Massless Majorana bispinors and two-qubit entangled states

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This is a pedagogical paper, where bispinors solutions to the four-dimensional massless Dirac equation are considered in relativistic quantum mechanics and in quantum computation, taking advantage of the common mathematical description of four-dimensional spaces. First, Weyl and massless Majorana bispinors are shown to be unitary equivalent, closing a gap in the literature regarding their equivalence. A discrepancy in the number of linearly independent solutions reported in the literature is also addressed. Then, it is shown that Weyl bispinors are algebraically equivalent to two-qubit direct product states, and that the massless Majorana bispinors are algebraically equivalent to maximally entangled sates (Bell states), with the transformations relating the two bispinors types acting as entangling gates in quantum computation. Different types of entangling gates are presented, highlighting a set that fulfills the required properties for Majorana zero mode operators in topological quantum computation. Based on this set, a general topological quantum computation model with four Majorana operators is presented, which exhibits all the required technical and physical properties to obtain entanglement of two logical qubits from topological operations.

Keywords: Massless bispinors; entanglement; majorana zero modes.

1. Introduction

A Majorana fermion is a spin 1/2 particle that is its own antiparticle. They were first proposed in 1937 by E. Majorana [1] in the context of particle physics. As an elementary particle, the only fundamental candidate for a Majorana fermion is the massive neutrino. It could also be a Dirac particle, although the Majorana alternative is theoretically preferred [2,3]. The experimental verification of the Majorana nature of the neutrino, through the observation of neutrinoless double beta decay processes, is still an open question.

Majorana fermions arise also in condensed matter systems [4-7]. Here they are not elementary particles, but rather localized zero-energy bound states (Bogoliubov quasiparticles) of electrons and holes, better known as Majorana zero modes[i], (MZMs). In this case the Majorana condition is satisfied through the use of Hermitian operators to describe MZMs. The composite objects consisting of Majorana bound states coupled to topological defects, such as vortices, obey non-Abelian statistics and are known as Ising anyons, [9,10] which constitute a particular type of non-Abelian anyons. Examples of 2-d systems admitting Ising anyons are the $\nu = 5/2$ fractional quantum Hall state, [9,11] $p + ip$ superconductors, [12,13] and the surface of topological insulators, [14] among others.

The interest in Ising anyons, from the perspective of quantum computation, is because they provide a means for fault-tolerant quantum computation [7,15-17]. In a system with localized anyons quantum information can be stored non-locally in pairs, or in general n-tuplets, with n even, of anyons. Computations are performed by adiabatically braiding the anyons worldlines. These braiding operations constitute the logical quantum gates acting on the states and, up to a phase, depend only on the topology of the trajectories, in turn classified by the braid group. A topological quantum computation (TQC) model is specified [10] by providing the Hilbert space, the initial state, the braid operators and the measurable observables.

It has been shown that the operators representing the MZMs can be given in terms of Dirac gamma matrices [11,18,19] and, in particular, in Refs. [18] and [19] it is shown that the Clifford algebra of the Majorana operators, for a 2-d system with four vortices, can be realized by elements of the 4-d spacetime Clifford algebra. This result suggests that a common mathematical description can be given for the four-component spinors (bispinors) and the relevant particle states in TQC, namely Weyl[ii] and massless Majorana states.

In this paper, we study massless Majorana bispinors, that is solutions to the 4-d massless Dirac equation satisfying the Majorana condition, in two different settings: relativistic quantum mechanics (RQM) and quantum computation (QC). In the first instance, besides showing explicit general solutions to the equation, which are difficult to find in the literature, if at all, we complete the known equivalence between massless Majorana and Weyl free field operators by showing that it also holds for c-number bispinors.

We also address an inconsistency in the number of linearly independent solutions to the massless Dirac equation reported in the literature where, up to differences in normalization and sign factors, it is stated that positive and negative energy solutions are proportional in momentum space [21-24]. This statement is made in terms of four-component Dirac spinors, and it then follows that only two linearly independent bispinor solutions exist in the massless case. Taking the limit $m \to 0$ in massive solutions, as is done in Refs. [25-27], also leads to an incomplete set of solutions. How-

ever, from a pure mathematical viewpoint, the massless Dirac equation in momentum space is an algebraic equation, whose solutions are the eigenvectors of a 4×4 Hermitian matrix, so that four independent solutions must exist. Indeed, this result is also found in the literature for the special case of momentum along the direction of $\hat{\mathbf{z}}$ [28,29].

In the QC context, we establish an algebraic equivalence between Weyl bispinors and bipartite qubit states. We show that the unitary transformations relating the Weyl and Majorana bispinors in RQM play the role of entangling two-qubit gates in QC, and that maximally entangled states (Bell states) are algebraically equivalent to massless Majorana bispinors[iii]. Different types of entangling gates are discussed, providing a list not meant to be exhaustive. A set of the entangling gates fulfills the requirements for MZMs operators, and we use it to construct a TQC toy model with four MZMs from the bottom up, showing how to obtain operators and states, as well as entanglement of two logical qubits from braiding.

The organization is as follows: In Sec. 2 we obtain bispinor solutions to the massless Dirac equation and show that they are unitarily equivalent. The completeness of the solutions is also discussed. In Sec. 3 we establish the algebraic equivalence between massless bispinors and two-qubit states and discuss the entangling gates. In Sec. 4 we provide a TQC toy model based on a set of the entangling gates. Finally, concluding remarks are given.

2. Massless *c*-number bispinors

2.1. Weyl

Let us begin by considering four-component Weyl bispinors with four-momentum $p^{\mu} = (\pm |\mathbf{p}|, \mathbf{p})$, respectively for positive- and negative-energy $p^{0} = \pm E = \pm |\mathbf{p}|$, which are solutions to the massless Dirac equation

$$i\gamma^{\mu}\partial_{\mu}\Psi = 0. \qquad (1)$$

The gamma matrices $\gamma^{\mu} = (\gamma^{0}, \boldsymbol{\gamma})$ obey the Clifford algebra relation

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}, \qquad (2)$$

with $g^{\mu\nu}$ the metric tensor with signature $\mathrm{diag}(1, -1, -1, -1)$, and the Weyl representation

$$\gamma^{0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad (3)$$

with $\boldsymbol{\sigma} = (\sigma^{1}, \sigma^{2}, \sigma^{3})$ the standard Pauli matrices will be used throughout. Using the plane waves

$$\Psi = u(\mathbf{p}) \exp\{i(\pm Et - \mathbf{x} \cdot \mathbf{p})\}, \qquad (4)$$

and the matrices

$$\gamma^{5} \equiv i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\boldsymbol{\Sigma} \equiv \gamma^{5}\gamma^{0}\boldsymbol{\gamma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \qquad (5)$$

TABLE I. Eigenvalues of the canonical frame Weyl bispinors.

	$u^{(1)}(p_z)$	$u^{(2)}(p_z)$	$u^{(3)}(p_z)$	$u^{(4)}(p_z)$
Energy	+	+	-	-
Helicity	1	-1	-1	1
Chirality	1	-1	1	-1

Eq. (1) is rewritten as

$$\boldsymbol{\Sigma} \cdot \hat{\mathbf{p}}\, u(\mathbf{p}) = \pm\gamma^{5}\, u(\mathbf{p}), \qquad (6)$$

with $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$. Thus, the bispinors $u(\mathbf{p})$ are eigenvectors of both helicity $\boldsymbol{\Sigma} \cdot \hat{\mathbf{p}}$ and chirality γ^{5} operators, and Eq. (6) expresses the known result that chirality equals the helicity for massless, positive-energy bispinors, while it is opposite for negative-energy ones. Taking the direction of $\mathbf{p}$ along $\hat{\mathbf{z}}$ (from now on called the canonical frame) in Eq. (6) one obtains the four independent solutions[iv], with their eigenvalues given in Table I.

$$u^{(1)}(p_z) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u^{(2)}(p_z) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$u^{(3)}(p_z) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad u^{(4)}(p_z) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \qquad (7)$$

To obtain solutions for general three-momentum we use spherical polar coordinates

$$\hat{\boldsymbol{p}} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta), \qquad (8)$$

and the transformation

$$\Lambda(\theta, \varphi) = \exp\left\{-\frac{\theta}{2}\left(\gamma^{1}\cos\varphi + \gamma^{2}\sin\varphi\right)\gamma^{3}\right\}, \qquad (9)$$

which is actually a rotation since it is unitary and of unit determinant. Applying Eq. (9) to the bispinors in Eq. (7) we have

$$\Lambda(\theta, \varphi)\, u^{(i)}(p_z) = u^{(i)}(\mathbf{p}), \qquad i = 1, \ldots, 4, \qquad (10)$$

with the general momentum bispinors, in two-block notation, given by

$$u^{(1)}(\mathbf{p}) = \begin{pmatrix} 0 \\ \chi_{+}(\mathbf{p}) \end{pmatrix}, \quad u^{(2)}(\mathbf{p}) = \begin{pmatrix} \chi_{-}(\mathbf{p}) \\ 0 \end{pmatrix},$$

$$u^{(3)}(\mathbf{p}) = \begin{pmatrix} 0 \\ \chi_{-}(\mathbf{p}) \end{pmatrix}, \quad u^{(4)}(\mathbf{p}) = \begin{pmatrix} \chi_{+}(\mathbf{p}) \\ 0 \end{pmatrix}, \qquad (11)$$

where $\chi_{\pm}(\mathbf{p})$ are the two-component helicity eigenspinors

$$
\begin{aligned}
\chi_{+}(\mathbf{p}) &= \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}, \\
\chi_{-}(\mathbf{p}) &= \begin{pmatrix} -e^{-i\varphi}\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix},
\end{aligned}
\tag{12}
$$

satisfying the equation

$$
\boldsymbol{\sigma} \cdot \hat{\mathbf{p}}\, \chi_{\pm}(\mathbf{p}) = \pm\chi_{\pm}(\mathbf{p}). \tag{13}
$$

The bispinors in Eq. (11) are orthonormal

$$
u^{\dagger(i)}(\mathbf{p}) u^{(j)}(\mathbf{p}) = \delta_{ij}, \tag{14}
$$

with a normalization that is adequate for massless spinors, as the Dirac adjoint $\overline{u} \equiv u^{\dagger}\gamma^0$ is not needed in this case. Another useful, Lorentz invariant normalization is to re-scale them to $\sqrt{2E}$. These bispinors are also solutions to Eq. (6), which in Hamiltonian form reads

$$
\begin{aligned}
\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}\, u^{(s)}(\mathbf{p}) &= +\,u^{(s)}(\mathbf{p}), \\
\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}\, u^{(s+2)}(\mathbf{p}) &= -\,u^{(s+2)}(\mathbf{p}),
\end{aligned}
\quad s = 1, 2 \tag{15}
$$

making explicit that $u^{(1)}(\mathbf{p})$ and $u^{(2)}(\mathbf{p})$ are positive-energy bispinors, while $u^{(3)}(\mathbf{p})$ and $u^{(4)}(\mathbf{p})$ are negative-energy ones. The helicity and chirality eigenvalues are the same as in Eq. (7). Energy projection operators are obtained from the spin sums

$$
\begin{aligned}
\Lambda_{+} &\equiv \sum_{s=1,2} u^{(s)}(\mathbf{p}) u^{\dagger(s)}(\mathbf{p}) = \frac{1}{2}\left(\mathbb{1} + \boldsymbol{\alpha} \cdot \hat{\mathbf{p}}\right), \\
\Lambda_{-} &\equiv \sum_{s=1,2} u^{(s+2)}(\mathbf{p}) u^{\dagger(s+2)}(\mathbf{p}) = \frac{1}{2}\left(\mathbb{1} - \boldsymbol{\alpha} \cdot \hat{\mathbf{p}}\right). \tag{16}
\end{aligned}
$$

They satisfy the required properties for projection operators

$$
\begin{aligned}
\Lambda_{\pm}^2 &= \Lambda_{\pm}, \\
\Lambda_{+}\Lambda_{-} &= \Lambda_{-}\Lambda_{+} = 0, \\
\Lambda_{+} + \Lambda_{-} &= 1,
\end{aligned}
\tag{17}
$$

and from the second and third properties it is readily seen that the bispinors in Eq. (11) constitute a complete and orthogonal set of solutions to the massless Dirac equation. The energy projection operators in Eq. (16) can be found in the literature [32-37], although without reference to the bispinors and the spin sums.

2.2. Majorana

Using the canonical frame bispinors in Eq. (7) we define the following Majorana bispinors

$$
\begin{aligned}
u_M^{(1)}(p_z) &= \frac{1}{\sqrt{2}}\left(u^{(2)}(p_z) + i\gamma^2 u^{*(2)}(p_z)\right), \\
u_M^{(2)}(p_z) &= \frac{1}{\sqrt{2}}\left(u^{(1)}(p_z) - i\gamma^2 u^{*(1)}(p_z)\right), \\
u_M^{(3)}(p_z) &= \frac{1}{\sqrt{2}}\left(u^{(3)}(p_z) - i\gamma^2 u^{*(3)}(p_z)\right), \\
u_M^{(4)}(p_z) &= \frac{1}{\sqrt{2}}\left(u^{(4)}(p_z) + i\gamma^2 u^{*(4)}(p_z)\right),
\end{aligned}
\tag{18}
$$

where the asterisk denotes complex conjugation, even though it is superfluous in this case because the $u^{(i)}(p_z)$ are real. The bispinors in Eq. (18) are eigenstates of the standard charge conjugation operator [38,39]

$$
\mathcal{C} \equiv CK \equiv i\gamma^2 K, \tag{19}
$$

where $C = i\gamma^2$ is the charge conjugation matrix, and K stands for the operation of complex conjugation to the right. We then have

$$
\begin{aligned}
\mathcal{C} u_M^{(1,4)}(p_z) &= +\,u_M^{(1,4)}(p_z), \\
\mathcal{C} u_M^{(2,3)}(p_z) &= -\,u_M^{(2,3)}(p_z),
\end{aligned}
\tag{20}
$$

and it is in this sense that they fulfill the Majorana condition. These Majorana bispinors are also solutions to Eq. (15), implying a unitary transformation must exist relating them to the Weyl bispinors in Eq. (7). Among several possibilities, to be discussed in the next section, we choose

$$
R_3 = \exp\left(\frac{\pi}{4}\gamma^0\gamma^1\gamma^3\right), \tag{21}
$$

as the transformation matrix, which besides being unitary is also of unit determinant, therefore a rotation. Thus, we have the following equivalence between the bispinors in Eqs. (7) and (18)

$$
\begin{aligned}
R_3 u^{(1)}(p_z) &= -u_M^{(1)}(p_z), \\
R_3 u^{(2)}(p_z) &= +u_M^{(2)}(p_z), \\
R_3 u^{(3)}(p_z) &= +u_M^{(4)}(p_z), \\
R_3 u^{(4)}(p_z) &= -u_M^{(3)}(p_z).
\end{aligned}
\tag{22}
$$

It is now straightforward to generalize this result to arbitrary momentum bispinors. Using the ones in Eq. (11) we obtain the generalization of Eq. (18)

$$u_M^{(1)}(\boldsymbol{p}) = \frac{1}{\sqrt{2}}\left(u^{(2)}(\boldsymbol{p}) + i\gamma^2 u^{(2)*}(\boldsymbol{p})\right),$$

$$u_M^{(2)}(\boldsymbol{p}) = \frac{1}{\sqrt{2}}\left(u^{(1)}(\boldsymbol{p}) - i\gamma^2 u^{(1)*}(\boldsymbol{p})\right),$$

$$u_M^{(3)}(\boldsymbol{p}) = \frac{1}{\sqrt{2}}\left(u^{(3)}(\boldsymbol{p}) - i\gamma^2 u^{(3)*}(\boldsymbol{p})\right),$$

$$u_M^{(4)}(\boldsymbol{p}) = \frac{1}{\sqrt{2}}\left(u^{(4)}(\boldsymbol{p}) + i\gamma^2 u^{(4)*}(\boldsymbol{p})\right). \tag{23}$$

These Majorana bispinors are obtained from the canonical frame ones in Eq. (18) by the same rotation in Eq. (9)

$$\Lambda(\theta,\varphi)\,u_M^{(i)}(p_z) = u_M^{(i)}(\mathbf{p}), \qquad i = 1,\ldots,4. \tag{24}$$

Then defining the rotation

$$\Omega(\theta,\varphi) \equiv \Lambda(\theta,\varphi)\,R_3\Lambda^\dagger(\theta,\varphi), \tag{25}$$

Eqs. (10) and (22) yield

$$\Omega(\theta,\varphi)\,u^{(1)}(\boldsymbol{p}) = -u_M^{(1)}(\boldsymbol{p}),$$

$$\Omega(\theta,\varphi)\,u^{(2)}(\boldsymbol{p}) = +u_M^{(2)}(\boldsymbol{p}),$$

$$\Omega(\theta,\varphi)\,u^{(3)}(\boldsymbol{p}) = +u_M^{(4)}(\boldsymbol{p}),$$

$$\Omega(\theta,\varphi)\,u^{(4)}(\boldsymbol{p}) = -u_M^{(3)}(\boldsymbol{p}). \tag{26}$$

Observing that $\Omega(\theta,\varphi)$ and $\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}$ commute, it is readily verified that the bispinors in Eq. (23) are solutions to the massless Dirac equation

$$\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\,u_M^{(s)}(\mathbf{p}) = +\,u_M^{(s)}(\mathbf{p}),$$

$$\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\,u_M^{(s+2)}(\mathbf{p}) = -\,u_M^{(s+2)}(\mathbf{p}). \qquad s = 1,2, \tag{27}$$

They also satisfy the Majorana condition

$$\mathcal{C}u_M^{(1,4)}(\boldsymbol{p}) = +\,u_M^{(1,4)}(\boldsymbol{p}),$$

$$\mathcal{C}u_M^{(2,3)}(\boldsymbol{p}) = -\,u_M^{(2,3)}(\boldsymbol{p}). \tag{28}$$

Accordingly, Eq. (26) establishes an equivalence between Weyl and massless Majorana bispinors. This relation is the c-number analogue of the known equivalence between Weyl and massless Majorana *field operators*, related by a Pauli-Gursey transformation [40-42]. In this sense this result completes the equivalence between massless Majorana and Weyl fermions, which is now seen to hold for both quantum fields and c-number spinors.

2.3. Completeness and degrees of freedom

There is a subtle but important matter regarding the negative-energy Weyl bispinors $u^{(3,4)}(\mathbf{p})$ in Eq. (11). If one substitutes the complete wavefunctions $\Psi^{(3,4)}(x) = u^{(3,4)}(\mathbf{p})e^{ip.x}$ in Eq. (1) it is found that $\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\,u^{(3,4)}(\mathbf{p}) = u^{(3,4)}(\mathbf{p})$, in contradiction with Eq. (15). Let us contrast this situation with

the standard massive case [38,43] where, following the Feynman - Stuckelberg prescription for antiparticles, the negative-energy bispinors are redefined as $v_m^{(1,2)}(\mathbf{p}) \equiv u_m^{(4,3)}(-\mathbf{p})$ (the subscript m is just to make explicit that these are massive bispinors). The momentum flip is necessary so that solutions with four-momentum $(-E,-\mathbf{p})$ are interpreted (E is always positive) as antiparticle solutions with four-momentum $(E,\mathbf{p})$, and the coordinate dependence is obtained from the positive-energy one $e^{-ip.x}$ by making the replacements $E \to -E$ and $\mathbf{p} \to -\mathbf{p}$. Also, the spinors indexes are relabeled to implement hole theory in the rest frame.

In the massless case there is no rest frame, but one can use the canonical frame instead, with helicity replacing spin in hole theory. Hence, the absence of a negative-energy solution with positive (negative) helicity, and therefore negative (positive) chirality, is to be interpreted as the presence of a positive energy solution with negative (positive) helicity, and the same chirality. The momentum flip is still necessary for the antiparticle interpretation, and in fact it is already implied for the plane wave $e^{ip.x}$, with $p^0 = E \equiv |\mathbf{p}|$, but combined with a simple relabeling of the spinor indexes, as in the massive case, is not enough to satisfy helicity invariance. Hence, both spin and momentum of the negative-energy solutions must be reversed. However these operations just produce the same bispinors up to a phase. To see it, it suffices to consider the spinors in Eq. (12). The momentum flip is accomplished through the substitution $(\theta,\varphi) \to (\pi-\theta,\phi+\pi)$, leading to

$$\chi_\pm(-\mathbf{p}) = \mp e^{\pm i\varphi}\chi_\mp(\mathbf{p}), \tag{29}$$

while the spin flip is done via [39]

$$-i\sigma_2\chi_\pm^*(\mathbf{p}) = \pm\chi_\mp(\mathbf{p}). \tag{30}$$

Thus, Eqs. (29) and (30) produce

$$-i\sigma_2\chi_\pm^*(-\mathbf{p}) = e^{\mp i\varphi}\chi_\pm(\mathbf{p}). \tag{31}$$

As for the coordinate dependence, and starting from $\exp\{-i(-Et-\boldsymbol{p}\cdot\boldsymbol{x})\}$, the operations of complex conjugating and flipping the momentum result in the positive-energy case $e^{-ip.x}$.

In the literature, the above discrepancy is expressed in terms of incompatible statements about the completeness of solutions to the massless Dirac equation. On one hand, in Refs. [21-27] it is concluded, following different approaches, that there are only two independent solutions to the equation, with the negative-energy bispinors being proportional to the positive-energy ones. On the other hand, the massless Dirac equation in momentum space is a 4×4 Hermitian matrix, so there must be four independent solutions, as already given in Eqs. (7) and (11), and expressed in the completeness relations in Eq. (16). The resolution of this problem lies in the degrees of freedom: A Majorana bispinor, either massless or massive, possesses two degrees of freedom because of the Majorana condition, and these are half the degrees of freedom of a Dirac bispinor. In view of the results of the last

subsection, this is also true for the Weyl bispinors. Thus, even if formally four independent solutions exist for the massless Dirac equation (a complete set in the mathematical sense), only two make sense physically.

At the level of c-number wave functions one could, in principle, either give up the Feynman - Stuckelberg interpretation for negative-energy states and keep the complete set of solutions, or maintain the conceptually useful antiparticle interpretation and disregard mathematical completeness, since ultimately it is the quantized theory (second quantization) the one that is expected to be free of ambiguities. Indeed, in a classic paper [44] Weinberg has shown that, under the general assumption of Lorentz invariance of the S matrix, massless fermionic field operators must be given by

$$\psi_-(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E}} \left(a_-(\boldsymbol{p}) e^{-ip\cdot x} + b_+^\dagger(\boldsymbol{p}) e^{ip\cdot x} \right)$$
$$\times \sqrt{2E}\chi_-(\mathbf{p}), \tag{32}$$

$$\psi_+(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E}} \left(a_+(\boldsymbol{p}) e^{-ip\cdot x} + b_-^\dagger(\boldsymbol{p}) e^{ip\cdot x} \right)$$
$$\times \sqrt{2E}\chi_+(\mathbf{p}), \tag{33}$$

where the subscripts $\pm$ respectively represent positive and negative helicity, and the spinors $\chi_\pm(\boldsymbol{p})$ are given in Eq. (12). These massless fields can be readily expressed in terms of bispinors by making the substitutions $\chi_\pm(\mathbf{p}) \rightarrow u^{(1,2)}(\mathbf{p})$, with the latter given in Eq. (11). There is no use for the complete set of massless bispinors in the field operator expansion.

3. Majorana condition and maximal entanglement

3.1. Massless bispinors as bipartite qubits

In quantum computation the quantum analogue of a classical bit, a qubit, is given by a complex linear combination of the basis states of a two-level quantum system, known as the computational basis. Denoting the basis states by $|0\rangle$ and $|1\rangle$, for spin-1/2 systems they can be chosen as the eigenstates of σ^3

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{34}$$

In this basis, the helicity spinors in Eq. (12) are given by the general pure-state qubits

$$|\chi_+\rangle = \cos\left(\frac{\theta}{2}\right) |0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right) |1\rangle,$$
$$|\chi_-\rangle = -e^{-i\varphi}\sin\left(\frac{\theta}{2}\right) |0\rangle + \cos\left(\frac{\theta}{2}\right) |1\rangle, \tag{35}$$

which are antipodal in the unit Bloch sphere representation, [45,46] with the three-momentum in Eq. (8) taken as the Bloch vector (Fig. 1).
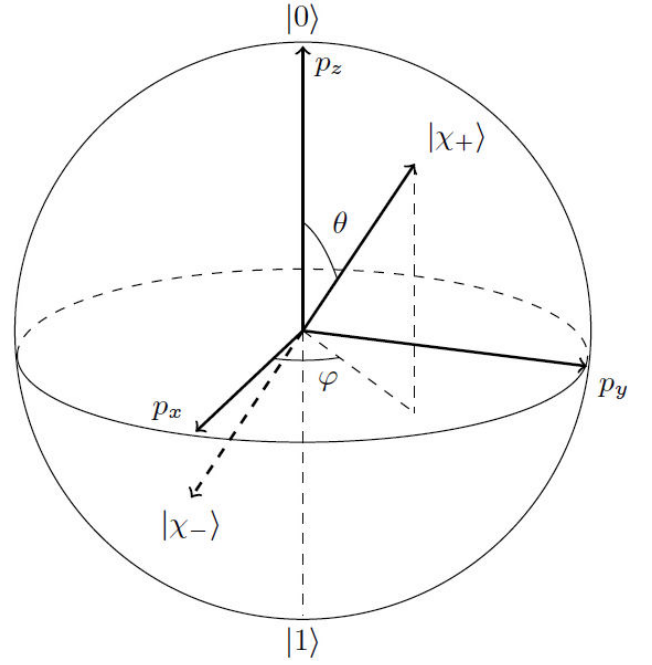

FIGURE 1. Unit Bloch Sphere. The computational basis is mapped to the north and south poles of the sphere. The orthogonal pure states $|\chi_+\rangle$ and $|\chi_-\rangle$ are antipodal and correspond to helicity eigenspinors if the Bloch vector is taken as the three-momentum.

The computational basis for the space of two pure-state qubits is then given by the set $\{|0\rangle, |1\rangle\} \otimes \{|0\rangle, |1\rangle\}$, whence, upon using Eq. (34) and the notation $|00\rangle \equiv |0\rangle \otimes |0\rangle$ and so on, we obtain the explicit representation

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$|10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{36}$$

and we see that the elements of the basis are just the canonical frame Weyl bispinors in Eq. (7)

$$|00\rangle = u^{(4)}(p_z), \quad |01\rangle = u^{(2)}(p_z),$$
$$|10\rangle = u^{(1)}(p_z), \quad |11\rangle = u^{(3)}(p_z). \tag{37}$$

Another basis for this space is provided by the Bell states, which are maximally entangled states

$$\left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle),$$

$$\left|\Phi^-\right\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle),$$

$$\left|\Psi^+\right\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle),$$

$$\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle). \tag{38}$$

Using either of Eqs. (7) or (36), explicit representations of the Bell states, as well as the massless Majorana bispinors in Eq. (18), are directly obtained, and upon comparing the two sets we arrive at the interesting result that the Bell states are algebraically equivalent to the massless Majorana bispinors in the canonical frame

$$u_M^{(1)}(p_z) = |\Psi^-\rangle, \qquad u_M^{(2)}(p_z) = |\Psi^+\rangle,$$
$$u_M^{(3)}(p_z) = -|\Phi^-\rangle, \quad u_M^{(4)}(p_z) = |\Phi^+\rangle. \tag{39}$$

This result is generalized to arbitrary momentum defining the general-momentum Bell states

$$\left|\Phi^+(\mathbf{p})\right\rangle = \frac{1}{\sqrt{2}}\left(u^{(4)}(\mathbf{p}) + u^{(3)}(\mathbf{p})\right),$$

$$\left|\Phi^-(\mathbf{p})\right\rangle = \frac{1}{\sqrt{2}}\left(u^{(4)}(\mathbf{p}) - u^{(3)}(\mathbf{p})\right),$$

$$\left|\Psi^+(\mathbf{p})\right\rangle = \frac{1}{\sqrt{2}}\left(u^{(2)}(\mathbf{p}) + u^{(1)}(\mathbf{p})\right), \tag{40}$$

$$\left|\Psi^-(\mathbf{p})\right\rangle = \frac{1}{\sqrt{2}}\left(u^{(2)}(\mathbf{p}) - u^{(1)}(\mathbf{p})\right),$$

then, from Eqs. (11), (12), and (23) we get

$$u_M^{(1)}(\mathbf{p}) = |\Psi^-(\mathbf{p})\rangle, \qquad u_M^{(2)}(\mathbf{p}) = |\Psi^+(\mathbf{p})\rangle,$$
$$u_M^{(3)}(\mathbf{p}) = -|\Phi^-(\mathbf{p})\rangle, \quad u_M^{(4)}(\mathbf{p}) = |\Phi^+(\mathbf{p})\rangle. \tag{41}$$

Thus, we conclude that for massless bispinors obeying the Dirac equation, the Majorana condition is equivalent to maximal entanglement.

3.2. Entangling gates

Operations on qubits are given by unitary quantum gates, and from Eqs. (22) and (39) we see that the rotation in Eq. (21) serves as a two-qubit gate that produces entanglement. We now provide a list, not meant to be exhaustive, of other entangling gates and their properties.

The common procedure for producing entanglement in quantum computation is by a combination of a CNOT (controlled not) gate and a Hadamard gate. The latter is a one-qubit gate given by

$$H = \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|\right)$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{42}$$

while the former is a two-qubit gate, with the most common realization given by

$$C_{NOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{43}$$

Then it is easy to see that, say, the combination $C_{NOT}(H \otimes \mathbb{1}_2)$ produces the Bell states in Eq. (38)

when acting on the computational basis in Eq. (36), *e.g*, $C_{NOT}(H \otimes \mathbb{1}_2)|00\rangle = |\Phi^+\rangle$. The CNOT is a universal gate [46] in the sense that any quantum circuit can be simulated with arbitrary accuracy by a combination of a CNOT and one-qubit gates (the latter usually taken as the Hadamard and the $\pi/8$ phase gates). It has also been shown, for the two-qubit case, that the relevant property for universality is entanglement [47], and so any quantum circuit can be simulated with arbitrary accuracy by a combination of an entangling two-qubit gate and suitable one-qubit gates. It is also worth noticing that the CNOT gate is not a rotation, since it has determinant -1, a feature that difficults actual implementations.

Another set of entangling gates, denoted by R_i, $i = 1,\ldots,4$, consists of the rotations

$$R_1 = \exp\left(\frac{\pi}{4}\gamma^1\right),$$
$$R_2 = \exp\left(-\frac{\pi}{4}\gamma^1\right),$$
$$R_3 = \exp\left(\frac{\pi}{4}\gamma^0\gamma^1\gamma^3\right), \tag{44}$$
$$R_4 = \exp\left(-\frac{\pi}{4}\gamma^0\gamma^1\gamma^3\right).$$

They also have the interesting property of being solutions to the algebraic Yang-Baxter equation [48].

$$(R_i \otimes \mathbb{1}_2)(\mathbb{1}_2 \otimes R_i)(R_i \otimes \mathbb{1}_2) = (\mathbb{1}_2 \otimes R_i)$$
$$\times (R_i \otimes \mathbb{1}_2)(\mathbb{1}_2 \otimes R_i). \tag{45}$$

These matrices have been studied by Kauffman *et al* [49] in connection with knot theory and topological linking. The gate R_3 (used in Eq. (25)) was introduced by Kauffman and Lomonaco [50], while the matrices R_1 and R_2 appear, respectively, in Refs. 51 and 52. The action of these gates on the computational basis is summarized in Table II.

Yet another set of entangling gates, denoted by $\hat{R}_i$, $i = 1,\ldots,4$, is given by the rotations

$$\hat{R}_1 = \frac{i}{\sqrt{2}}\gamma^3\left(\mathbb{1} + \gamma^1\right), \quad \hat{R}_2 = \frac{i}{\sqrt{2}}\gamma^2\left(\mathbb{1} + \gamma^1\right),$$

$$\hat{R}_3 = \frac{1}{\sqrt{2}}\gamma^0\left(\mathbb{1} + \gamma^1\right), \quad \hat{R}_4 = \frac{i}{\sqrt{2}}\left(\gamma^0\gamma^2\gamma^3 + i\gamma^5\right). \tag{46}$$

TABLE II. Action of the entangling gates in Eq. (44) on the computational basis in Eq. (36). The table is read so that the gates in the first column act on the basis states in the top first row and produce the given Bell state in the intersection.

	$|10\rangle$	$|01\rangle$	$|11\rangle$	$|00\rangle$
R_1	$|\Psi^+\rangle$	$|\Psi^-\rangle$	$|\Phi^+\rangle$	$|\Phi^-\rangle$
R_2	$-|\Psi^-\rangle$	$|\Psi^+\rangle$	$-|\Phi^-\rangle$	$|\Phi^+\rangle$
R_3	$-|\Psi^-\rangle$	$|\Psi^+\rangle$	$|\Phi^+\rangle$	$|\Phi^-\rangle$
R_4	$|\Psi^+\rangle$	$|\Psi^-\rangle$	$-|\Phi^-\rangle$	$|\Phi^+\rangle$

TABLE III. Action of the entangling gates in Eq. (46) on the computational basis in Eq. (36). The table is read so that the gates in the first column act on the basis states in the top first row and produce the given Bell state in the intersection.

	$|10\rangle$	$|01\rangle$	$|11\rangle$	$|00\rangle$
$\hat{R}_1$	$i\,|\Phi^+\rangle$	$-i\,|\Phi^-\rangle$	$-i\,|\Psi^+\rangle$	$i\,|\Psi^-\rangle$
$\hat{R}_2$	$-\,|\Psi^+\rangle$	$|\Psi^-\rangle$	$|\Phi^+\rangle$	$-\,|\Phi^-\rangle$
$\hat{R}_3$	$|\Phi^+\rangle$	$-\,|\Phi^-\rangle$	$|\Psi^+\rangle$	$-\,|\Psi^-\rangle$
$\hat{R}_4$	$|\Psi^-\rangle$	$|\Psi^+\rangle$	$|\Phi^-\rangle$	$|\Phi^+\rangle$

They are also Hermitian and therefore square to the identity matrix

$$\hat{R}_i = \hat{R}_i^\dagger, \quad \hat{R}_i^2 = \mathbb{1}, \quad i = 1, \ldots, 4. \tag{47}$$

They do not obey Eq. (45), but instead satisfy the anti-commutation (Clifford algebra) relations

$$\left\{ \hat{R}_i, \hat{R}_j^\dagger \right\} = 2\delta_{i,j}\,\mathbb{1}, \tag{48}$$

which, in contrast, are not obeyed by the gates in Eq. (44). These matrices are all orthogonal to each other, as is verified with the inner product

$$\mathrm{Tr}\left(\hat{R}_i^\dagger \hat{R}_j \right) = 0, \quad i,j = 1, \ldots, 4, \quad i \neq j, \tag{49}$$

hence, they are linearly independent. Using the 16 elements of the 4-d gamma matrices Clifford algebra it can be verified that no other matrix exists with these characteristics that fulfils Eq. (47) and also closes the algebra in Eq. (48). In this sense the set in Eq. (46) is complete. Their action on the computational basis is shown in Table III.

4. Quantum computational toy model with four Majorana zero modes

The properties in Eqs. (47) and (48), obeyed by the $\hat{R}_i$ gates, are the same as the ones satisfied by Majorana zero mode operators [5-7] in topological quantum computation. Hence, we will regard them as such and present a general model with four Majorana bound states that admits entanglement from braiding. We mostly follow Ref. [10].

The set $\mathcal{M}$ of particle types is given by

$$\mathcal{M} = \{\mathbf{1}_{\mathrm{vac}}, \sigma, \psi\}, \tag{50}$$

consisting of the vacuum $\mathbf{1}_{\mathrm{vac}}$, anyons σ, and Weyl fermions ψ, with the standard fusion rules

$$g \times \mathbf{1}_{\mathrm{vac}} = g, \quad \forall\ g \in \mathcal{M}, \quad \sigma \times \psi = \sigma,$$

$$\psi \times \psi = \mathbf{1}_{\mathrm{vac}}, \quad \sigma \times \sigma = \mathbf{1}_{\mathrm{vac}} + \psi. \tag{51}$$

To define braid operators we take the branch cuts of the Majorana zero modes, described by the Majorana operators,

in the same direction, and order them in a way that exchanging $\hat{R}_i$ and $\hat{R}_{i+1}$ clockwise ensures that $\hat{R}_i$ crosses solely the branch cut of $\hat{R}_{i+1}$, with no other operator crossing any other branch cut. Then the local (nearest-neighbor) braid operators are given by

$$B_{12} = \exp\left(-\frac{\pi}{4}\hat{R}_1\hat{R}_2 \right), \quad B_{23} = \exp\left(-\frac{\pi}{4}\hat{R}_2\hat{R}_3 \right),$$

$$B_{34} = \exp\left(-\frac{\pi}{4}\hat{R}_3\hat{R}_4 \right). \tag{52}$$

They are unitary by construction, and satisfy the required properties for braiding operators, [5,10,12] namely the Yang-Baxter equations

$$B_{12}B_{23}B_{12} = B_{23}B_{12}B_{23},$$

$$B_{23}B_{34}B_{23} = B_{34}B_{23}B_{34}, \tag{53}$$

and commutation relations

$$[B_{12}, B_{34}] = 0, \quad [B_{12}, B_{23}] = \hat{R}_1\hat{R}_3,$$

$$[B_{23}, B_{34}] = \hat{R}_2\hat{R}_4. \tag{54}$$

We also have the non-local braid operators

$$B_{13} = \exp\left(-\frac{\pi}{4}\hat{R}_1\hat{R}_3 \right),$$

$$B_{14} = \exp\left(-\frac{\pi}{4}\hat{R}_1\hat{R}_4 \right), \tag{55}$$

$$B_{24} = \exp\left(-\frac{\pi}{4}\hat{R}_2\hat{R}_4 \right),$$

connected to the local ones in Eq. (52) through the operations

$$B_{13} = B_{23}B_{12}B_{23}^\dagger,$$

$$B_{14} = B_{34}B_{23}B_{12}B_{23}^\dagger B_{34}^\dagger, \tag{56}$$

$$B_{24} = B_{34}B_{23}B_{34}^\dagger.$$

The relevant operator to obtain entanglement is B_{23} in Eq. (52), since it cannot be written as the tensor product of two 2×2 matrices, and therefore is an entangling gate. This also holds for all three operators in Eq. (56). B_{12} and B_{34}, on the other hand, are separable

$$B_{12} = \mathbb{1}_2 \otimes R_x\left(\pi/2 \right),$$

$$B_{34} = R_y\left(\pi/2 \right) \otimes \mathbb{1}_2, \tag{57}$$

where $R_x\left(\pi/2 \right)$ and $R_y\left(\pi/2 \right)$ are the one-qubit gates (rotation matrices)

$$R_x\left(\pi/2 \right) = \exp\left(i\frac{\pi}{4}\sigma^1 \right),$$

$$R_y\left(\pi/2 \right) = \exp\left(i\frac{\pi}{4}\sigma^2 \right). \tag{58}$$

Thus, leaving out the identity, the braid gates of the model form the set

$$\{R_x\left(\pi/2 \right), R_y\left(\pi/2 \right), B_{23}\}. \tag{59}$$

Acting on the Majorana operators in Eq. (46), the braid operators in Eqs. (52) and (55) yield

$$B_{pq}\hat{R}_k B_{pq}^\dagger = \begin{cases} \hat{R}_k & \text{if } k \notin \{p,q\}, \\ \hat{R}_q & \text{if } k = p, \\ -\hat{R}_p & \text{if } k = q. \end{cases} \tag{60}$$

We also specify the observables F_{pq}

$$F_{pq} = -i\hat{R}_p\hat{R}_q, \qquad p < q , \tag{61}$$

which are the fermion parity operators for the pair of Majoranas pq, and the total parity operator Q (topological charge)

$$Q = F_{12}F_{34} = -\hat{R}_1\hat{R}_2\hat{R}_3\hat{R}_4. \tag{62}$$

It can be verified that Q commutes with all braid operators and observables, in compliance with the superselection rules for total topological charge conservation [10].

To complete the model a computational basis needs to be specified. We choose to fuse the anyons in the pairs 1, 2 and 3, 4, so we consider the fermionic operators

$$f_{12} = \frac{1}{2}\left(\hat{R}_1 + i\hat{R}_2\right),$$
$$f_{34} = \frac{1}{2}\left(\hat{R}_3 + i\hat{R}_4\right), \tag{63}$$

producing the states

$$|\bar{0}\bar{0}\rangle, \qquad |\bar{1}\bar{0}\rangle = f_{12}^\dagger |\bar{0}\bar{0}\rangle,$$
$$|\bar{0}\bar{1}\rangle = f_{34}^\dagger |\bar{0}\bar{0}\rangle, \quad |\bar{1}\bar{1}\rangle = f_{34}^\dagger f_{12}^\dagger |\bar{0}\bar{0}\rangle, \tag{64}$$

where $|\bar{0}\bar{0}\rangle$ is such that $f_{12}|\bar{0}\bar{0}\rangle = f_{34}|\bar{0}\bar{0}\rangle = 0$, and the over bar is used to distinguish them from the canonical states in Eq. (36). Explicitly

$$|\bar{0}\bar{0}\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -i \\ i \end{pmatrix}, \qquad |\bar{1}\bar{0}\rangle = \frac{e^{i\frac{\pi}{4}}}{2}\begin{pmatrix} 1 \\ 1 \\ e^{-i\frac{\pi}{2}} \\ e^{-i\frac{\pi}{2}} \end{pmatrix},$$
$$|\bar{0}\bar{1}\rangle = \frac{e^{i\frac{\pi}{4}}}{2}\begin{pmatrix} e^{-i\frac{\pi}{2}} \\ -e^{-i\frac{\pi}{2}} \\ 1 \\ -1 \end{pmatrix}, \qquad |\bar{1}\bar{1}\rangle = \frac{1}{2}\begin{pmatrix} -i \\ -i \\ 1 \\ 1 \end{pmatrix}. \tag{65}$$

These states are separable as is readily checked. The first digit in the kets corresponds to the occupation number of the fermion operator f_{12}, while the second digit to that of the f_{34} operator. This is verified by acting on the basis with the fermion parity operators in Eq. (61), giving

$$F_{12}|\bar{0}\bar{0}\rangle = |\bar{0}\bar{0}\rangle,$$
$$F_{12}|\bar{1}\bar{0}\rangle = -|\bar{1}\bar{0}\rangle,$$
$$F_{12}|\bar{0}\bar{1}\rangle = |\bar{0}\bar{1}\rangle, \tag{66}$$
$$F_{12}|\bar{1}\bar{1}\rangle = -|\bar{1}\bar{1}\rangle,$$

$$F_{34}|\bar{0}\bar{0}\rangle = |\bar{0}\bar{0}\rangle,$$
$$F_{34}|\bar{1}\bar{0}\rangle = |\bar{1}\bar{0}\rangle,$$
$$F_{34}|\bar{0}\bar{1}\rangle = -|\bar{0}\bar{1}\rangle, \tag{67}$$
$$F_{34}|\bar{1}\bar{1}\rangle = -|\bar{1}\bar{1}\rangle,$$

with the plus eigenvalue corresponding to the vacant slot $\bar{0}$ and the minus sign to the occupied state $\bar{1}$. The total parity operator gives

$$Q|\bar{0}\bar{0}\rangle = |\bar{0}\bar{0}\rangle,$$
$$Q|\bar{1}\bar{1}\rangle = |\bar{1}\bar{1}\rangle,$$
$$Q|\bar{0}\bar{1}\rangle = -|\bar{0}\bar{1}\rangle, \tag{68}$$
$$Q|\bar{1}\bar{0}\rangle = -|\bar{1}\bar{0}\rangle.$$

The model is now complete and the system can be initiated in any pair of the basis states with the same Q parity, due to total parity conservation. The last two states in Eq. (68) correspond to the fusion rule $\sigma \times \sigma = \psi$, while the first ones to $\sigma \times \sigma = \mathbf{1}_{\text{vac}}$ and $\sigma \times \sigma \times \sigma \times \sigma = \mathbf{1}_{\text{vac}}$, respectively. Whatever the initial states are, braiding anyons two and and three, with the B_{23} operator, produces the states

$$B_{23}|\bar{0}\bar{0}\rangle = \frac{1}{\sqrt{2}}\left(|\bar{0}\bar{0}\rangle + i|\bar{1}\bar{1}\rangle\right),$$
$$B_{23}|\bar{0}\bar{1}\rangle = \frac{1}{\sqrt{2}}\left(|\bar{0}\bar{1}\rangle - i|\bar{1}\bar{0}\rangle\right),$$
$$B_{23}|\bar{1}\bar{0}\rangle = \frac{1}{\sqrt{2}}\left(-i|\bar{0}\bar{1}\rangle + |\bar{1}\bar{0}\rangle\right), \tag{69}$$
$$B_{23}|\bar{1}\bar{1}\rangle = \frac{1}{\sqrt{2}}\left(i|\bar{0}\bar{0}\rangle + |\bar{1}\bar{1}\rangle\right),$$

which conserve total parity and are maximally entangled. The former is directly seen from Eq. (68), while the latter can be established by their Schmidt decomposition, e. g., for $B_{23}|\bar{0}\bar{0}\rangle$ we have $B_{23}|\bar{0}\bar{0}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0\rangle + i|1\rangle \otimes |1\rangle\right)$, with $|0\rangle$, $|1\rangle$ given in Eq. (34). Similar relations hold for the rest of the states in Eq. (69). On the other hand, the braid operators B_{12} and B_{34} produce the same state multiplied by a phase of the type $\exp(\pm i\pi/4)$ when acting on the basis in Eq. (65), as expected from their Abelian nature expressed in the first relation of Eq. (54). The states in Eq. (69) correspond to the fusion rule $\sigma \times \psi = \sigma$. Finally, we also verify that these maximal entangled states satisfy the Majorana condition

$$i\gamma^2 (B_{23}|\bar{0}\bar{0}\rangle)^* = -iB_{23}|\bar{0}\bar{0}\rangle,$$
$$i\gamma^2 (B_{23}|\bar{0}\bar{1}\rangle)^* = -B_{23}|\bar{0}\bar{1}\rangle,$$
$$i\gamma^2 (B_{23}|\bar{1}\bar{0}\rangle)^* = -B_{23}|\bar{1}\bar{0}\rangle, \tag{70}$$
$$i\gamma^2 (B_{23}|\bar{1}\bar{1}\rangle)^* = -iB_{23}|\bar{1}\bar{1}\rangle.$$

5. Concluding remarks

The methods and results presented regarding bispinor solutions to the massless Dirac equation are of pedagogical value on their own, and this value can only be enhanced by the connection to QC, *e.g.*, after discussing massless bispinors one can readily introduce logical two-qubit states and entangling gates, or vice-versa. Calculations in QC could also benefit from the use of relativistic spinors and the Clifford algebra of the Dirac gamma matrices. Particularly, the TQC model presented, where operators and states are readily obtained departing from the set of entangling gates in Eq. (46), provides a suitable playground to test and understand how Majorana zero modes and topological braiding work, both in the technical and physical assumptions.

i. As is nicely put in Ref. [8] an MZM is "*a quasiparticle that is its own hole*."

ii. Weyl spinors are always massless, and in the condensed matter literature massless fermions are usually referred to as Dirac fermions, but being massless they are indeed Weyl fermions.

iii. In Ref. [30] it is already established a connection between entanglement and invariance under charge conjugation for two-qubit density matrices.

iv. This set of solutions is reported in Refs. [31], with variations due to the gamma matrices representation employed.

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