A novel approach to the Child-Langmuir law

G. González[a,b] and F.J. González[b]
[a] *Cátedras CONACyT, Universidad Autónoma de San Luis Potosí, San Luis Potosí, 78000 México.*
e-mail: gabriel.gonzalez@uaslp.mx
[b] *Coordinación para la Innovación y la Aplicación de la Ciencia y la Tecnología, Universidad Autónoma de San Luis Potosí, San Luis Potosí, 78000 México.*

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We analyze the motion of charged particles in a vacuum tube diode using a new set of variables. We obtain the space charge limited current for a charged particle moving non-relativistically in one dimension for the case of zero and non zero initial velocity without solving a nonlinear differential equation. We introduce what we call the *microscopic* Child-Langmuir law which is valid for the classical and relativistic cases that allows to determine the space charge limited current without solving a nonlinear differential equation.

Keywords: Child-Langmuir law; Mott-Gurney Law; space charge current.

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1. Introduction

The Child-Langmuir (CL) law is one of the most well known and often applied laws of plasma physics which states that the behavior of the current density in a planar vacuum tube diode is proportional to the three-halves power of the bias potential and inversely proportional to the square of the gap distance between the electrodes. For more than 100 years the CL law has been obtained through the solution of a second-order nonlinear differential equation. More recently, a method for estimating the space charge limited current in a vacuum planar tube diode without solving a nonlinear differential equation was described in Ref. 1 by considering the vacuum capacitance of the gap. However, the vacuum capacitance model is only an approximation and is not valid in the relativistic regime.

In this article we present a novel approach to this fundamental law which avoids the need of solving a nonlinear differential equation and provides a microscopic physical insight into the origins of the CL law. Our approach is based by using a new set of variables and requires only what we call the *microscopic* CL law to derive the space charge limited current. We use the *microscopic* CL law to obtain the exact analytical result for the relativistic and non-relativistic regimes.

The article is organized as follows, first we review the traditional solution to the CL law. We then present our new approach where we show how one can obtain the CL law without solving a nonlinear differential equation and introduce what we call the *microscopic* CL law. We also show that the *microscopic* CL law is valid for the relativsitic regime, which makes it a fundamental law for the electron dynamics inside a planar vacuum tube diode. At the end we summarize our conclusions.

2. Traditional approach to the Child-Langmuir law

For infinite planar electrodes having a potential difference V and separated by a distance D, the CL law is obtained by solving Poisson's equation

$$\frac{d^2V}{dz^2} = -\frac{\rho}{\epsilon_0} \tag{1}$$

where V is the electrostatic potential, ρ is the volume charge density and ϵ_0 is the permittivity of free space [2]. We can define the current density by

$$J(z) = \rho(z)v(z) = -J_{CL} \tag{2}$$

where v is the velocity of the electrons. By charge conservation the current density can not vary with z, hence the current density is constant. We can obtain the velocity using conservation of energy, *i.e.*

$$\frac{mv^2}{2} - eV = 0 \tag{3}$$

where m and e are the electron's mass and charge. In Eq.(3) we have assumed that the electron is initially at rest in the grounded cathode. Solving Eq. (3) for the velocity and substituting in Eq. (2) we obtain the volume charge density

$$\rho(z) = -\frac{J_{CL}}{\sqrt{2eV/m}} \tag{4}$$

Substituting Eq. (4) into Eq. (1) we have a second-order nonlinear differential equation for the electrostatic potential

$$\frac{d^2V}{dz^2} = \frac{J_{CL}}{\epsilon_0\sqrt{2eV/m}} \tag{5}$$

with the following boundary conditions

$$\left.\frac{dV}{dz}\right|_{z=0} = 0 \quad \text{and} \quad V(z)\Big|_{z=0} = 0 \tag{6}$$

The solution for Eq. (5) is given by

$$V(z) = V_0 \left(\frac{z}{D}\right)^{4/3} \tag{7}$$

and the volume charge density in the gap is

$$\rho(z) = -\frac{4\epsilon_0 V_0}{9D^2}\left(\frac{D}{z}\right)^{2/3} \tag{8}$$

substituting Eq. (7) and Eq. (8) into Eq. (4) we find that the space charge limited current density is given by

$$J_{CL} = \frac{4\epsilon_0}{9D^2}\sqrt{\frac{2e}{m}}V_0^{3/2} \tag{9}$$

Equation (9) is the well known Child-Langmuir law [3,4]. Since the derivation of this fundamental law many important and useful variations on the classical CL law have been investigated to account for special geometries, [5–7] relativistic electron energies, [8] non zero initial electron velocities, [9, 10] quantum mechanical effects, [11–13] nonzero electric field at the cathode surface, [14] slow varying charge density, [15] and quadratic damping [16].

3. New Approach

Consider now that the electrostatic potential is given as a function of the volume charge density, *i.e.* $V = V(\rho)$. This means that the electric field is given by

$$E = -\frac{dV}{dz} = -\frac{dV}{d\rho}\frac{d\rho}{dz} \tag{10}$$

and Gauss law is given by

$$\frac{\rho}{\epsilon_0} = \frac{dE}{d\rho}\frac{d\rho}{dz} \tag{11}$$

Using Eq. (11) we obtain

$$z(\rho) = \int_{-\infty}^{\rho} \frac{\epsilon_0}{\rho}\frac{dE}{d\rho}d\rho \tag{12}$$

Combining Eq. (10) and Eq. (11) we have

$$\epsilon_0 E\frac{dE}{d\rho} = -\rho\frac{dV}{d\rho} \tag{13}$$

Solving Eq. (13) for the electrostatic potential we have

$$V(\rho) = -\int \frac{1}{\rho}\frac{d}{d\rho}\left(\frac{\epsilon_0}{2}E^2\right)d\rho \tag{14}$$

If we know the electric field as a function of the volume charge density we can use Eq. (14) and Eq. (12) to obtain the electrostatic potential as a function of position. Our task then is to find $E = E(\rho)$, to do this we use Poisson's equation

$$\frac{d^2V}{dz^2} = -\frac{\rho}{\epsilon_0} = -\frac{J}{\epsilon_0\sqrt{2/m}}\frac{1}{\sqrt{K_0 + eV}} \tag{15}$$

where $K_0 = mv_0^2/2$ is the initial kinetic energy. Multiplying Eq. (15) by dV/dz and integrating from zero to z we have

$$\frac{1}{2}E^2 = -\frac{2J}{e\epsilon_0\sqrt{2/m}}\sqrt{K_0 + eV} + C \tag{16}$$

where $C = E_0^2/2 + Jmv_0/e\epsilon_0$ is a constant of integration given as a function of the value of the electrostatic field E_0 and velocity v_0 at $z = 0$. Substituting Eq. (15) into Eq. (16) we have

$$\left(\frac{\epsilon_0}{2}E^2 - \frac{\epsilon_0}{2}E_0^2\right)\rho = -\frac{mJ^2}{e} + \frac{Jmv_0}{e}\rho \tag{17}$$

Using the relation $J = \rho v$ in Eq. (17) we end up with

$$\Delta\delta_E = -\frac{J}{e}\Delta p \tag{18}$$

where $\delta_E = \epsilon_0 E^2/2$ is the electrostatic energy density and $p = mv$ is the linear momentum. Equation (18) is what we call the *microscopic* Child-Langmuir law, which states that the change in electrostatic energy density is proportional to the change in linear momentum. For the case when $E_0 = v_0 = 0$ we have

$$\delta_E = -\frac{J}{e}mv = -\frac{J^2m}{e\rho} \tag{19}$$

Substituting Eq. (19) into Eq. (14) and integrating we obtain the electrostatic potential

$$V = \frac{J^2m}{2e\rho^2} = \frac{m}{2e}v^2 \tag{20}$$

Note that the electrostatic potential in Eq. (20) equals the kinetic energy per unit charge. Substituting Eq. (19) into Eq. (12) and integrating we obtain $z = z(\rho)$

$$z = \frac{2}{3}\sqrt{\frac{\epsilon_0 J^2 m}{2e}}(-\rho)^{-3/2} \tag{21}$$

Solving Eq. (21) for ρ and substituting into Eq. (20) we end up with

$$V = \left(\frac{9Jz^2}{4\epsilon_0}\sqrt{\frac{m}{2e}}\right)^{2/3} \tag{22}$$

If we evaluate Eq. (22) when $z = D$ and solve for the charge current density we find the space charge limited current density which is given in Eq. (9).

An interesting case is when the initial velocity at $z = 0$ is non zero, *i.e.* $v_0 \neq 0$, for this case the microscopic Child-Langmuir law is given by

$$\delta_E = -\frac{J}{e}(mv - mv_0) = -\frac{J^2m}{e\rho} + \frac{Jmv_0}{e} \tag{23}$$

The electrostatic potential will be given by

$$V = \frac{m}{2e}v^2 - \frac{K_0}{e} = \frac{J^2m}{2e\rho^2} - \frac{K_0}{e} \tag{24}$$

Substituting Eq. (23) into Eq. (12) we have

$$z = \epsilon_0 \int_{-\infty}^{\rho} \frac{1}{\rho}\sqrt{\frac{-mJ}{2e\epsilon_0}}\left(\frac{J}{\rho} - v_0\right)^{-1/2}\frac{J}{\rho^2}d\rho \qquad (25)$$

The integral given in Eq. (25) can be transformed by the change of variable $v = J/\rho$ to a more suitable form

$$z = \sqrt{\frac{-m\epsilon_0}{2eJ}}\int_{v_0}^{v}\frac{vdv}{\sqrt{v-v_0}}$$

$$= \frac{2}{3}\sqrt{\frac{-m\epsilon_0}{2eJ}}\sqrt{v-v_0}(v+2v_0) \qquad (26)$$

Solving for v in Eq. (26) we have

$$v = -v_0 + \frac{\sqrt[3]{2}v_0^2}{\left(a^2 + 2v_0^3 + \sqrt{a^4 + 4a^2v_0^3}\right)^{1/3}} \qquad (27)$$

$$+ \frac{\left(a^2 + 2v_0^3 + \sqrt{a^4 + 4a^2v_0^3}\right)^{1/3}}{\sqrt[3]{2}}$$

where $a = 3z\sqrt{-2eJ/m\epsilon_0}/2$. Substituting Eq. (27) into Eq. (24) we have the electrostatic potential as a function of z.

However, if we evaluate Eq. (26) when $z = D$ and use the fact that $v(z = D) = \sqrt{v_0^2 + 2eV_0/m}$ we can solve for the charge current density directly, which is given by

$$J = -\frac{2m\epsilon_0}{9eD^2}\left[\sqrt{v_0^2 + \frac{2eV_0}{m}} - v_0\right]$$

$$\times\left[\sqrt{v_0^2 + \frac{2eV_0}{m}} + 2v_0\right]^2 \qquad (28)$$

Note that equation (28) reduces to the Child-Langmuir result when $v_0 = 0$. Equation (28) can be rewritten in the following form

$$J = J_{CL}\left[\sqrt{1 + \frac{K_0}{eV_0}} - \sqrt{\frac{K_0}{eV_0}}\right]$$

$$\times\left[\sqrt{1 + \frac{K_0}{eV_0}} + 2\sqrt{\frac{K_0}{eV_0}}\right]^2 \qquad (29)$$

One can see from Eq. (29) that a non zero initial velocity has a strong influence on the space charge limited current.

There have been other expressions proposed for the space charge limited current in a planar vacuum tube diode with nonzero initial velocity, the one given by Liu and Dougal [17]

$$J_{BF} = J_{CL}\left[\left(1 + \frac{K_0}{eV_0}\right)^{3/4} + \left(\frac{K_0}{eV_0}\right)^{3/4}\right]^2 \qquad (30)$$

and the one given by Jaffé [10]

$$J_{SCL} = J_{CL}\left[\sqrt{1 + \frac{K_0}{eV_0}} + \sqrt{\frac{K_0}{eV_0}}\right]^3 \qquad (31)$$

It has been shown that Eq. (30) represents the current at the bifurcation point and hence do not represents the space charge limit correctly [18]. On the other hand, Eq. (31) is obtained by imposing a boundary condition for the current, *i.e.* "the number of electrons entering the discharge space must be equal or smaller than a given number N_0 per cm^2 per sec., and is for each potential as high as the potential permits" [10]. Our result given by Eq. (29) is obtained by using a integral of motion of the electron dynamics inside the vacuum tube diode given by the energy momentum relation in Eq. (18) and only assumes that the volume charge density does not depends explicitly on time, *i.e.* it only depends on the coordinates.

In Fig. (1) we show all three expressions as a function of K_0/eV_0. Note how our expression given by Eq. (29) follows closely the relation given by Jaffé for small values of K_0/eV_0, but for large values of K_0/eV_0 the expressions given by Eq. (30) and Eq. (31) grow more rapidly than ours.

For the case of relativistic velocities the equation of motion of the electron in one dimension is given by

$$m\frac{dv}{dt} = -eE\left(1 - \frac{v^2}{c^2}\right)^{3/2} \qquad (32)$$

if we substitute the velocity in Eq. (32) by $v = J/\rho$ we obtain

$$-\frac{mJ^2}{\rho^3}\frac{d\rho}{dz} = -eE\left(1 - \frac{J^2}{\rho^2c^2}\right)^{3/2} \qquad (33)$$

If we multiply by $dE/d\rho$ and use Gauss law in Eq. (33) we have

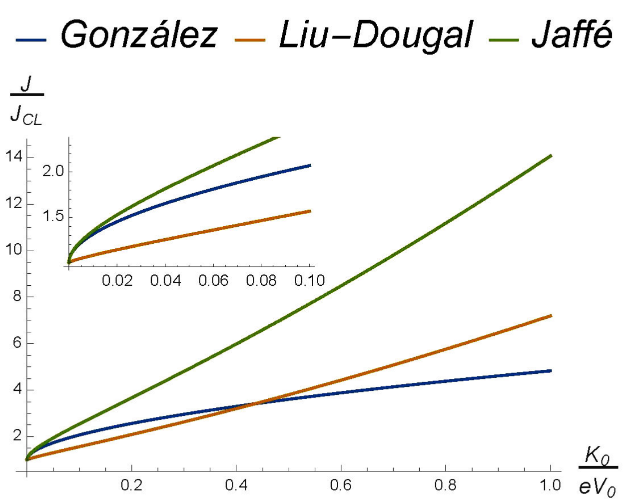

FIGURE 1. The figure shows the space charge limited currents as a function of K_0/eV_0 obtained by Liu and Dougal, Jaffé and González for explicit comparison between them. We see in the inset figure that our expression follows closely the relation given by Jaffé for small values of K_0/eV_0.

$$\frac{mJ^2}{\epsilon_0\rho^2} = eE\frac{dE}{d\rho}\left(1-\frac{J^2}{\rho^2c^2}\right)^{3/2} \tag{34}$$

separating variables in Eq. (34) we have

$$-\frac{mJ^2}{e}d\left(\frac{1}{\rho\sqrt{1-J^2/\rho^2c^2}}\right) = d\left(\frac{\epsilon_0}{2}E^2\right) \tag{35}$$

Integrating Eq. (35) and using the relation $J=\rho v$ we have

$$-\frac{J}{e}\Delta p = \Delta\delta_E \tag{36}$$

where $p = mv/\sqrt{1-v^2/c^2}$. We see that the *microscopic* Child-Langmuir law also holds for the relativistic regime. Thus, for the case when $v_0 = E_0 = 0$, and considering that the charge density times the velocity remains constant, then the integral of motion is given by

$$\frac{\epsilon_0}{2}\left(-\frac{dV}{dz}\right)^2 + \frac{Jmv}{e\sqrt{1-v^2/c^2}} = 0 \tag{37}$$

substituting $v = \sqrt{eV(2+eV/mc^2)/m}/(1+eV/mc^2)$ in Eq. (37) and separating variables we have

$$\int_0^V \frac{dV}{(eV/m)^{1/4}(2+eV/mc^2)^{1/4}} = -\sqrt{\frac{-2mJ}{e\epsilon_0}}\int_0^z dz \tag{38}$$

Equation (38) is the same as Eq. (10) given in Ref. 8 with the change of variable $\omega^4 = U^2+2U$, where $U = eV/mc^2$. If we evaluate Eq. (38) when $z=D$ and use $V=V_0$ we can solve for the charge current density directly, which is

$$J = J_{CL}\left({}_2F_1\left(\frac{1}{4},\frac{3}{4},\frac{7}{4},-\frac{eV_0}{2mc^2}\right)\right)^2 \tag{39}$$

where ${}_2F_1(a,b;c;z)$ is the hypergeometric function [19]. Equation (39) reduces to the classical Child-Langmuir law for $c\to\infty$.

4. Conclusions

In summary, we have shown a new method of deriving the Child-Langmuir law for the case of the electron motion inside a planar vacuum tube diode which avoids the need of solving a nonlinear differential equation and presents a new insight into the way of approaching the problem of the charge dynamics inside a planar vacuum tube diode. We found what we call the *microsocopic* Child-Langmuir law, which states that the change in electrostatic energy density is proportional to the change in linear momentum. We have shown that the *microscopic* Child-Langmuir law is valid for the classical and relativistic regimes.

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