Anomaly of non-locality and entanglement in teaching quantum information

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Non-locality and quantum entanglement constitute two special features of quantum systems of enormous relevance in quantum information theory (QIT). Historically regarded as identical or equivalent for many years, they constitute different concepts. We want to stress in the present contribution such difference which actually may guide the instructor to cover the most essential topics of QIT in any section of some introductory course. We shall address the simplest possible case of that of two qubits. The material may be of interest to instructors teaching high-undergraduate quantum mechanics, as well as students.

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1. Introduction

In teaching quantum mechanics (QM) to upper-year undergraduates a bit of the basics of QIT, it is quite common to study the singlet state as a paradigm. When the two spin-$(1/2)$ particles are moving apart, towards two distant observers –*Alice* and *Bob*, a quite common terminology borrowed from information theory– their spins will be measured *locally* (usually along the z-axis for simplicity). Students are told that whenever Alice measures either $\hbar/2$ or $-\hbar/2$, she *instantly* knows that Bob will obtain the opposite result upon measuring the other particle's spin along the same direction. Then it is mentioned that the two particles are *entangled* or, in other words, they display a quantum correlation which is called *non-locality*. This zeroth approximation to QIT for quantum mechanics students is extremely common in non-specialized textbooks. The discussion then goes on to address if causality is violated –which is not– and how it is possible that after the measurement on one particle, the state of the other becomes *immediately* well-defined.

Schrödinger's reply [1] to the paradox posed by Einstein *et al*. [2] motivated the modern notion of entanglement in a quantum system. Schrödinger, opposing EPR, did not recognize any conflict in their argument and regarded entanglement, or the impossibility of describing two particles separately from each other, as *the* characteristic feature of QM.

The above description explained in terms of spins is the Bohm's simplified version [3] of the original argument by Einstein, Podolsky and Rosen (EPR) [2] in 1935 (for about fifteen years following EPR!). EPR suggested a description of nature, called "local realism", which assigned independent properties to distant parties in a composite system to conclude that QM was incomplete. They used instead the Heisenberg position-momentum uncertainty principle $\Delta x \cdot \Delta p \geq \hbar/2$.

The existing literature regarding entanglement is vast. The role played by entanglement [4, 4–13] in quantum systems for two parties and in the context of general many-body systems has been the subject of extensive work (See [14] and references therein). The advent of quantum-information theory (QIT) boosted the interest for entanglement since it lies at the basis of some of the most important processes and applications studied by QIT such as quantum cryptographic key distribution [9], quantum teleportation [10], superdense coding [11] and quantum computation [12,13], among many others which possess no classical counterpart.

Likewise, extensive research has also been made on the issue of non-locality, ranging from its contextualization in QM [15–17], regarding experimental set-ups in the undergraduate laboratory [18], and also somehow against the usual *mainstream* [19]. We want to highlight that our contribution is not to review entanglement and non-locality, but to cope with the best way to teach them at the undergraduate level without confusion.

There are two key assumptions that are misunderstood in the usual first attempts to address QIT by studying two qubits. On the one hand, any two particles governed by a given statistics, let us say two fermions, are **not** entangled for the sake of being anti-symmetric. They just possess a state which is anti-symmetric, that is all. These "statistical correlations" are not the quantum correlations that are required in QIT. On the other hand, it is usually taken for granted that *entanglement* and *non-locality* are equivalent, which is partially true in the case of two qubits, but not in the general case. In point of fact, entanglement and non-locality constitute two different concepts.

The clarification of the former second issue constitutes the aim of the present contribution. Actually, the process itself of addressing similarities between the two notions will unfold basic issues in QIT that will clarify many ideas to the students. We shall assume that the student is already familiar with the definition of a two qubit state, which is the only instance we will address here.

2. Entanglement and Bell inequalities

The modern definition of entanglement is based in partitioning the corresponding Hilbert space of the system, which was provided by Werner [20] in 1989 for the bipartite case: a state of a composite quantum system constituted by the two subsystems A and B is called entangled if it can not be represented as a convex linear combination of product states. In other words, the density matrix $\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ represents an entangled state if it *cannot* be expressed as as the mixture of product states

$$\rho_{AB} = \sum_k p_k \, \rho_A^{(k)} \otimes \rho_B^{(k)} = \sum_k p_k |\psi_A^k\rangle\langle\psi_A^k| \otimes |\psi_B^k\rangle\langle\psi_B^k|, \qquad (1)$$

with $0 \leq p_k \leq 1$ and $\sum_k p_k = 1$ (convex sum). On the contrary, states of the form (1) are called separable or unentangled. The above definition is physically meaningful because entangled states (unlike separable states) should not be "created" by acting on any subsystem individually. The set of those operations are called Local Operations and Classical Communications (LOCC operations). An example of a LOCC operation is provided by

$$\rho\prime = (U_A \otimes U_B)\, \rho\, (U_A \otimes U_B)^{\dagger}, \qquad (2)$$

where U_A(U_B) represents a local action (a unitary transformation) acting on subsystem A(B).

Usually, the preferred basis for two qubit states is the so called computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Also, it may be at some point convenient to employ the so called Bell basis of maximally correlated states, which are of the form

$$|\Phi^{\pm}\rangle = \frac{(|00\rangle \pm |11\rangle)}{\sqrt{2}}, \qquad |\Psi^{\pm}\rangle = \frac{(|01\rangle \pm |10\rangle)}{\sqrt{2}}, \qquad (3)$$

with $|ij\rangle = |i_A\rangle \otimes |j_B\rangle$.

If the state we are studying is pure, then $\rho_{AB} = |\Psi\rangle\langle\Psi|$. One measure that quantifies how entangled (pure and mixed) states are is given by the concurrence, defined for pure states in the following fashion: when a pure state $|\psi\rangle$ is written in the Bell basis, as $|\psi\rangle = \alpha_1|\Phi^+\rangle + \alpha_2|\Phi^-\rangle + \alpha_3|\Psi^+\rangle + \alpha_4|\Psi^-\rangle$, the concurrence measure is actually defined as $C(|\psi\rangle) = |\sum_i \alpha_i^2|$. The quantity C ranges from zero to one. Another way of computing it is given by the relation $C^2 = 4\text{det}(\rho_A) = 4\text{det}(\rho_B)$. ρ_A and ρ_B are the reduced density matrices of the original state ρ_{AB} obtained by tracing out the degrees of freedom of the other subsystem $\rho_{A,B} = \text{Tr}_{B,A}\rho_{AB}$. That is, $\rho_A = \langle 0_B|\rho_{AB}|0_B\rangle + \langle 1_B|\rho_{AB}|1_B\rangle$ and similarly for ρ_B. The computation of the concurrence for mixed states is more involved. One widely used measure for entanglement is the so called *entanglement of formation*. For two-qubits systems is given by Wootters' expression [21], $E[\rho] = h\left(1 + \sqrt{1 - C^2}/2\right)$, where $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$, and C stands for the *concurrence* of the two-qubits state ρ. The concurrence is given by $C = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \lambda_i, \ (i = 1, \ldots 4)$ being the square roots, in decreasing order, of the eigenvalues of the matrix $\rho\tilde{\rho}$, with $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$. The above expression has to be evaluated by recourse to the matrix elements of ρ computed with respect to the product basis.

Now, what about entanglement and EPR? The most significant progress toward the resolution of the EPR debate was made by Bell [22] in 1960s. Bell showed that what is called local realism, mathematically in the form of local variable models (LVM), implied constraints (Bell inequalities) on the predictions of spin correlations. That is, separated observers sharing an entangled state and performing measurements could induce (nonlocal) correlations which cannot be mimicked by local means (violate Bell inequalities). This physical limitation is actually exploited for implementing QIT tasks. In other words, Bell's crucial contribution was that he devised a way to experimentally test the structure of QM.

Let us put into simple words how to understand the trinomial quantum correlations - entanglement - non-locality. Entanglement is the potential of quantum states to exhibit correlations that cannot be accounted for classically. Using Bohm's argument within the framework of two distinguishable parties A and B sharing a two-qubit singlet state, either party's (A or B) measurement outcome will immediately unveil the other side's result, and outcome that cannot be communicated user faster-than-light means. EPR where strongly against this *non-local* aspect of QM, for their view was that nothing that can be measured *locally* can affect a distant party. In short, LVM constitute mathematical attempts to mimic the experimental results of the quantum theory by introducing random (and, of course, *local*) variables and averaging them so as to obtain an equivalent result.

Most of our knowledge on Bell inequalities and their quantum mechanical violation is widely based on the Clauser-Horne-Shimony-Holt (CHSH) inequality [23]. We have to stress the fact that Bell's original mathematical inequality was different. With two dichotomic observables per party, it is the simplest [24] (up to local symmetries) nontrivial Bell inequality for the bipartite case with binary inputs and outcomes. Let A_1 and A_2 be two possible operators or observables acting on A side whose outcomes are $a_j \in \{-1, +1\}$, and similarly for the B side. Mathematically, it can be shown, following LVM, that the averge of the classical operator $\mathcal{B}_{CHSH}^{LVM}$ representing the Bell inequality

$$\left|\mathcal{B}_{CHSH}^{LVM}(\lambda)\right| = |a_1b_1 + a_1b_2 + a_2b_1 - a_2b_2| \leq 2. \quad (4)$$

The letter λ represents a random variable belonging to a certain LVM probability distribution $\mu(\lambda)$. Since $a_1(b_1)$ and $a_2(b_2)$ cannot be measured simultaneously (only one outcome at a time, therefore they ought to be measured at different times), instead one estimates after randomly chosen measurements the average value of the

$$\mathcal{B}_{CHSH}^{LVM} \equiv \sum_{\lambda} \mathcal{B}_{CHSH}^{LVM}(\lambda)\mu(\lambda) = E(A_1, B_1) + E(A_1, B_2) + E(A_2, B_1) - E(A_2, B_2), \quad (5)$$

where $E(\cdot)$ represents the expectation value. Therefore the CHSH inequality reduces to

$$\left|\mathcal{B}_{CHSH}^{LVM}\right| \leq 2. \quad (6)$$

Quantum mechanically, since we are dealing with qubits, these observables reduce to $\mathbf{A_j}(\mathbf{B_j}) = \mathbf{a_j}(\mathbf{b_j}) \cdot \sigma$, where $\mathbf{a_j}(\mathbf{b_j})$ are unit vectors in $\mathbb{R}^3$ and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the usual Pauli matrices. Therefore the quantal prediction for (6) reduces to the expectation value of the operator

$$\mathcal{B}_{CHSH} = \mathbf{A_1}\otimes\mathbf{B_1} + \mathbf{A_1}\otimes\mathbf{B_2} + \mathbf{A_2}\otimes\mathbf{B_1} - \mathbf{A_2}\otimes\mathbf{B_2}. \quad (7)$$

Tsirelson showed [25] that CHSH inequality (6) is maximally violated by a multiplicative factor $\sqrt{2}$ (Tsirelson's bound) on the basis of quantum mechanics. In fact, it is true that $|Tr(\rho_{AB}\mathcal{B}_{CHSH})| \leq 2\sqrt{2}$ **for all observables $\mathbf{A_1}$, $\mathbf{A_2}$, $\mathbf{B_1}$, $\mathbf{B_2}$, and all states** ρ_{AB}. Notice that $Tr(\rho\mathcal{B}_{CHSH})$ is the general way of computing the expected value of a Bell inequality for a quantum state ρ, pure or mixed. In general, it is not known how to calculate the best such bound for an arbitrary Bell inequality.

Non-locality is now to be understood as that property displayed by quantum states whose quantum predictions (violation of Bell inequalities) cannot be reproduced by LVM (hence the name *non-local*).

Ever since Bell's contribution, entanglement and non-locality were essentially regarded as the same thing. With the advent QIT, interest in entanglement dramatically increased over the years [26,27].

How do quantum correlations relate to non-locality after-all, that is, the violation of a Bell inequality? We can ascertain that if a state violates a Bell inequality, it is because it possesses correlations that are of quantum nature, that is, cannot be imitated by an alternative local theory. In a way, the violation of a Bell inequality indicates the "quantumness" of the state of a system. The converse, however, it is not true. One can encounter states that are genuinely entangled but do not violate a Bell inequality. This confusion is precisely discussed in the following Section.

3. Gisin theorem

Confusion between non-locality and entanglement arose when the usefulness of quantum correlations was put in doubt (see [28]). The nonlocal character of entangled states, however, is clear for pure states since *all entangled pure states of two qubits violate the CHSH inequality and are therefore nonlocal*. This is the celebrated Gisin's Theorem [29] that we are about to show.

Suppose we are given an arbitrary general pure state of two qubits as

$$|\psi(\theta)\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle. \quad (8)$$

Quantum mechanically, we have

$$E(\mathbf{A_j}, \mathbf{B_j}) = \langle\psi(\theta)|(\vec{a}_i \cdot \vec{\sigma}) \otimes (\vec{b}_j \cdot \vec{\sigma})|\psi(\theta)\rangle = a_z^i b_z^j + \sin(2\theta)\left(a_x^i b_x^j - a_y^i b_y^j\right). \quad (9)$$

We then gather these expressions and maximize the ensuing quantity over all possible orientations of the settings for Alice and Bob. It is not difficult to show that the arrangement $\{\vec{a}_1 = \hat{z}, \vec{a}_2 = \hat{x}, \vec{b}_1 = (1/\sqrt{2})(\hat{z}+\hat{x}), \vec{b}_2 = (1/\sqrt{2})(\hat{z}-\hat{x})\}$ returns the optimal value

$$\mathcal{B}_{CHSH}^{\max} = 2\sqrt{1 + \sin^2(2\theta)}, \quad (10)$$

which is always greater than two, unless we have an unentangled state ($|00\rangle = |0_A\rangle\otimes|0_B\rangle$ or $|11\rangle = |1_A\rangle\otimes|1_B\rangle$). Now, if we recall the definition for the concurrence measure of entanglement for a two qubit pure state, $C^2 = 4$det(ρ_A)=4det(ρ_B), and ρ_A =diag($\cos^2\theta, \sin^2\theta$), we obtain

$$C^2 = 4\,det\rho_A = 4\cos^2\theta\sin^2\theta = \sin^2(2\theta). \quad (11)$$

Thus, combining (11) and (10), we obtain that $\mathcal{B}_{CHSH}^{\max} = 2\sqrt{1 + C^2}$ for any pure state of two qubits. Since $0 \leq C \leq 1$, we see that all entangled pure states of two qubits violate the CHSH inequality, and are therefore non-local.

However, the situation became more involved when Werner [20] discovered that while entanglement is necessary for a state to be nonlocal, for mixed states is not sufficient. Let us recall that entanglement is necessary for a pure state to display non-local features. After introducing the states which are now called Werner states

$$\rho_W = p|\Psi^-\rangle\langle\Psi^-| + (1-p)\frac{I}{4}, \quad (12)$$

where $|\Psi^-\rangle$ is the singlet state and I is the 4×4 identity, there is a range for p where the state (12) is entangled but does not violate the CHSH Bell inequality.

Summing up, we have to clarify concepts even in the first approach to QIT in QM: entanglement is to be related to the possibility or not of having a state written as a convex sum of a tensor product of individual parties, whereas non-locality is related to the impossibility of LVM to describe or reproduce the predictions associated to that state.

4. Gisin theorem for mixed states?

The study of new Bell inequalities as well as new correlation measures constitutes a subject for actual research, but not for two qubits states only. The study of multipartite quantum correlation is matter of intense current research. We could wonder if the Gisin result holds for general mixed states. The answer is no and the Werner state (12) is just one well-known counterexample.

However, we can ask ourselves whether a particular class of two qubit mixed states exists such that entanglement and non-locality imply each other. If those states exist, they have to violate the CHSH maximally. In other words, we have to look for those mixed states whose quantum correlations are more concentrated. One well-known result [30, 31] is that Bell diagonal states,

$$\rho_{\text{Bell}}^{(\text{diag})} = \lambda_1|\Phi^+\rangle\langle\Phi^+| + \lambda_2|\Phi^-\rangle\langle\Phi^-| + \lambda_3|\Psi^+\rangle\langle\Psi^+| + \lambda_4|\Psi^-\rangle\langle\Psi^-|, \quad (13)$$

do concentrate quantum correlations as well as possessing non-zero entanglement. ρ_{Bell} indicates that it is given in the Bell basis. Actually, if we arrange their eigenvalues λ_i in decreasing order, we obtain

$$\max_{\mathbf{a_j},\mathbf{b_j}} Tr(\rho_{\text{Bell}}^{(\text{diag})}\mathcal{B}_{CHSH}) = 2\sqrt{2} \times \sqrt{(\lambda_1-\lambda_4)^2+(\lambda_2-\lambda_3)^2}. \quad (14)$$

Recall that $2\sqrt{2}$ is the maximum value allowed by quantum mechanics, and fulfilled by any state of the Bell basis.

Let us describe the most general only class two qubit states ρ_G that do fullfill the equivalent Gisin theorem for mixed states. First, we need to know what matrix elements intervene in the computation of Tr($\rho_{Bell}\mathcal{B}_{CHSH}$). Given both ρ_{Bell} and $\mathcal{B}_{CHSH}$ in the Bell basis, we separate [31] the elements of ρ into two contributions, namely

$$\rho = \rho_{\|} + \rho_{\perp} = \begin{pmatrix} \rho_{11} & i\rho_{12}^I & i\rho_{13}^I & \rho_{14}^R \\ -i\rho_{12}^I & \rho_{22} & \rho_{23}^R & i\rho_{24}^I \\ -i\rho_{13}^I & \rho_{23}^R & \rho_{33} & i\rho_{34}^I \\ \rho_{14}^R & -i\rho_{24}^I & -i\rho_{34}^I & \rho_{44} \end{pmatrix} + \begin{pmatrix} 0 & \rho_{12}^R & \rho_{13}^R & i\rho_{14}^I \\ \rho_{12}^R & 0 & i\rho_{23}^I & \rho_{24}^R \\ \rho_{13}^R & -i\rho_{23}^I & 0 & \rho_{34}^R \\ -i\rho_{14}^I & \rho_{24}^R & \rho_{34}^R & 0 \end{pmatrix}. \quad (15)$$

This separation is motivated because only terms in $\rho_{\|}$ contribute to $Tr(\rho B_{CHSH})$. In other words, $Tr(\rho B_{CHSH}) = Tr(\rho_{\|}B_{CHSH}) + Tr(\rho_{\perp}B_{CHSH}) = Tr(\rho_{\|}B_{CHSH})$.

Given the following rank-2 state [30, 31] in the computational basis (any square arrangement of non-zero entries would do)

$$\rho_G = \begin{pmatrix} a & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b^* & 0 & 0 & 1-a \end{pmatrix}, \quad (16)$$

once it is expressed in the Bell basis, it reads as

$$\rho_G = \begin{pmatrix} A & B & 0 & 0 \\ B^* & 1-A & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (17)$$

with $A = (1/2) + \Re(b)$ and $B = a - (1/2) + i\Im(b)$.

Direct comparison between (17) and (15) points out the elements that intervene in the computation of $\mathcal{B}_{CHSH}^{\max}$, being almost diagonal in the Bell basis. In point of fact, when performing the optimization calculations, $\mathcal{B}_{CHSH}^{\max}$ for states (17) is very similar to (14), with one additional term inside the square root. The final outcome is

$$\mathcal{B}_{CHSH}^{\max} = 4\sqrt{1/4 + |b|^2}. \quad (18)$$

This expression is always greater than 2 provided that $|b|^2 \neq 0$. Calculating the concurrence for states ρ_G is straightforward ($C = 2|b|$), leading *exactly to the same result for pure states*, that is, $\mathcal{B}_{CHSH}^{\max} = 2\sqrt{1+C^2}$. We have thus found, apparently, the only family of mixed states for which entanglement and non-locality are equivalent. This result is, to our knowledge, the first time to be reported in the literature. We have to stress the fact that any state with rank greater than two will not display the expected result. This is so because it has been checked that if we let other entries in (15) to be non-zero (that is, higher ranks) the relationship between $\mathcal{B}_{CHSH}^{\max}$ and C is no longer be given in analytic fashion.

5. Conclusions

We have highlighted the fact that non-locality and entanglement should be taught as separate concepts in introductory courses of QM on the subject of QIT. By pointing out the differences and similarities in bipartite systems, the student is more capable of grasping the subtleties associated to entanglement and Bell inequalities. In such a way, the student will regard non-locality and entanglement as different resources from the very beginning. Also, we have provided the only family of states ρ_G of mixed two qubits (not being Bell diagonal) for which both resources imply each other. This result may be of interest to researchers as well as students learning the first tenets of QIT.

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