What is the most “non-point” gravitating or electrically charged object?

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In this paper we search the shape of an aspherical body and the direction in space, for which the greatest deviations from the point mass field (the difference from the inverse-square law) take place for large distances from the field source. It turns out to be a system of two equal point-like masses at the poles of a fixed sphere (giving the greatest positive deviations from the point mass field) and uniform distribution of point-like masses (discrete or continuous) around the sphere equator (giving the greatest negative deviations from the point mass field). In these cases the extremal direction of the field measurement respectively passes through point-like particles and coincides with the axis of symmetry of a ring, which is perpendicular to its plane. Our numerical estimations show that any body can be considered with reasonable accuracy (the relative error in the determination of the field strength is less than 5%) as point-like mass if the distance to the observation point is more than an order of magnitude larger than its characteristic sizes. The problem considered in this paper can help readers to probe the limits of applicability of the field point source model.

Keywords: Point-like mass; multipole expansion; quadrupole term; inertia moment.

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1. Introduction

According to the field theory under the point-like mass (or point charge) we understand a body, the sizes of which are much smaller than the distance to the point of its gravitational (or electrostatic) field observation. In this case the inverse-square law [1] will be a good approximation; that is, the field strength is inversely proportional to the square of the distance between the source and observation points. The inverse-square law can be used to derive Gauss's law, and vice versa. We note that the field of any uniform body of spherical shape can be considered as the field of point-like mass placed at its center at any distance to the observation point. In the general case of an aspherical body, the object field will depend on its shape, sizes, the choice of the observation point, the direction in space, and we should do some integration or use Gauss's law [2] to get the spatial distribution of the field. For example, the very interesting problem of the determination of the optimal shape of an object for generating maximum gravity field at a given point in space is considered in Ref. [3].

In this paper we find the shape of an aspherical body and the direction in space, for which the greatest deviations from the point mass field (the difference from the inverse-square law) take place for large distances from the field source. There are two motivations for this work. First, there are many objects in the Universe, whose shape is very different from spherical (e.g., galaxies, interstellar dust, and clouds). In the second one, this consideration can help readers to probe the limits of the applicability of the field point source model.

Let us consider a homogeneous body of fixed mass m (or the total charge $q \neq 0$; q can be both positive and negative in the electrostatic case) that is within a ball of fixed radius R. It is convenient to make further analysis using the concept of multipole decomposition of field potential.

2. A multipole expansion

In the case of continuous mass distribution a multipole expansion for the gravitational (or electrostatic) potential can be represented in such a form (see Eq. (41.9) in Ref. [4]):

$$V(r) = -G\frac{m}{r}\left(1 - \frac{\vec{p}\cdot\vec{r}}{mr^2} - \frac{Q}{2mr^2} + ...\right), \qquad (1)$$

where r is the distance from reference point (the center of a ball) to the point of observation P;

$$\vec{p} = \iiint\limits_V \vec{r'}\,\mathrm{d}m \qquad (2)$$

is the dipole moment of system;

$$Q = \iiint\limits_V r'^2(3\cos^2\theta - 1)\mathrm{d}m. \qquad (3)$$

Here r' is the distance from the reference point to the point, where a mass element $\mathrm{d}m$ is placed; θ is the angle between vectors $\vec{r}$ and $\vec{r'}$ (Fig. 1).

The series (1) converges quickly if the distance to the observation point is much larger than radius R. Three first terms in Eq. (1) are known as a monopole, dipole and quadrupole terms. The monopole term via relation $\vec{g} = -\nabla V$ (here $\vec{g}$ is the field strength) gives us the inverse-square law. If we choose the reference point to be the center of mass of the system, then the dipole moment (2) (and respectively the second term in Eq. (1)) will be vanished (we can always cancel this term for the system with $q \neq 0$ as well; the case of an electrically neutral system should be excluded from our consideration, since it is not even qualitatively described by the

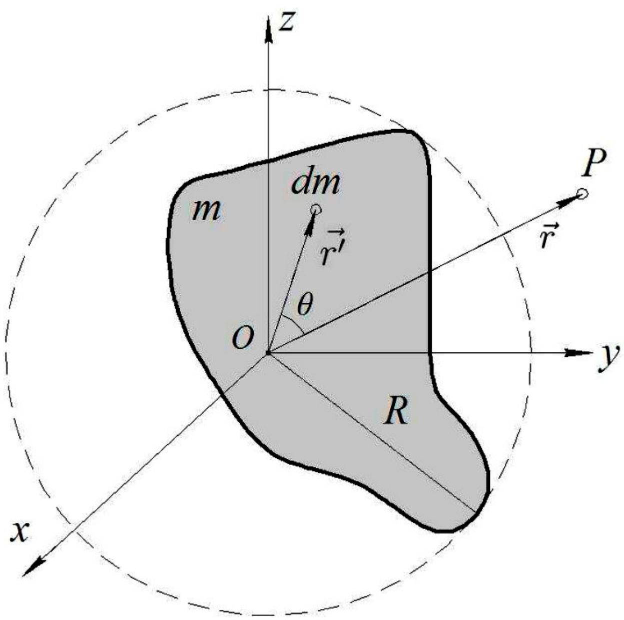

FIGURE 1. Geometry of the problem.

model of field point source (the monopole term vanishes)) and we should extremize only specific quadrupole term Q/m.

Since a body density is constant and $\vec{p} = 0$, our object should have the center of symmetry at O. Below we limit ourselves to the case of an axisymmetric (around Oz' axis) body. Thus, our figure has axial and central symmetry (the point group symmetry is $D_{\infty h}$).

3. The McCullagh's formula

We can rewrite Eq. (3) in the following form:

$$Q = \iiint_V r'^2(3\cos^2\theta - 1)\mathrm{d}m = \iiint_V r'^2(2 - 3\sin^2\theta)\mathrm{d}m$$
$$= (I_{x'x'} + I_{y'y'} + I_{z'z'} - 3I), \tag{4}$$

where

$$I_{x'x'} = \iiint_V (y'^2 + z'^2)\mathrm{d}m,$$

$$I_{y'y'} = \iiint_V (x'^2 + z'^2)\mathrm{d}m,$$

$$I_{z'z'} = \iiint_V (x'^2 + y'^2)\mathrm{d}m$$

are the components of the inertia tensor of the body relative to Ox', Oy' and Oz' axes respectively;

$$I = \iiint_V r'^2 \sin^2\theta \mathrm{d}m$$

is the inertia moment relative to the axis that coincides with $\vec{r}$-direction (we assume that in general case Oz'-axis does not coincide with $\vec{r}$-direction; in this case distance $d = r'\sin\theta$ equals to perpendicular dropped from $\mathrm{d}m$ to this direction). Using Eq. (4) and putting $\vec{p} = 0$ we can rewrite Eq. (1) in the following form:

$$V(r) \approx -G\frac{m}{r} - \frac{G}{2r^3}(I_{x'x'} + I_{y'y'} + I_{z'z'} - 3I). \tag{5}$$

Eq. (5) is known as McCullagh's formula [5].

Now we require that the greatest deviations from the point mass potential are towards z'-direction ($I = I_{z'z'}$). In view of axial symmetry $I_{y'y'} = I_{x'x'}$. Then $Q = 2(I_{x'x'} - I_{z'z'})$. In order to maximize Q one needs to minimize $I_{z'z'}$ and simultaneously maximize $I_{x'x'}$. The former is achieved by distributing the mass as close to the symmetry axis as possible, while the latter is attained by putting the mass a far away from the equatorial plane as possible. This directly leads to two equal point-like masses $m/2$ at the poles of a fixed sphere (extremely prolate body; $Q/m = 2R^2$). Likewise, using a similar argument, one obtains the equatorial (placed in $x'y'$-plane) filamentary ring as the shape that minimizes Q (extremely oblate body; $Q/m = -R^2$). In these cases the extremal direction of the field measurement respectively passes through point-like particles and coincides with the axis of symmetry of a ring, which is perpendicular to its plane (see Fig. 2).

Let us find specific quadrupole term Q/m for the discrete system of N equal point-like masses m/N located at N vertices of a regular polygon inscribed in the circle of radius R, when $I = I_{z'z'}$ and Oz'-axis is perpendicular to the plane of the system and passes through the center of mass of this

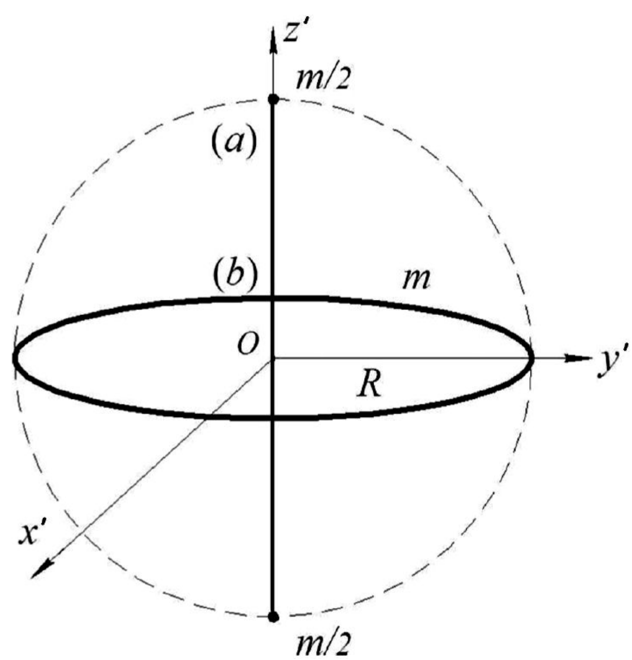

FIGURE 2. The system with: (a) the biggest positive quadrupole moment Q; (b) the biggest negative quadrupole moment Q. The extremal direction of the field measurement coincides with Oz'-axis.

system. We have: $I_{z'z'} = NmR^2/N = mR^2$. Applying the perpendicular axis theorem [6], we get: $I_{x'x'} = I_{y'y'} = I_{z'z'}/2$. Then using Eq. (12) we derive: $Q/m = -R^2$. Therefore, this system gives the greatest negative deviations from the point mass field in z'-direction alike an infinitely thin ring placed in $x'y'$-plane.

Above we have considered the case, where Oz'-axis coincides with $\vec{r}$-direction ($I = I_{z'z'}$). If it is not so, then [4]

$$Q = \frac{Q_0(3\cos^2\Theta - 1)}{2}, \tag{6}$$

where Θ is the angle between rotation Oz'-axis of body and $\vec{r}$-direction; Q_0 is the value of Q at $\Theta = 0$. Interestingly this quantity varies from Q_0 at $\Theta = 0$ to 0 at $\Theta = \Theta_{\rm cr}$ ($\cos\Theta_{\rm cr} = 1/\sqrt{3}$); then it changes the sign reaching the value of $-Q_0/2$ at $\Theta = \pi/2$.

4. Approximate versus exact solutions

In this section we compare the numerically spatial distribution of the field for two found above asymptotic cases using the exact and the approximate expressions. As an example of the system giving the greatest positive deviations from the point mass field, we may mention the model of a binary star system with equal masses. Using the field superposition principle and the expression for the point mass field, we can get the exact formula for the field strength in z'-direction for the system of two equal point-like masses in such a form:

$$g(\rho) = \frac{g_0}{2}\left[\frac{1}{(\rho-1)^2} + \frac{1}{(\rho+1)^2}\right], \tag{7}$$

where $g_0 = Gm/R^2$, $\rho = r/R$ is the dimensionless distance from the reference point to the point of observation. An approximate expression for the field strength can be found using Eq. (5), equality $Q/m = 2R^2$ and relation $\vec{g} = -\nabla V$:

$$g_{\rm approx}(\rho) \approx g_0\left(\frac{1}{\rho^2} + \frac{3}{\rho^4}\right). \tag{8}$$

The examples of ring-like gravitational objects are given in Ref. [7]. An exact expression for the field strength of an infinitely thin ring is obtained in Ref. [8]. In our notations it has the following form:

$$g(\rho) = g_0\frac{\rho}{(\rho^2+1)^{3/2}}. \tag{9}$$

The corresponding approximate formula that takes into account quadrupole term $Q = -mR^2$ is as follows:

$$g_{\rm approx}(\rho) \approx g_0\left(\frac{1}{\rho^2} - \frac{3}{2\rho^4}\right). \tag{10}$$

In Fig. 3 we plot the dependencies of relative error $\delta_g = |g - g_{\rm approx}|/g$ of ρ for both asymptotic cases.

It is seen that the quadrupole approximation is quite correct for $\rho > 3$, whereas the point mass model gives reasonable results for $\rho > 10$ (the relative error in the determination of the field strength is less than 5%). Therefore, we conclude that any body can be considered with reasonable accuracy as point-like mass if the distance to the observation point is more than an order of magnitude larger than its characteristic sizes.

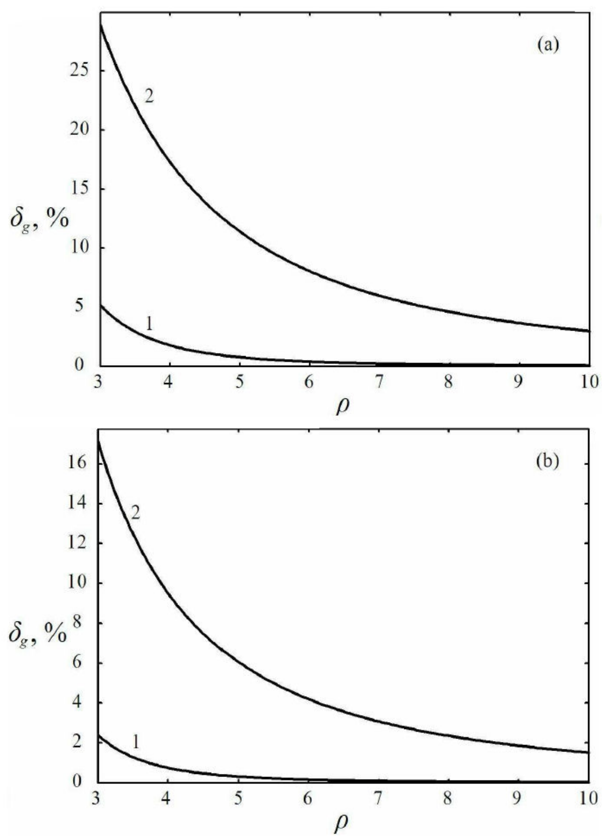

FIGURE 3. Dependence of relative error $\delta_g = |g - g_{\rm approx}|/g$ of the dimensionless distance $\rho = r/R$ from the center of mass along z'-direction for: (a) a system of two equal point-like masses; (b) an infinitely thin ring. (1) the quadrupole approximation (both terms in Eqs. (8), (10) are taken into account); (2) the point-like mass approximation (only the first term in Eqs. (8), (10) is taken into account).

5. Conclusions

In this paper we search the shape of an aspherical body and the direction in space, for which the greatest deviations from the point mass field (the difference from the inverse-square law) take place for large distances from the field source. Using the McCullagh's formula we find that these are a system of two equal point-like masses at the poles of a fixed sphere (giving the greatest positive deviations from the point-like mass field) and uniform distribution of point-like masses (discrete or continuous) around the sphere equator (giving the greatest negative deviations from the point mass field). In these cases the extremal direction of the field measurement respectively passes through point-like particles and coincides with the axis of symmetry of a ring, which is perpendicular to its plane. Thus, the objects found differ from the sphere not only in form but also in topology and have infinitely large density. Our findings relate to the gravitational field or the electric field of positively charged systems. If these objects are negatively charged, then such deviations should change the sign. Using numerical estimations, we also conclude that any body can be considered with reasonable accuracy (the relative error in the determination of the field strength is less than 5 %) as point-like mass if the distance to the observation point is more than an order of magnitude larger than its characteristic sizes.

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