigenvectors, the transformation matrix P and the similar matrix. The obtained results are shown in Fig. 3.

Note that using this software you can obtain the similar matrices by applying:

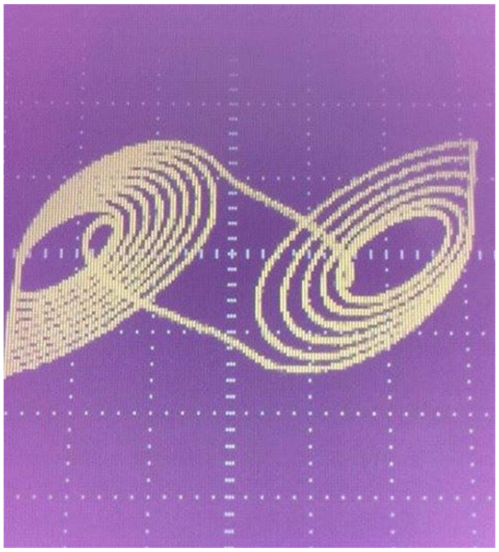

FIGURE 6. Phase portrait with two attractors.

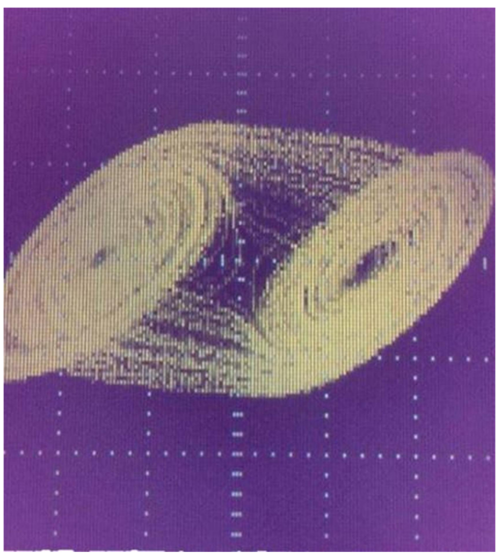

FIGURE 7. Phase portrait for $t \to \infty$.

$$\begin{aligned} \bar{A} &= PAP^{-1} \\ \bar{B} &= PB \\ \bar{C} &= CP^{-1} \\ \bar{D} &= D. \end{aligned} \tag{8}$$

This way you are able to work with the similar system in its internal representation.

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u, \\ y &= \bar{C}\bar{x} + \bar{D}u \end{aligned} \tag{9}$$

FIGURE 8. One attractor.

On the other hand, as a complement to the theory of linear systems, it is shown that the model used in this work is a chaotic circuit (see Fig. 4). In order to analyze this kind of systems, we have to find the equilibrium points of the dynamical system. Equilibrium points are obtained by equaling to zero the system (4) and replacing parts of the Eq. (5) concerned; it is assumed that the system has no entry (*i.e.*) in Fig. 1, $V_m = 0$. If the calculations are carried out, the following equilibrium points are obtained:

$$PE = \{(-3.09, 0, -1.8), (0, 0, 0), (3.09, 0, 1.8)\} \tag{10}$$

Figure 5 illustrates the ball of dissipation of the system, in this case it is a topological circle. In other words: for any given point x_0 in the xy plane there exists a number $t > 0$ such that $A^t(x_0) = A(A(A(A(...(A(x_0)))))$ forms a path that goes into the ball of dissipation. Once inside it, it cannot go out and it is confined to oscillate between the points of balance, where A is the matrix system. See demonstration [8].

Figure 6 depicts a phase delay of the equilibrium points, which behave as attractors and repulsors. Even if the number of oscillations is enormous, the system will never leave the dissipation ball. Mathematically, this situation has a starting point and the number $t > 0$ is small.

For Fig. 7 the number $t > 0$ is large enough to generate sufficient oscillations.

Figure 8 illustrates the case when the system is only able to orbit in one attractor, and it does not have the necessary amount of energy to leave one attractor and go to the other.

5. Conclusions

Different software tools can be used to find the solution of the matrix equations. In this paper, we used Mathematics 10.0 running on the Ubuntu-Linux platform. Matlab can also be used to solve this sort of problems, however, neither Mathematics 10.0 nor Matlab are open access utilities, hence their purchase requires an investment, making it difficult for students to use on their simulations and models. Luckily, freely distributed options are available, packages like Octave and Scilab software can be used as auxiliaries in teaching physics and mathematics.

Another important point of this work is the agreement of the experimental results with the theoretical predictions. This fact is very appealing to students, since the application of theory on real-life problems adds great value to their development as professionals.

As future work, we will attempt to mask an information signal through this circuit using the potential v_m marked in Fig. 1, which could be used on signal-processing related issues.

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