Strong magnetic fields in gauge theories at finite temperature I

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Received 18 December 2019; accepted 22 March 2021

The problem of propagation of photons and currents in a medium at finite temperature and in presence of a strong magnetic field in the frame of quantum electrodynamics is discussed in the present paper. Its first part is devoted to introduce the reader to the formalism of quantum field theory at finite temperature and density. The basic Schwinger-Dyson equations are obtained, by using functional methods and path integrals. It is discussed the meaning of the zero temperature and zero density limit. The breaking of the spatial symmetry by the magnetic field determine the existence of a set of basic vectors and tensors which must satisfy the relativistic, gauge and CPT invariance of quantum electrodynamics. The charge symmetric and non-symmetric cases are discussed. Also a chiral current arises, associated to a pseudo-eigenvector of the polarization operator, due to the breaking of the spatial symmetry by the external magnetic field. As a chiral effect in photons, the Faraday effect is discussed.

Keywords: Magnetic fields; QED; standard model

DOI: https://doi.org/10.31349/RevMexFisE.18.020209

1. Introduction

The influence of magnetic fields in relativistic quantum systems, like electron-positron plasma and quark-gluon plasma [1–10], is an important subject in high energy physics.

The present paper is the first of a couple devoted to describe methods for the study problems of gauge field theories at finite temperature and density. It is assumed that the reader has a minimal background in relativistic quantum theory and quantum field theory. We shall start with statistical quantum electrodynamics, which in the case of zero temperature T and chemical potential μ reproduce usual results of standard quantum electrodynamics, but formulated in Euclidean variables. Of particular interest are the phenomena which occur in presence of strong magnetic fields. We start by establishing the basic formalism of quantum field theory at finite temperature, using the Green functions method, as developed by E. S. Fradkin [11], and devote some space to describe the necessary tools, for instance the functional differentiation, Grassmann variables and path integrals of interacting Bose and Fermi fields. We find a set of functional equations leading to get the Schwinger-Dyson equations in quantum statistical electrodynamics, whose solutions are usually among the main expected results. This is followed by a presentation of the non-relativistic and relativistic problem of motion of charged fermions in a magnetic field. The expressions for the Green functions tensor structure are written exactly, independently of any order of approximation, based on the conditions imposed by assuming relativistic, $U(1)$ gauge, and CPT invariances. The approximate quantities are in general scalars multiplying the tensors, obtained by perturbation theory. The dispersion equations for photon propagation as well as the properties of currents parallel to the magnetic field are discussed. The chiral effects on photons and electrons, leading respectively to Faraday and Chiral Magnetic effects are discussed. The dispersion equations for propagation orthogonal to the magnetic field as well as the basic equations for the quantum statistics in the non-Abelian case of the electroweak plasma will be discussed in the second part of the present work, to be published in a next paper.

2. The density matrix

The role of the wave function Ψ in quantum theory is played in quantum statistics by the density matrix ρ. Apart from the intrinsic probabilistic nature of quantum theory, it is necessary to introduce in quantum statistical physics an additional ignorance about the quantum state of the system under study. This means another statistics, which is provided by the density matrix. This is usually done by conceiving an “ensemble” or infinite replica of our system where each one of the members satisfies the known macroscopic conditions of the given system and differs on the microscopic state compatible with the macroscopic conditions. In non-relativistic quantum mechanics for each of the members of the ensemble we have the set of wave functions Ψ^e, which can be expressed in terms of a complete set of orthogonal eigenfunctions of the observables of the system, which can be chosen as the two conserved operators: the Hamiltonian $\hat{H}$ and the number of particles $\hat{N}$. In place of the total number of particles, in relativistic quantum theory, as it is quantum electrodynamics (QED), it is usually taken as conserved the net electric charge Q, that is, the difference between the number of particles and antiparticles. Quantum electrodynamics satisfies Lorentz invariance and the discrete CPT (Charge, Parity and Time) symmetries.

Now, we consider our system as a subsystem of a larger system, which is described by a wavefunction Ψ^e which can be expanded in terms of an orthonormal basis $\{\Phi_n(q)\}$ in Hilbert space. The upper index e is to be taken over the states corresponding to the ensemble. Then we can write

$$\Psi^e = \sum_n c_n \Phi_n. \tag{1}$$

The quantum average of an operator corresponding to a physical quantity $\hat{P}$ is written as:

$$\langle \hat{P} \rangle = \langle \Psi^e | \hat{P} | \Psi^e \rangle = \sum_{n,m} c_n^{*e} c_m^e P_{nm}, \tag{2}$$

where

$$P_{nm} = \int \Phi_m^*(q) P \Phi_m(q) dq, \tag{3}$$

are the matrix elements of $\hat{P}$. The quantum statistical average over Ψ^e states implies the ensemble mean value, which is taken over c_n^e values, since to each microstate of the ensemble corresponds a Ψ^e state

$$\langle\langle \hat{P} \rangle\rangle = \langle\langle \Psi^e | \hat{P} | \Psi^e \rangle\rangle = \sum_{n,m} \langle c_n^{*e} c_m^e \rangle P_{nm}, \tag{4}$$

Let us call them as $\langle c_n^{*e} c_m^e \rangle = \rho_{mn}$. The quantum statistical average value of P may be written now as

$$\langle\langle \hat{P} \rangle\rangle = \sum_{n,m} \rho_{mn} P_{nm}, \tag{5}$$

where we call ρ_{mn} the density matrix. It must be a function of the parameters characterizing the system under study, as it is the energy. Thus, we can take $\hat{H} = \hat{P}$ [12]. If we take also the number of particles, as the system must be in thermodynamic equilibrium with the medium, exchanging energy and particles with it, it must depend also from the temperature T and chemical potential μ. In that case it is used the so-called grand canonical ensemble, which in the non relativistic case use mostly the Hamiltonian operator $\hat{H}$ and the number of particles $\hat{N}$. Both quantities are simultaneously observable, thus the commutator $[\hat{H}, \hat{N}] = 0$ and we can write the expression

$$\hat{\rho} = e^{-\beta(\hat{H} - \mu \hat{N})}, \tag{6}$$

where $\beta^{-1} = kT$ and $k = 1.38 \times 10^{-23} J/K$ is the Boltzmann constant, and T is the absolute temperature. We may take the set of functions Φ_n as the common eigenfunctions of $\hat{H}$ and $\hat{N}$. In the frame of high energy physics, the density matrix operator is defined in terms of the Hamiltonian $\hat{H}$ and some conserved quantity, whose operator commutes with $\hat{H}$, as the net electric charge $\hat{Q}$, or the lepton and baryon numbers $N_{l,b}$ (assumming that baryon and lepton numbers are conserved independently. Models based on the conservation of the quantity $N_b - N_l$, are out of our the scope of the present paper). The equilibrium of energy exchange is characterized by the temperature T and the exchange of particles by the corresponding chemical potential μ.

$$\rho_{mn} = e^{-\beta(\varepsilon_n^{(N)} - \mu N_n)} \delta_{mn}. \tag{7}$$

The grand partition function is defined by

$$Z = \mathrm{Tr}\, e^{-\beta(\hat{H} - \mu \hat{N})} = \sum_{N=0}^{\infty} \sum_n e^{-\beta(\varepsilon_n^{(N)} - \mu N_n)}, \tag{8}$$

where $\varepsilon_n^{(N)}$ is the energy of the state with N_n particles and the thermodynamic potential $\Omega = -PV$ is given by

$$\Omega = -kT \ln Z, \tag{9}$$

where P is the pressure and V the volume of the system. From Ω one can know the thermodynamic properties of the system, because:

$$d\Omega = -SdT - Nd\mu, \tag{10}$$

where S is the entropy, then

$$S = -\left(\frac{\partial \Omega}{\partial T}\right)_{\mu=\text{const}}, \quad N = -\left(\frac{\partial \Omega}{\partial \mu}\right)_{T=\text{const}}, \tag{11}$$

$$U = -\mu \left(\frac{\partial \Omega}{\partial \mu}\right) - T \left(\frac{\partial \Omega}{\partial T}\right) + \Omega, \tag{12}$$

where U is the internal energy.

For ideal quantum gases we have:

a) Bosons

$$\Omega_B = kT \sum_p \ln\left(1 - e^{-\beta(\varepsilon_p - \mu)}\right). \tag{13}$$

b) Fermions

$$\Omega_F = -kT \sum_p \ln\left(1 + e^{-\beta(\varepsilon_p - \mu)}\right). \tag{14}$$

From it the average number of particles is:

$$N = \sum_p n_p \tag{15}$$

where

$$n_p = \frac{1}{e^{\beta(\varepsilon_p - \mu)} - 1}, \qquad \text{for bosons}, \tag{16}$$

$$n_p = \frac{1}{e^{\beta(\varepsilon_p - \mu)} + 1}, \qquad \text{for fermions}. \tag{17}$$

3. Quantum field theory at finite temperature

All the previous equations are valid for systems of identical non-interacting particles. A more powerful theory is necessary for interacting systems of fields at finite temperature and conserved charge (or number of particles).

Starting from the definition Eq. (6) for the grand canonical ensemble we get the Bloch equation

$$-\frac{\partial \hat{\rho}}{\partial \beta} = (\hat{H} - \mu \hat{N})\hat{\rho}, \tag{18}$$

which has an analogy to the Schrodinger's equation, allowing to a parallelism to usual quantum theory, but using an imaginary time variable x_4 such that $-\beta \leq x_4 \leq \beta$ [13–15]. By defining the fields $\psi(\mathbf{x}, 0)$ as the Schrödinger representation, we may define a new "Heisenberg representation", $\psi'(\mathbf{x}, x_4) = \rho(-x_4)\psi(\mathbf{x}, 0)\rho(x_4)$ leading to the equation of motion

$$\frac{\partial \psi'(\mathbf{x}, x_4)}{\partial x_4} = [\hat{H} - \mu \hat{N}, \psi'(\mathbf{x}, x_4)]. \tag{19}$$

If we know, for example, the Lagrangian of a system we can build $\hat{H}$, $\hat{N}$ and $\hat{\rho}$, and from it, to get Z, Ω, and obtain the thermodynamical properties of the system by usual methods. The Eq. (18) gives the possibility of developing a formalism analogous to that of quantum field theory, where the imaginary time variable is now replaced by the Euclidean variable $x_4 \to ict$ such that $0 \leq x_4 \leq \beta$. What we do is to construct a gauge theory, in Euclidean four dimensional space, which has physical meaning when the system as a whole is at rest (the four velocity space components are zero). The four imaginary time coordinate x_4 is defined in the interval $[0, \beta]$ where $\beta = 1/kT$. This is of particular interest in both high energy physics and the physics of condensed matter. In the limit $\beta \to \infty$ (or $T \to 0$) and $\mu \to 0$ one gets quantum field theory in Euclidean variables. The analytical continuation $x_4 \to ict$ leads to an analogy between relativistic statistical physics and quantum field theory. The method of finite temperature Green functions in quantum statistics arised, starting from a basic work of Matsubara [16], by Abrikosov, Gorkov and Dzhialoshinski [17] in condensed matter and by Fradkin [11], Martin and Schwinger [18] in a wider scope, among other authors.

In the present paper we shall use several results due to Fradkin. At this point we must stress that the analogy which we shall study is valid only in a reference frame at rest, since in this respect, in field theory this is not important in many cases, unless it is specified. This is due to the fact that in the ground state the vacuum eigenvalue of the four-momentum vector is not important in the absence of external fields because

$$p_\mu \mid 0\rangle = 0. \tag{20}$$

However, in the temperature case the ground state is not isotropic and homogeneous in time as it is in quantum vacuum, and on the opposite,

$$p_\mu \mid \varepsilon_n, N_n\rangle \neq 0. \tag{21}$$

An important consequence of Eq. (21) is that the four velocity of the system $u_\mu = p_\mu/M \neq 0$ (if the total mass of the system is M, its four momentum vector is $p_\mu = Mu_\mu$,). It has an important role in the temperature case, although in most cases calculations of physical interest are done in the system at rest, where the only non zero component is $u_4 = i/c$. In the system $\hbar = c = 1$, obviously $u_4 = i$.

Our model would be useful not only to describe equilibrium states but also certain non-equilibrium states (quantum kinetics) in which is possible to deal simultaneously with time and temperature. The temperature Green's functions play a central role in describing systems in thermodynamical equilibrium and can be extended to states close to equilibrium, like the dynamics of processes of emission and absorption of electromagnetic radiation, the description of systems of particles in a medium, the lifetime of quasi-particles and others, where can be described by functions of temperature T and chemical potential μ.

As pointed out earlier, in the present paper we shall deal with finite temperature quantum field theory, providing models which can be used to describe phenomena and systems in a wide range of interest, from condensed matter to astroparticle physics. We shall consider first the general case of systems without external fields, and at the end study the case in which the system is under the influence of a strong external magnetic field. We obtain the basic equations in the free and interacting field cases, and the corresponding Schwinger-Dyson equations. Later, by starting from the non-relativistic electron dynamics in an external magnetic field, and continuing with the relativistic case, we obtain the Green function for an electron-positron gas placed in an external magnetic field **B**. The chemical potentials μ_e (μ_p) of electrons (positrons) are different from zero. The photon chemical potential satisfies the equation $\mu_\gamma = \mu_e + \mu_p = 0$, in correspondence to the fact that the number of photons is not a conserved quantity. Thus, we have the property $\mu = \mu_e = -\mu_p$. To guarantee the stability in the charge non-symmetrical case $\mu \neq 0$, one must assume that there exists an ion background which compensates the leptonic electric charge and whose effect is not considered in every other respect. We shall find the photon equation of motion, and determine the modes of propagation of electromagnetic waves (after taking $i\omega = k_4$ and $it = x_4$; we are using natural units in which $\hbar = c = 1$). Eventually, if natural units are used and T is given in energy units, it is equivalent to take the Boltzmann constant as $k = 1$. By knowing the polarization tensor $\Pi_{\mu\nu}$ in the temperature case it is possible to calculate the thermodynamical potential of the photon system, in equilibrium with an electron-positron background. This case occurs for instance in the black body problem at temperatures larger than twice the electron rest energy.

We want at this point to suggest to readers (especially those unfamiliar with quantum field theory based on functional methods), to do a first lecture aimed to grasp the main new ideas and definitions given below and to go directly to the basic formulae for non-interacting and interacting systems at non-zero temperature T . Once we have the partition functional Z, it is easy to obtain the thermodynamical potential Ω. Having Ω, we can obtain the thermodynamic properties of the whole system. For a slight departure from equilibrium (notice that strict thermodynamic equilibrium is not compatible with macroscopic flow of matter or energy), so that we can study the propagation of specific particles, as electrons, positrons and photons moving in the medium at finite temperature T, we get its dynamics described by the analytical continuation $k_4 = i\omega t$, $p_4 = i\varepsilon t$ made on the corresponding Green functions. We then get quantities depending on time, and the Schwinger-Dyson equations allow us to find the poles of the Green functions of the interacting particles. Real values for the energy lead to stable states, complex solutions to instabilities, due for instance to absorptive processes in the medium.

3.1. Functional derivatives

Let us consider the sum defining the function $S(\varphi)$

$$S(\varphi) = \sum_i I(\varphi_i), \tag{22}$$

where $i = 1, 2, 3, ..., N$ is a discrete variable. The integral defining the functional $S(\varphi)$,

$$S = \int dx I(\varphi(x)), \tag{23}$$

can be interpreted as the sum over the continuous variable x. In Eq. (22) we define the partial derivative $\partial S/\partial\varphi_k$ by

$$\frac{\partial S}{\partial \varphi_k} = \lim_{\epsilon\to 0} \frac{\sum_i I(\varphi_i + \epsilon\delta_{ik}) - \sum_i I(\varphi_i)}{\epsilon} = \frac{\partial I}{\partial \varphi_k}. \tag{24}$$

Analogously we may define in Eq. (23) the functional derivative of $S(\varphi)$ with regard to $\varphi(y)$ as

$$\frac{\delta S}{\delta\varphi(y)} = \lim_{\varepsilon\to 0} \frac{\int dx I[\varphi(x) + \epsilon\delta(x-y)] - \int dx I(\varphi(x))}{\epsilon}$$
$$= \frac{\delta I}{\delta\varphi(y)}. \tag{25}$$

3.2. Path integrals

The use of path integrals is very important, since it is the method to be used in the non-Abelian gauge field theories. In the Abelian case, as it is in quantum electrodynamics, it becomes very simple. Let us start from the one dimensional Gaussian

$$G(a) = \int dx e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}}. \tag{26}$$

For N degrees of freedom it is:

$$G(A) = \int dx_1...dx_N \; e^{-\frac{1}{2}x_i a_{ij} x_j}, \tag{27}$$

where a_{ij} are the elements of a symmetric real matrix $(x_i a_{ij} x_j = X^T AX, A^T = A)$. Let R be the matrix which diagonalizes A, that is $R^T AR = D$, with $R^T = R^{-1}$. Then $X^T AX = Y^T DY$, where $Y = RX$. As the Jacobian of the transformation $Y = RX$ is unity, we have

$$G(A) = \int dy_1...dy_N \; e^{-\frac{(Y^T DY)}{2}}$$
$$= G(d_1)G(d_2)...G(d_N), \tag{28}$$

where d_i are the elements of the diagonal matrix D. Finally $G(A) = (2\pi)^{N/2}(\det\ A)^{-1/2}$. If we define the measure as $(dx) = d^n x(2\pi)^{-N/2}$ we can write

$$(\det\ A)^{-1/2} = (2\pi)^{-N/2}G(A)$$
$$= \int (dx)\; e^{-\frac{1}{2}X^T AX}. \tag{29}$$

From Eq. (28) and Eq. (29) when $N\to\infty$ by using

$$\ln\det\ M = \text{Tr}\ \ln M = \sum_i \ln d_i, \tag{30}$$

it results

$$\ln\int (dx)\; e^{-\frac{1}{2}X^T AX} = -\frac{1}{2}\ln\det\ A$$
$$= -\frac{1}{2}\int dt \ln d(t). \tag{31}$$

Let us consider the expression

$$F(A,\omega) = \int \prod_{i=1}^{N} (dx_i) e^{-\frac{1}{2}X^T AX + \omega^T X}. \tag{32}$$

If A^{-1} exists, by calling $X' = X - A^{-1}\omega$, it can be shown easily that

$$F(A,\omega) = e^{\frac{1}{2}\omega^T A^{-1}\omega} \int \prod_{i=1}^{N} dx_i'\; e^{-\frac{1}{2}X'^T AX'}$$
$$= e^{\frac{1}{2}\omega^T A^{-1}\omega} (\det A)^{-1/2}. \tag{33}$$

Let us consider the case in which the Gaussian has A singular. Let us assume that the n eigenvalues d_i vanish (from $N - n + 1$ to N). Let us define the restricted Gaussian

$$G_{\text{rest}}(A) = \int dy_1...dy_{N-n}\; e^{X^T(y)AX(y)}, \tag{34}$$

in which we integrate only on the variables having non-zero eigenvalues. However, the dependence of $G_{\text{rest}}(A)$ with regard to the variables y makes this representation rather complicated. It is desirable to have the integrals on the variables

x_i. To solve this difficulty, we shall introduce new variables $y_{N-n+1},...y_N$ and rewrite Eq. (34) as

$$G_{\text{rest}}(A) = \int dy_1...dy_{N-n+1}...dy_N \times \delta(y_{N-n+1})...\delta(y_N)\; e^{-X^T(y)AX(y)}, \tag{35}$$

and after it, we change the variables from y to x by means of the Jacobian of the transformation of coordinates

$$dy_1...dy_N = dx_1...dx_N \text{ Det}\left|\frac{\partial y}{\partial x}\right|, \tag{36}$$

to obtain finally

$$G_{\text{rest}}(A) = \int \prod_{i=1}^{N} dx_i \times \text{Det}\left|\frac{\partial y}{\partial x}\right| \prod_{a=N-n+1}^{N} \delta(y_a) e^{-X^T AX}. \tag{37}$$

This is a well-defined integral, and the set of functions $y_a(x)$ and the factors

$$\text{Det}\left|\frac{\partial y}{\partial x}\right| \prod_{a=N-n+1}^{N} \delta(y_a)$$

in the measure restrict the integration from an $N-$dimensional space to another one having $N-n$ dimensions. $G_{\text{rest}}(A)$ does not depend on the specific form of $y_a(a \geq N-n)$. This problem appears in the formulation of gauge field theories by means of path integrals, which are necessary when passing from infinite discrete to continuous variables. The previous procedure can be applied to gauge theories, where there are constraints due to the dependence among momenta and fields, and also due to gauge invariance. It is necessary to fix a gauge condition through a parameter, to avoid singularites due to the gauge invariant field tensor. This can be handled according to the procedure established by Faddeev and Popov, which is very simple in the case of quantum electrodynamics. The gauge fixing parameter does not appear in the physical results. A more detailed study of the path integral method and gauge theories can be found in [19–21].

3.3. Grassmann variables

For commuting quantities (bosonic particles), the transition amplitude $\langle q''(t'')|q'(t')\rangle$ is expressed as a path integral on classical dynamic variables. In the case of bosonic fields, the condition is the same; the dynamic variables are in this case classical fields. For fermion fields, however, it is necessary to introduce anticommuting classical variables. These variables are called Grassmann variables or numbers. Grasmann numbers are defined by the anticonmutation relation

$$\{\Theta_i, \Theta_j\} = 0, \qquad i,j = 1...N. \tag{38}$$

It implies that a Grassmann variable satisfies $\Theta_i^2 = 0$. If we choose $i = 1$, then any function $f(\Theta)$ has the general form

$$f(\Theta) = \alpha + \beta\Theta, \tag{39}$$

where $\alpha, \beta \in \mathbb{C}$ are complex numbers (independent of Θ), that is, the exponent of θ can be only $0, 1$ as in Eq. (39). Define the left derivative

$$\frac{\partial f}{\partial \Theta} = \frac{\partial}{\partial \Theta}(\alpha + \beta\Theta) = \frac{\partial}{\partial \Theta}(\Theta\beta) = \beta, \tag{40}$$

and the right derivative

$$f\frac{\partial}{\partial \Theta} = (\alpha + \beta\Theta)\frac{\partial}{\partial \Theta} = -\beta, \tag{41}$$

in general, for

$$f = \alpha + \beta_i\Theta_i + C_{ij}\Theta_i\Theta_j, \tag{42}$$

we have

$$\frac{\partial f}{\partial \Theta_k} = \beta_k + C_{kj}\Theta_j - C_{ik}\Theta_i, \tag{43}$$

and similarly for the right derivative. Then, we obtain that

$$\int d\Theta = 0, \qquad \int d\Theta\; \Theta = 1. \tag{44}$$

(it acts as a derivative operator). For instance

$$\int d\Theta_1 \int d\Theta_2 \Theta_1 \Theta_2 = -\int d\Theta_1 \Theta_1 \times \int d\Theta_2 \Theta_2 = -1. \tag{45}$$

Let us consider the general integral

$$I_N(M) = \int d\Theta_1...d\Theta_N\; e^{-\Theta^T M \Theta}, \tag{46}$$

with N even and M antisymmetric matrix, with elements m_{ij}. We consider $N = 2$, then

$$I_2(M) = -\int d\Theta_1 d\Theta_2 [\Theta_1 m_{12} \Theta_2 - \Theta_2 m_{12} \Theta_1] = 2m_{12} = 2\sqrt{\text{Det } M}. \tag{47}$$

But for any even N one has

$$I_N(M) = 2^{N/2}\sqrt{\text{Det } M}. \tag{48}$$

For the limit $N \to \infty$, by using the measure $(d\Theta_i) = d\Theta_i/\sqrt{2}$, we have

$$\sqrt{\text{Det } M} = \lim_{N\to\infty} \int \prod_i^N (d\Theta_i) e^{\Theta^T M \Theta} = \int D\Theta\; e^{\int \Theta(x)M(x,y)\Theta(y)dxdy}. \tag{49}$$

Let us consider now the integral

$$I_N(M,X) = \int d\Theta_1 ... d\Theta_N \; e^{-\Theta^T M \Theta + X^T \Theta}. \tag{50}$$

By changing the variables $\Theta' = \Theta - M^{-1}X$, it is easy to show that

$$I_N(M,X) = e^{-\frac{1}{2}X^T M^{-1} X} I_N(M), \tag{51}$$

and by defining the $n-$components Grassman vectors $\boldsymbol{\theta} = \theta_i....\theta_n$ and $\boldsymbol{\eta} = \eta_i....\eta_n$ and a real square matrix A_{ij} of rank n, we have

$$\int d\boldsymbol{\theta} d\boldsymbol{\eta} e^{[-\theta_i A_{ij} \eta_j]} = \text{Det } A. \tag{52}$$

We shall used below the Grassmann variables in quantum electrodynamics.

4. Statistical Quantum Electrodynamics

Physically we are going to describe the interaction of an electron-positron gas with a photon. We may consider two cases. First, the neutral case, when there is an equal number of electrons and positrons. The chemical potential is zero. This is the case of the black body at a temperature such that $kT = mc^2$ (where m is the electron mass). It may correspond for instance, to the astrophysical scenario, as may be the magnetosphere of neutron stars, when a large number of gamma rays decay in electron-positron pairs, and both systems of fermions and bosons coexist in equilibrium for some time. The second case, when there is an excess of electrons or positrons leading to a nonzero chemical potential and there is, for instance, a positively charged background of ions compensating the negative charge of the electron-positron gas. The dynamics of such background may be ignored in several cases, except to guarantee the electric neutrality of the system. The ions have masses of order 10^3 times the mass of the electrons and positrons which allows frequently, for simplicity, to reduce their role to the contribution of their rest energy. Its full incorporation into the model, however, can be done without difficulties.

The problem of photons in equilibrium with the electron–positron gas may have interest in astroparticle physics. In any case, it has academic interest to construct a model for the interaction of the electron-positron system with electromagnetic radiation in a medium on which can be defined a uniform temperature and the average densities of particles. We start by writing quantum electrodynamics in Euclidean variables, where $\overline{\psi}, \psi$ are four spinor functions describing the electron-positron field and A_μ is the electromagnetic field four-vector. We have the correspondence $it \to x_4$ (we are taking natural units $\hbar = c = 1$), $i\gamma_0 \to \gamma_4$, $ik_0 \to k_4$. Let us start with the Lagrangian density

$$L = -\overline{\psi}(\gamma_\mu \partial_\mu + m)\psi + ie\overline{\psi}\gamma_\mu A_\mu \psi - \frac{1}{4} F_{\mu\nu} F_{\nu\mu}. \tag{53}$$

The conserved Noether current density is $j_\mu = (\partial L/\partial(\partial_\mu \psi))\psi = e\overline{\psi}\gamma_\mu \psi$, which integrating on an appropriate volume V leads to the current

$$\mathbf{J}_\mu = e \int_V d^3\mathbf{x} \overline{\psi}(x) \gamma_\mu \psi(x). \tag{54}$$

We shall use $\mathbf{J}_\mu$ for the field theory electromagnetic current and J_μ for the external current. Also along this paper we use α as an arbitrary gauge parameter and not as the fine structure coupling constant. In CGS units it is $\alpha = e^2/\hbar c = 1/137$. According to the system of units we are using, $\hbar = c = 1$. Thus e is not the usual electron charge, and $e^2 = 1/137$.

The electromagnetic field satisfy the gauge invariance $A_\mu \to A_\mu + \partial_\mu \lambda$, as can be shown by substituting the shifted A_μ in the antisymmetric electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We can write $F_{\mu\nu}F_{\nu\mu} = -2(\partial_\eta^2 \delta_{\mu\nu} - \partial_\mu \partial_\nu) A_\mu A_\nu$. If we want to integrate on the field A_μ as a Gaussian, we have a difficulty since the determinant multiplying $A_\mu A_\nu$ is singular. To overcome it we must fix a gauge, and add it to the Lagrangian. The Faddeev-Popov determinant Det M is defined as the derivative of the gauge conditions with regard to the gauge parameters, in our case only one, λ. Below we will take the gauge as the term $\alpha(\partial_\mu A_\mu(x))$, and it implies that Det $M = \alpha(\partial/\partial\lambda(\partial_\mu A_\mu + \partial_\mu^2 \lambda)) = \alpha\partial_\mu^2$, and it depends only on the coordinates (in momentum space it is Det $M(k) = -\alpha k_\mu^2$). In consequence, it can be put out of the functional integral. From Eqs. (6-8), by using the quantum mechanical canonical transformations, as the Lagrangian and the Hamiltonian are related by the equation $p_i \dot{q}_i - H = L$, one can show that the partition functional can be written as

$$Z = N(\beta)\text{Det } M \int Dp_A Dp_\psi D\bar{\psi} \times D\psi DA_\mu \delta(C_n) e^{\int_\beta d^4x [p_\psi \dot{\psi} - H + \mu N]}, \tag{55}$$

where C_n are constraints which link the momenta with the corresponding fields (we remind that here the dots mean derivatives with regard to x_4 and not derivatives with regard to time). One can write the terms $\delta(C_{1,2})$, respectively, as $\delta(p_\mu^A - F_{4\mu})$ and $\delta(p_\psi - \overline{\psi}\gamma_4)$. After the path integration over the momenta $p_A \; p_\psi$ is done, another constraint is needed to fix the gauge condition, which we shall introduce by adding to the effective Lagrangian density the term

$$\frac{1}{2}\int ds s^2 \delta(s - \alpha(\partial_\mu A_\mu(x))) = \frac{1}{2}\alpha^2 (\partial_\mu A_\mu(x))^2.$$

This gauge fixing allows us to write the path integral of the effective Lagrangian which depends from the electron and positron fields $\psi, \bar{\psi}$ as well as the electromagnetic field four vector A_μ, which are functions of the coordinates $\mathbf{x}, x_4$. Here $(\mathbf{x} = x_1, x_2, x_3)$. The term $\mu N = (\mu/e)\mathbf{J}_4$ subtracted from the Hamiltonian H acts equivalently to a shift of the vector field component $A_4 \to A_4 - i\mu/e$. This is due to the fact that

N is the net density of electric charge (the average density of particles minus antiparticles). The electromagnetic field tensor $F_{\mu\nu}$ does not change under that shift, due to gauge invariance. By introducing external field sources for the electron-positron fields, $\eta, \bar{\eta}$ and the external current interacting with the electromagnetic field J_μ we get the effective QED generating functional in quantum statistical electrodynamics. For us is of primary importance the functional Z, which serves as a generating functional of Green functions, and leads also to the partition function of the system, from which we may obtain the thermodynamic potential. We shall write

$$Z(\overline{\eta},\eta,J_\mu,\alpha) = N(\beta)\text{Det } M\int D\bar{\psi}D\psi DA_\mu \times e^{\int_\beta d^4x[L_{\text{eff}}+\bar{\eta}(x)\psi(x)+\bar{\psi}(x)\eta(x)+J^\mu(x)A_\mu(x)-\frac{\alpha^2}{2}(\partial_\mu A_\mu(x))^2]}, \tag{56}$$

where we have used the procedure of the quantum field theory, but changing some details, for instance by making the usual factor for the path integral N as a temperature dependent constant $N(\beta)$ and $J_\mu, \eta, \bar{\eta}$ playing the role of external currents, which will be taken as zero afterwards, and the fourth component of the electromagnetic field in the effective Lagrangian $L_{eff} = L_{A_4\to A_4 - i\frac{\mu}{e}}$. (Notice that μ is used frequently as a vector index, and also as a chemical potential. It is easy to distinguish, since in the second case it is always a thermodynamic variable). We shall show that after integrating on the field variables $\bar{\psi}, \psi, A_\mu$, we can write the following expression

$$Z(\overline{\eta},\eta,J_\mu,\alpha) = N(\beta)\text{Det }[-\alpha\Box^2](\text{Det } D^\alpha_{\mu\nu})^{1/2}\text{Det } G \times e^{ie\gamma_\mu\int_\beta d^4x\frac{\delta}{\delta\bar{\eta}(x)}\frac{\delta}{\delta\eta(x)}\frac{\delta}{\delta J_\mu(x)}}Z_0(\bar{\eta},\eta,J,\alpha), \tag{57}$$

and write from now on $Z_F = \text{Det }[-\alpha\Box^2](\text{Det } D^\alpha_{\mu\nu})^{1/2}$ Det G, the free term, resulting after dividing Z in Eq. (57) by $N(\beta)Z_F$ and taking the coupling constant $e = 0$. It contains the integrated terms over the field variables of the unperturbed Lagrangian,(that is, the free fields, by ignoring the terms multiplied by the coupling constant). Obviously, the exponential operator containing e and the functional derivative operators, shall act on the functional Z_0 which contains the terms due to the external perturbative currents

$$Z_0(\overline{\eta},\eta,J,\alpha) = e^{\int_\beta d^4xd^4x'\ \frac{1}{2}J_\mu(x)D^\alpha_{\mu\nu}(x-x')J_\nu(x')+\overline{\eta}(x)G(x-x')\eta(x')}. \tag{58}$$

Notice that due to the integration of a Gaussian boson field A_μ which depends linearly on the external source J_μ, it leads to a new Gaussian function dependent on the J_μ vector, and to the tensor function $D^\alpha_{\mu\nu}(x-x')$, which is the photon propagator. Also, a term containing the product of fermion fields $\psi, \bar{\psi}$ and linear terms depending on η and $\bar{\eta}$ as external sources, after integration, leads to a bilinear function on Grassmann variables, producing the spinor matrix $G(x-x')$, which is the electron-positron field propagator (the propagator is usually defined as the quantum average of the product of two time-ordered field operators, for instance, $< T\{\psi(x,t)\,\bar{\psi}(x',t')\} >$), see the Appendix A). These propagators are defined also as the solutions of the Green function equations generated by the action of their inverse operators on them, and the first (the photon one) must be combined with the (trivial in the present case) Faddeev-Popov determinant, as will be shown later. We have for the photon the inverse operator

$$D^\alpha_{\mu\nu}(x)^{-1} = \Box\delta_{\mu\nu} - \partial_\mu\partial_\nu(1-\alpha^2), \tag{59}$$

(notice that this term comes from the Lagrangian Eq. (53) last term accounting for the electromagnetic field tensor minus the gauge fixing term $[\Box\delta_{\mu\nu} - (1-\alpha^2)\partial_\mu\partial_\nu]A_\mu A_\nu$).

For the electron-positron field, the inverse operator is

$$G(x)^{-1} = \gamma_\mu\partial^*_\mu + m, \tag{60}$$

where $(\partial^*_\nu = \partial_\nu - \mu\delta_{\nu4})$ and $(x_4, x'_4) \in [0,\beta]$.

One can write the temperature Green functions equations corresponding to the previously defined operators describing free particles

$$[\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu(1-\alpha^2)]D^\alpha_{\mu\nu}(x-x') = \delta(x_4 - x'_4) \times \delta^3(\mathbf{x}-\mathbf{x}'),$$
$$[\gamma_\mu\partial^*_\mu + m]G(x-x') = \delta(x_4 - x'_4) \times \delta^3(\mathbf{x}-\mathbf{x}'). \tag{61}$$

In the case of the system being in an external field (as we shall discuss below for the magnetic field case) the unperturbed part of the Lagrangian may contain the external field term as $-\bar{\psi}[\gamma_\mu(\partial_\mu + ieA^{ext}_\mu) + m]\psi$. Then $G^{-1} = \gamma_\mu(\partial^*_\mu + ieA^{ext}_\mu) + m$. The term $J_\mu A_\mu$ is then the product of the external current J_μ by the radiation field A_μ, and all other perturbative terms refer also to it.

We write below the properties on which the previous equations for bosons are based

$$\frac{\delta}{\delta J_\mu(x)}e^{\int_\beta d^4xJ_\mu(x)\phi(x)} = \phi(x)e^{\int_\beta d^4xJ_\mu(x)\phi(x)}, \tag{62}$$

$$\left(\frac{\delta}{\delta J_\mu(x)}\right)^n e^{\int_\beta d^4xJ_\mu(x)\phi(x)} = [\phi(x)]^n \times e^{\int_\beta d^4xJ_\mu(x)\phi(x)}. \tag{63}$$

In general

$$F\left(\frac{\delta}{\delta J_\mu(x)}\right)e^{\int_\beta d^4xJ_\mu(x)\phi(x)} = F(\phi(x)) \times e^{\int_\beta d^4xJ_\mu(x)\phi(x)}, \tag{64}$$

and in particular

$$e^{-\int_\beta d^4xV(\delta/\delta J_\mu(x))} \times e^{\int_\beta d^4xJ_\mu(x)\phi(x)} = e^{-\int_\beta d^4x(V(\phi(x))-J_\mu(x)\phi(x))}. \tag{65}$$

Then we can write the generating functional for the boson part of the electromagnetic field Green functions

$$Z_B(J_\mu) = N(\beta) Z_F e^{-\int_\beta d^4x V(\delta/\delta J_\mu(x))} \times e^{-(1/2)\int_\beta J_\mu(x') D_{\mu\nu}(x',x'') J_\nu(x'') d^4x' d^4x''}, \quad (66)$$

where $D_{\mu\nu}(x', x'')$ is the photon propagator. Similar properties, taking into account that η and $\bar{\eta}$ are Grassmann variables, are valid for fermions.

4.1. Calculations of the non interacting Z term

Let us return back to use the expression Eq. (61), which after taking their Fourier transform it is easy to show that physical results are independent from α. Let us consider the boson sector in the previous expression for the Z_F term. Notice that we can write

$$(\text{Det } D^\alpha_{\mu\nu})^{1/2} = \left(\left[\left\{k_4^2 + \mathbf{k}^2\right\}^4 \alpha^2\right]^{-1}\right)^{1/2} = \left(\left[k_4^2 + \mathbf{k}^2\right]^2 \alpha\right)^{-1}. \quad (67)$$

We must deal with the quantity

$$\frac{1}{2}\text{Tr } \ln[\text{Det} D^\alpha_{\mu\nu}] = \text{Tr } \ln\left(\left[k_4^2 + \mathbf{k}^2\right]^2 \alpha\right)^{-1} = \text{Tr } \ln \frac{1}{(k_4^2 + \mathbf{k}^2)^2 \alpha}, \quad (68)$$

which contributes with a factor dependent on α. It will be canceled by the factor from $\ln(-\alpha\Box^2)$ whose Fourier expression is $\ln \alpha(k_4^2 + \mathbf{k}^2)$. Thus the sum of both logarithms cancel α and gives a term gauge independent. We have finally

$$\frac{1}{2}\text{Tr } \ln[\text{Det} D^\alpha_{\mu\nu}(k_4^2 + \mathbf{k}^2)\alpha] = \text{Tr } \ln \frac{1}{(k_4^2 + \mathbf{k}^2)}, \quad (69)$$

where Tr of a continuous determinant means the integral on the continuous variables and sum over discrete ones. In our case of temperature field theory, where the discrete variable plays an essential role, in addition to integrate over spatial variables, it is necessary to sum over discrete integers $\sum_n k_4 = \sum_n (2\pi n/\beta)$.

To obtain results corresponding to thermodynamic quantities, it is necessary to calculate the sum over n and the integral over the momenta p or k. Let us describe the method used for the sum over k_4. Let us consider the sum $(1/\beta)\sum_n F(...k_4)$ and the auxiliar functions

$$f^\pm(z) = \frac{\pm i\beta}{1 - e^{\mp i\beta z}}, \quad (70)$$

having their poles in the points $z_n = (2\pi n/\beta)$, with $n = 0, \pm 1, \pm 2...$, and residue equal to unity. We shall choose F so that the product $F(...z) f^\pm(z)$ converge in a circumference of infinite radius

$$\frac{1}{\beta}\oint F(z) f(z) dz = 0. \quad (71)$$

Equation (71) can be written as

$$\frac{1}{\beta}\sum_n F(...z = k_4) Res f^\pm + \frac{1}{\beta}\sum_k f^\pm(z_k) \times \text{Res } F(...z_k) = 0, \quad (72)$$

from which

$$\frac{1}{\beta}\sum_n F(\ldots z = k_4) = -\frac{1}{\beta}\sum_k f^\pm(z_k) \times \text{Res } F(\ldots z_k), \quad (73)$$

where z_k are the poles of F. We shall introduce a parameter mass squared λ^2 added to $\mathbf{k}^2$, and later, will put $\lambda = 0$. In Eq. (69) we have

$$\text{Tr } \ln \frac{1}{(k_4^2 + \mathbf{k}^2) + \lambda^2} = -\frac{1}{(2\pi)^3}\sum_{k_4} 2 \times \int d^3\mathbf{k} \ln(\mathbf{k}^2 + k_4^2 + \lambda^2), \quad (74)$$

By differentiating with regard to λ^2 and using the auxiliary function $f^+(z)$, we get

$$-\frac{1}{(2\pi)^3}\sum_{k_4} 2\int d^3\mathbf{k} \ln(\mathbf{k}^2 + k_4^2 + \lambda^2) = -\frac{\beta}{(2\pi)^3} 2\int \frac{d^3\mathbf{k}}{\varepsilon_\mathbf{k}}\left(\frac{e^{-\beta\varepsilon_\mathbf{k}}}{1 - e^{-\beta\varepsilon_\mathbf{k}}} + \frac{1}{2}\right). \quad (75)$$

Integrating on λ^2, and taking $\lambda = 0$ afterwards, it results

$$\ln Z_0 = -\int 2\frac{d^3\mathbf{k}}{(2\pi)^3}[\ln(1 - e^{-\beta\varepsilon_\mathbf{k}}) + \beta\varepsilon_\mathbf{k}/2]. \quad (76)$$

We replace $\varepsilon_\mathbf{k} = \omega_\mathbf{k}$ and introducing a factor β^{-1}, since the thermodynamical potential $\Omega = \beta^{-1}\ln Z$, get finally

$$\Omega_0 = -\pi^{-2}\beta^{-1}\int_0^\infty \omega^2 d\omega[\ln(1 - e^{-\beta\varepsilon_\mathbf{k}}) + \beta\varepsilon_\mathbf{k}/2], \quad (77)$$

which is the thermodynamic potential of the unperturbed system. Notice that the last term in the integrand is independent of the temperature $kT = \beta^{-1}$. This term is divergent, as it is the fermion sector we shall find below, in the limit $kT = 0, \mu = 0$. These divergent terms correspond to the quantum field theory in Euclidean variables limit. The divergences must be subtracted according to some specific prescriptions. We advice the interested reader to search for this topic in the Refs. [11,22]. Usually this term, which leads to the vacuum zero point energy, is discarded, but in presence of external fields it becomes very important.

In the present case, as it was taken $\lambda = 0$, and by taking CGS units, we have for photons $\varepsilon_k = \hbar k c$ and $k = \omega/c$, then we have

$$\Omega_0^\gamma = -kT(\pi^{-2}c^{-3}) \int_0^\infty \omega^2 d\omega \left(\ln\left[1 - e^{-\hbar\omega/kT}\right]\right)$$
$$= -\frac{(kT)^4}{3\pi^2\hbar^3 c^3} \int_0^\infty \frac{x^3 dx}{e^x - 1}, \tag{78}$$

which is the thermodynamic potential per unit volume of black body radiation. Taking into account that the entropy density is $S = -\partial\Omega/\partial T$, and $\Omega = -PV$, the energy density is $U = TS - PV = -3\Omega$.

For the calculation of $G(x - x')$ we must take into account that the sum over the fourth Fourier component p_4 is taken over odd integers. This is due to the property

$$G(x_4 = 0) = -G(x_4 = \pm\beta), \tag{79}$$

of fermion one-particle Green functions. One can define $G(x, x')$ as

$$G(x, x') = \begin{cases} \dfrac{\mathrm{Tr}\,(\rho\psi(x)\overline{\psi}(x'))}{\mathrm{Tr}\,[\rho]}, & \text{for} \quad x_4 > x_4' \\ -\dfrac{\mathrm{Tr}\,(\rho\psi(x)\overline{\psi}(x'))}{\mathrm{Tr}\,[\rho]}, & \text{for} \quad x_4 < x_4'. \end{cases} \tag{80}$$

The Eq. (80) can be written by using the spectral representation as

$$G(x, x') = \frac{1}{Z}\begin{cases} \sum_{m,n} \psi_{mn}(0)\overline{\psi}_{nm}(0) e^{-(\varepsilon_m - \mu N_m)\beta + [\varepsilon_m - \varepsilon_n + \mu(N_n - N_m)](x_4 - x_4') + i[(\mathbf{p}_m - \mathbf{p}_n)(\mathbf{x} - \mathbf{x}')]}, & \text{for} \quad x_4 > x_4' \\ -\sum_{m,n} \overline{\psi}_{nm}(0)\psi_{mn}(0) e^{-(\varepsilon_n - \mu N_n)\beta + [\varepsilon_m - \varepsilon_n + \mu(N_n - N_m)](x_4 - x_4') + i[(\mathbf{p}_m - \mathbf{p}_n)(\mathbf{x} - \mathbf{x}')]}, & \text{for} \quad x_4 < x_4' \end{cases}, \tag{81}$$

where the unity operator was inserted as the complete set $I = \sum_n |\psi_n\rangle\langle\psi_n|$.

For instance, if in the second equation in Eq. (81) we write $x_4 - x_4' = -\beta$, it coincides with the first one (in Eq. (81)) for $x_4 - x_4' = 0$, and changing its sign. From it one can write the Fourier expansion

$$G(x) = \frac{1}{\beta(2\pi)^3} \sum_{p_4} \int d^3\mathbf{p}\, G(p) e^{ip_\nu x_\nu}, \tag{82}$$

where $p_4 = (2n + 1)\pi/\beta$. Now from Eq. (60), results, by taking its Fourier transform

$$G(p) = \frac{1}{-i\gamma_\mu p_\mu^* + m} = \frac{-i\gamma_\mu p_\mu^* - m}{p^{*2} + m^2}, \tag{83}$$

where $p_\nu^* = p_\nu + i\mu\delta_{\nu 4}$. Then

$$G(x) = \begin{cases} (\gamma_\mu \partial_\mu^* - m) \int \dfrac{d^3\mathbf{p}}{(2\pi)^3 \varepsilon_p} \left([n_e(\varepsilon_p) - 1]\, e^{i\mathbf{p}\cdot\mathbf{x} - (\varepsilon_p - \mu)x_4} + n_p(\varepsilon_p) e^{-i\mathbf{p}\cdot\mathbf{x} + (\varepsilon_p + \mu)x_4}\right), & \text{for} \quad x_4 > 0 \\ (\gamma_\mu \partial_\mu^* - m) \int \dfrac{d^3\mathbf{p}}{(2\pi)^3 \varepsilon_p} \left(n_e(\varepsilon_p) e^{-i\mathbf{p}\cdot\mathbf{x} - (\varepsilon_p - \mu)x_4} + [n_p(\varepsilon_p) - 1]\, e^{i\mathbf{p}\cdot\mathbf{x} + (\varepsilon_p + \mu)x_4}\right), & \text{for} \quad x_4 < 0 \end{cases}, \tag{84}$$

where $n_{e,p}(\varepsilon_p) = (1 + e^{(\varepsilon_p \mp \mu)\beta})^{-1}$, with p the modulus of the spatial momentum vector $\mathbf{p}$.

From Eq. (83), we get

$$\frac{1}{2}\mathrm{Tr}\,\ln[\mathrm{Det}G(p)] = \mathrm{Tr}\,\ln\frac{1}{(p^{*2} + m^2)^2} = -2\mathrm{Tr}\,\ln(p^{*2} + m^2). \tag{85}$$

By differentiating with regard to m^2, we get

$$\frac{\partial}{\partial m^2}\mathrm{Tr}\,\ln[\mathrm{Det}G(p)] = -4\sum_{p_4}\int_0^\infty \frac{d^3\mathbf{p}}{(2\pi^3)}\frac{1}{(p^{*2} + m^2)}. \tag{86}$$

We shall use for fermions auxiliary functions of form $f^{\pm}(z) = \pm i\beta/1 + e^{\mp i\beta z}$, which have poles at points $z = (2n + 1)\pi/\beta$, with $n = 0, \pm 1, \pm 2...$, and residue equal to unity . By integrating over m^2 and following a procedure similar to the photon gas, we obtain finally

$$\Omega_0^{ep} = -\beta^{-1}\int_0^\infty \frac{p^2 dp}{\hbar^3 c^3 \pi^2}\left(\ln\left[1 + e^{-(\varepsilon_\mathbf{p} - \mu)\beta}\right]\left[1 + e^{-(\varepsilon_\mathbf{p} + \mu)\beta}\right] - \beta\varepsilon_\mathbf{p}\right). \tag{87}$$

The last term in the integrand, which is temperature independent, accounts for the electron-positron zero point energy, which is usually discarded. The total thermodynamic potential (in CGS units) for the non interacting photon-electron-positron system is

$$\Omega_0^\gamma + \Omega_0^{ep} = -\frac{(kT)^4}{3\pi^2\hbar^3c^3}\int_0^\infty \frac{x^3dx}{e^x-1} - \frac{kT}{\hbar^3c^3\pi^2}\int_0^\infty p^2dp \times\left(\ln\left[1+e^{-(\varepsilon_{\mathbf{p}}-\mu)/kT}\right]\left[1+e^{-(\varepsilon_{\mathbf{p}}+\mu)/kT)}\right]-\frac{\varepsilon_{\mathbf{p}}}{kT}\right). \quad (88)$$

If one introduces the variable $y = pc/kT$, by defining $\bar{\mu} = \mu/kT$ and $\bar{m} = m/kT$, the electron-positron term becomes proportional to $(kT)^4$. We want to remark that the chemical equilibrium between photons and electron-positrons guarantee that $\mu_\gamma = \mu_e + \mu_p = 0$, thus, $\mu_e = -\mu_p = 0$ in equilibrium. If we assume the limit of very high temperatures so that we can neglect fermion masses and chemical potential, one gets

$$\Omega_0^\gamma + \Omega_0^{ep} = -\frac{\pi^2k^4T^4}{45\hbar^3c^3} - \frac{7\pi^2k^4T^4}{360\hbar^3c^3} = -\frac{\pi^2k^4T^4}{24\hbar^3c^3}. \quad (89)$$

Let us neglect the temperature independent term in Eq. (87), and take the temperature-dependent part and sum it to Eq. (78). We call this sum $\Omega_0^{\gamma ep} = -P$ where P is the total pressure of the system. To get the total thermodynamic potential at temperature T, we must multiply by the volume of the system V. We get the total thermodynamic potential as $\Omega_t = V\Omega_0^{\gamma ep} = -PV$. From Ω_t we may get the entropy $S = -(\partial\Omega_t/\partial T)_\mu$, the number of particles $N = -(\partial\Omega_t/\partial\mu)_T$. Other thermodynamic quantities can be easily obtained. The temperature independendent term acounts for the so-called zero point energy of vacuum. In absence of external fields, we assume its contribution to thermodynamic quantities as unimportant, and its average contribution to momentum and energy is taken as zero.

4.2. Functional derivation of basic equations for interacting fields

We want to get also a set of equations by using functional differentiation, (valid in quantum field theory) as temperature field theory, after some simple changes, one can write

$$Z(\overline{\eta},\eta,J_\mu,\alpha) = e^{ie\mathrm{Tr}\,\gamma_\mu\int_\beta d^4x\frac{\delta}{\delta\overline{\eta}(x)}\frac{\delta}{\delta\eta(x)}\frac{\delta}{\delta J_\mu(x)}} \times Z_0(J_\mu,\overline{\eta},\eta,\alpha). \quad (90)$$

We have $Z(\overline{\eta},\eta,J_\mu,\alpha)$ as generating functional of Green functions. Let us introduce another very useful functional,

$$W(\overline{\eta},\eta,J_\mu,\alpha) = \ln Z(\overline{\eta},\eta,J_\mu,\alpha). \quad (91)$$

From it we can write the functional equations to one-particle Green functions G for the fermions and D for the bosons respectively

$$G(x,x') = \left.\frac{\delta^2 W}{\delta\overline{\eta}(x)\delta\eta(x')}\right|_{\overline{\eta}=\eta=0} = \langle T\{\psi(x)\overline{\psi}(x')\}\rangle = \frac{1}{Z}\left.\frac{\delta^2 Z}{\delta\overline{\eta}(x)\delta\eta(x')}\right|_{\overline{\eta}=\eta=0}, \quad (92)$$

$$D_{\mu\nu}(x,x') = \left.\frac{\delta^2 W}{\delta J_\mu(x)\delta J_\nu(x')}\right|_{\overline{\eta}=\eta=0} = \left.\frac{\delta^2 \ln Z}{\delta J_\mu(x)\delta J_\nu(x')}\right|_{\overline{\eta}=\eta=0} = \langle T\{A_\mu(x)A_\nu(x')\}\rangle - \langle A_\mu(x)\rangle\langle A_\nu(x')\rangle, \quad (93)$$

Let us remind that the $T\{A_\mu(x)A_\nu(x')\}$ means the chronological product of the two operators, in which earlier operators acts first, and $\langle A_\mu(x)\rangle$ is the average field.

Consider the derivative $\delta Z/\delta\overline{\eta}$. We have

$$\frac{\delta Z}{\delta\overline{\eta}} = e^{ieV\left(\frac{\delta}{\delta\overline{\eta}}\frac{\delta}{\delta\eta}\frac{\delta}{\delta J_\mu}\right)}\frac{\delta Z_0}{\delta\overline{\eta}}, \quad (94)$$

where we have written

$$V\left(\frac{\delta}{\delta\overline{\eta}}\frac{\delta}{\delta\eta}\frac{\delta}{\delta J_\mu}\right) = \mathrm{Tr}\,\gamma_\mu\int_\beta d^4x\frac{\delta}{\delta\overline{\eta}}\frac{\delta}{\delta\eta}\frac{\delta}{\delta J_\mu}.$$

Now Eq. (94) can be written as

$$\frac{\delta Z}{\delta\overline{\eta}} = \int_\beta d^4xG(x-z)\eta(z)Z(\overline{\eta},\eta,J_\mu,\alpha) = e^{ieV(\dots)}\int_\beta d^4xG(x-z)\eta(z)e^{-ieV(\dots)} \times Z(\overline{\eta},\eta,J_\mu,\alpha). \quad (95)$$

As we have $G^{-1}G(x-x') = \delta^4(x-x')$, where $G^{-1} = \gamma_\mu\delta_\mu^* + m$, after integrating (95) can be written

$$(\gamma_\mu\delta_\mu^* + m)\frac{\delta Z}{\delta\overline{\eta}} = e^{ieV(\dots)}\eta(x)e^{-ieV(\dots)}Z(\overline{\eta},\eta,J_\mu,\alpha) = \widetilde{\eta}(x)Z(\overline{\eta},\eta,J_\mu,\alpha). \quad (96)$$

By a similar calculation one can get, taking the Feynman gauge $\alpha = 0$,

$$-\Box^2_{\mu\nu}\frac{\delta Z}{\delta J_\mu}(x) = e^{ieV(\dots)}J_\nu e^{-ieV(\dots)}Z(\overline{\eta},\eta,J_\mu) = \widetilde{J_\nu}(x)Z. \quad (97)$$

Now we shall prove that

$$\widetilde{\eta}(x) = \eta(x) + ie\gamma_\mu\frac{\delta}{\delta\overline{\eta}(x)}\frac{\delta}{\delta J_\mu(x)}. \quad (98)$$

We shall start from the spinor (we shall write spinor indices in few cases below)

$$\widetilde{\eta}_s(\lambda,x) = e^{ie\lambda V}\eta_s(x)e^{-ie\lambda V}. \quad (99)$$

$$\frac{\delta\tilde{\eta}_s}{\delta\lambda} = iee^{ie\lambda V}[V, \eta_s(x)]e^{-ie\lambda V} \tag{100}$$

$$= iee^{ie\lambda V}\left[\gamma_{\mu lm}\int_\beta d^4z \frac{\delta}{\delta\overline{\eta}_l(z)}\frac{\delta}{\delta\eta_m(z)}\frac{\delta}{\delta J_\mu(z)}, \eta_s(x)\right]e^{-ie\lambda V} \tag{101}$$

$$\equiv iee^{ie\lambda V}\left[\gamma_{\mu lm}\int_\beta d^4z \frac{\delta}{\delta\overline{\eta}_l(z)}\frac{\delta}{\delta\eta_m(z)}\frac{\delta}{\delta J_\mu(z)}\eta_s(x) - \eta_s(x)\gamma_{\mu lm}\int_\beta d^4z \frac{\delta}{\delta\overline{\eta}_l(z)}\frac{\delta}{\delta\eta_m(z)}\frac{\delta}{\delta J_\mu(z)}\right]e^{-ie\lambda V}. \tag{102}$$

As $(\delta/\delta\eta_m(z))\eta_s(x) = \delta_{ms}\delta^4(x-z)$, the first (second) term in brackets gives the second (first) term below. After integrating over λ and taking $\lambda = 1$ afterwards, one gets

$$\widetilde{\eta}_s(x) = \eta_s(x) + ie\gamma_{\mu st}\frac{\delta}{\delta\overline{\eta}_t(x)}\frac{\delta}{\delta J_\mu(x)}. \tag{103}$$

After substituting in Eq. (96) we get

$$\left(\gamma_\mu\partial_\mu^* + m - ie\gamma_\mu\frac{\delta}{\delta J_\mu(x)}\right)\frac{\delta Z}{\delta\overline{\eta}(x)} = \eta(x)Z. \tag{104}$$

$$-\Box\frac{\delta Z}{\delta J_\mu(x)} = J_\mu(x)Z - ie\mathrm{Tr}\;\gamma_\mu\frac{\delta^2 Z}{\delta\overline{\eta}(x)\delta\eta(x)}, \tag{105}$$

which is the set of functional equations in statistical quantum electrodynamics, let us call

$$\langle A_\mu(x)\rangle = \frac{1}{Z}\frac{\delta Z}{\delta J_\mu(x)}\bigg|_{\eta=\overline{\eta}=0}, \tag{106}$$

we have from Eq. (105)

$$-\Box\langle A_\mu(x)\rangle = -\frac{ie}{2}\mathrm{Tr}\;\gamma_\mu$$
$$\times\left[G(\mathbf{x}, x_4; \mathbf{x}, x_4 - \epsilon) + G(\mathbf{x}, x_4; \mathbf{x}, x_4 + \epsilon)\right]_{\epsilon\to 0}. \tag{107}$$

Now, by differentiating Eq. (104) and Eq. (105) with regard to $\eta(x')$ and $J_\lambda(x')$ respectively and taking into account that

$$\frac{\delta}{\delta J_\nu(x)}ZG(x, x') = \langle A_\nu(x)\rangle ZG(x, x')$$
$$+ Z\frac{\delta G(x, x')}{\delta J_\nu(x)}, \tag{108}$$

we get

$$(\Box_\alpha\delta_{\mu\nu} - \partial_\mu\partial_\nu(1-\alpha^2))\langle A_\mu(x)\rangle = J_\mu(x) - j_\mu(x), \tag{109}$$

$$\left[\gamma_\mu\partial_\mu^* + m - ie\gamma_\mu\langle A_\mu(x)\rangle - ie\gamma_\mu\frac{\delta}{\delta J_\mu(x)}\right]$$
$$\times G(x, x') = \delta^4(x - x'), \tag{110}$$

$$\Box D_{\mu\lambda}(x, x') - ie\mathrm{Tr}\,\gamma_\mu\frac{\delta G(x, x)}{\delta J_\lambda(x')} = \delta_{\mu\lambda}\delta^4(x - x'), \tag{111}$$

where the electromagnetic four-current $j_\mu(x) = ie\mathrm{Tr}\,\gamma_\mu G(x, x)$ in Eq. (109). By integrating j_4 over x_i, $i = 1, 2, 3$ on a volume V, which is taken later $V \to \infty$, one gets the density of particles as

$$\rho_e = e\lim_{V\to\infty}V^{-1}\int d^3\mathbf{x}\mathrm{Tr}\,\gamma_4 G(x, x). \tag{112}$$

We shall define

$$-ie\gamma_\mu\frac{\delta G(x, x')}{\delta J_\mu(x)} = \int_\beta d^4y\Sigma(x, y)G(y, x'), \tag{113}$$

$$-ie\mathrm{Tr}\;\gamma_\mu\frac{\delta G(x, x)}{\delta J_\lambda(x')} = \int_\beta d^4y\Pi_{\mu\nu}(x, y)D_{\nu\lambda}(y, x'), \tag{114}$$

where $\Sigma(x, y)$ and $\Pi_{\mu\nu}(x, y)$ correspond to mass and polarization operators respectively. To understand what the quantities $\sum$ and $\Pi_{\mu\nu}$ mean, let us assume an equation (it is understood formally the integration over repeated indexes)

$$G^{-1}(y, z')G(z', z) = \delta(y - z), \tag{115}$$

then

$$\frac{\delta G^{-1}(y, z')}{\delta J_\mu(x)}G(z', z) + G^{-1}(y, z')\frac{\delta G(z', z)}{\delta J_\mu(x)} = 0, \tag{116}$$

and

$$\frac{\delta G(y', z)}{\delta J_\mu(x)} = -G(y', y)\frac{\delta G^{-1}(y, z')}{\delta J_\mu(x)}G(z', z), \tag{117}$$

writing

$$\frac{\delta G^{-1}(y, z')}{\delta J_\mu(x)} = \frac{\delta G^{-1}(y, z')}{\delta\langle A_\nu(z'')\rangle}\frac{\delta\langle A_\nu(z'')\rangle}{\delta J_\mu(x)}$$
$$= -ie\int_\beta d^4z''\Gamma_\nu(y, z', z'')D_{\nu\mu}(z'', x), \tag{118}$$

where we have defined the vertex function

$$\Gamma_\nu(y, z', z'') = -\frac{\delta G^{-1}(y, z')}{\delta ie\langle A_\nu(z'')\rangle}.$$

Now from Eq. (113), Eq. (117) and Eq. (118), we have

$$-ie\gamma_\mu \frac{\delta G(y',z)}{\delta J_\mu(y')} = e^2 \int_\beta d^4y d^4z'' d^4z' \gamma_\mu G(y',y) \times \Gamma_\nu(y,z',z'') D_{\nu\mu}(z'',x) G(z',z) = \int_\beta d^4z' \Sigma(y',z') G(z',z), \qquad (119)$$

and

$$\Sigma(y',z') = e^2 \int_\beta d^4y d^4z'' \gamma_\mu G(y',y) \times \Gamma_\nu(y,z',z'') D_{\nu\mu}(z'',y'). \qquad (120)$$

Similarly

$$-ie\mathrm{Tr}\,\gamma_\mu \frac{\delta G(x,x)}{\delta J_\lambda(x')} = e^2 \int_\beta d^4z'' d^4z' d^4y \mathrm{Tr}\,\gamma_\mu G(x,y) \times \Gamma_\eta(y,z',z'') G(z',x) D_{\eta\lambda}(z'',x') = \int d^4z'' \Pi_{\mu\eta}(x,z'') D_{\eta\lambda}(z'',x'),$$

and

$$\Pi_{\mu\eta}(x,z'') = e^2 \int d^4z' d^4y \mathrm{Tr}\,\gamma_\mu \times G(x,y) \Gamma_\eta(y,z',z'') G(z',x), \qquad (121)$$

$$\Gamma_\mu(x,y,z) = -\frac{\delta G^{-1}(x,y)}{\delta ie\langle A_\mu(z)\rangle} = \gamma_\mu \delta(x-y)\delta(x-z) - \frac{\delta\Sigma(x,y)}{\delta ie\langle A_\mu(z)\rangle}, \qquad (122)$$

so that finally one can write the set of temperature dependent Schwinger-Dyson equations as:

$$[\gamma_\mu \partial^*_\mu + m - ie\gamma_\mu \langle A_\mu(x)\rangle] G(x,x') + \int_\beta d^4z \Sigma(x,z) \times G(z,x') = \delta^3(\mathbf{x}\text{-}\,\mathbf{x}')\delta(x_4 - x_4'), \qquad (123)$$

$$\Box D_{\mu\lambda}(x,x') + \int_\beta d^4z \Pi_{\mu\eta}(x,z) D_{\eta\lambda}(z,x') = \delta_{\mu\lambda}\delta^4(x-x'). \qquad (124)$$

In a general gauge Eq. (124) looks

$$\left(\Box\delta_{\mu\nu} - \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}(1-\alpha^2)\right) D_{\mu\lambda}(x,x') + \int_\beta d^4z \Pi_{\mu\eta}(x,z) \times D_{\eta\lambda}(z,x') = \delta_{\mu\lambda}\delta^3(\mathbf{x}\text{–}\mathbf{x}')\delta(x_4 - x_4'). \qquad (125)$$

It can be shown that $\Pi_{\mu\nu}$ is gauge invariant and satisfies the transversality condition

$$k_\mu \Pi_{\mu\nu} = \Pi_{\mu\nu} k_\nu = 0. \qquad (126)$$

We shall assume a fixed gauge, and call the photon unperturbed propagator as $D^0_{\mu\nu}(x-x')$. It satisfies the equation

$$\left(\Box\delta_{\mu\nu} - \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}(1-\alpha^2)\right) D^0_{\mu\nu}(x-x') = \delta^3(\mathbf{x}\text{–}\mathbf{x}')\delta(x_4 - x_4').$$

5. Ward identities and generalization of Furry's theorem

In the p-representation and assuming an external electromagnetic field such that $< A_\mu >\neq 0$, the system of equations for the Green's functions simplifies considerably and takes the following form [11]

$$G^{-1}(p,p') = [i\gamma_\mu p_\mu - \gamma_4\mu + m]\delta(p-p') - ie\gamma_\mu\langle A_\mu(p-p')\rangle + \Sigma(p,p'), \qquad (127)$$

$$D_{\mu\nu}(k,k') = D^0_{\mu\nu}(k)\delta(k-k') - D^0_{\mu\eta}(k)\sum_{s_4}\int d^3s \Pi_{\eta\rho}(k,s) D_{\rho\nu}(s,k'), \qquad (128)$$

$$\langle A_\mu(k)\rangle = D^0_{\mu\nu}(k) J_\nu(k) - D^0_{\mu\nu}(k)\frac{ie}{(2\pi)^3\beta}\mathrm{Tr}\,\gamma\sum_{p_4}\int d^3p G(p+k,p), \qquad (129)$$

$$\Pi_{\mu\nu}(k,k') = \frac{e^2}{(2\pi)^3\beta}\mathrm{Tr}\sum\int d^3s_1 d^3s_2 d^3s \gamma_\mu G(k+s,s_1)\Gamma_\nu(s_1,s_2,k') G(s_2,s), \qquad (130)$$

$$\Sigma(p,p') = \frac{e^2}{(2\pi)^3\beta}\sum\int d^3s_1 d^3s_2 d^3s \gamma_\mu G(p+s,s_1)\Gamma_\nu(s_1,p',s_2) D_{\mu\nu}(s_2,s), \qquad (131)$$

$$\Gamma_\mu(p,p',k) = -\frac{\delta G^{-1}(p,p')}{\delta\langle ieA_\mu(k)\rangle} = \gamma_\mu\delta(p-p'-k) - \frac{\delta\Sigma(p,p')}{\delta\langle ieA_\mu(k)\rangle}. \qquad (132)$$

From Eq. (127) and Eq. (132), expanding all quantities in a perturbation-theory series, one can prove the following relation

$$G^{-1}(p-k,p') - G^{-1}(p,p'+k) = k_\mu \frac{\delta G^{-1}(p,p')}{\delta e\langle A_\mu(k)\rangle}. \quad (133)$$

The Eq. (133) is equivalent to

$$G^{-1}(p) - G^{-1}(p-k) = ik_\mu \Gamma_\mu(p, p-k, k), \quad (134)$$

$$\frac{\partial G^{-1}(p)}{\partial p_\mu} = i \lim_{\delta_\mu \to 0} \Gamma_\mu(p, p-\delta_\mu, \delta_\mu), \quad (135)$$

where δ_μ is a four-vector for which the μ-component is different from zero. The relation Eq. (135) is called a Ward identity. Taking into account that the chemical potential μ enters into $G^{-1}(p)$ linearly combined with ieA_4, one can show that the other limit for Γ, when $k_4 < |\mathbf{k}| \to 0$ coincides with $\delta G^{-1}/\delta\mu$, that is

$$-\frac{\partial G^{-1}(p)}{\partial \mu} = \Gamma_4(p,p,0). \quad (136)$$

One can also give a generalization of the well known Furry's theorem of quantum field theory. One can write the polarization operator as

$$\Pi_{\mu\nu}(k,k') = \frac{e^2}{2(2\pi)^3\beta} \mathrm{Tr}\left(\sum_{p_4} \int d^3p \gamma_\mu \frac{\delta}{\delta e\langle A_\nu(k')\rangle} \times [G(p+k,k) - G^c(p+k,k)] \right), \quad (137)$$

where G^c is the charge-conjugate of the electron Green function, which can be defined as $G^c(p,p_1|A_\nu,\mu) = G(p,p_1|-A_\nu,-\mu)$. Then,

$$\left[\frac{\delta \Pi_{\mu\nu}}{\delta\langle A(k_1)\rangle ... \delta\langle A(k_{2n+1})\rangle}\right]_{\langle A\rangle=0} \approx \mu^{2n+1}, \quad (138)$$

which vanishes for $\mu = 0$. Notice that if $\mu \neq 0$, we have $A_4 \to A_4 + \mu$ and formally the limit of $\langle A_\nu\rangle \to 0$ is now $\langle A_\nu\rangle \to \delta_{\nu 4}\mu = \mu$.

6. Dispersion equation for photons in vacuum and in a medium

Let us advance the following important fact: It can be shown that the temperature formalism we are using, when the analytic continuation $k_4 = i\omega$, $x_4 = it$ (in CGS units, $x_4 = ict$) is done, we get time-dependent equations for physical systems. Similarly, for electrons and positrons, also $p_4 \to i\varepsilon_p$ leads to the energy of the particles. This is very important in what follows.

In quantum electrodynamics we get from the Schwinger-Dyson equation the dispersion equation for the two transverse photon modes [11, 23–26] as the poles of the Green function $D_{\mu\nu}$ (we shall introduce a gauge fixing only to solve the dispersion equation), which leads to $D^{-1}_{0\mu\nu} - \Pi^R_{\mu\nu} = 0$,

$$k^2 T_{\mu\nu} - \Pi^R(k) T_{\mu\nu} = 0, \quad (139)$$

where

$$T_{\mu\nu} = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right), \quad (140)$$

is the four-dimensional transverse tensor: $k_\mu T_{\mu\nu} = 0$. In the case of a photon propagating in vacuum, the four-vector k_μ it is the only one available and $T_{\mu\nu}$ is the only transverse tensor which can be constructed. Notice that $T_{\mu\nu}T_{\nu\lambda} = T_{\mu\lambda}$, that is, it is idempotent. We have that the renormalized expression of the polarization operator in vacuum contains the scalar

$$\Pi^R(k) = \frac{e^2}{12\pi^2} k^4 \int_{\infty}^{4m^2} \frac{dz^2 \left(1 + \frac{2m^2}{z^2}\right)\sqrt{1 - \frac{4m^2}{z^2}}}{z^2(z^2+k^2)}, \quad (141)$$

which multiplied by $T_{\mu\nu}$ gives the polarization tensor in Euclidean variables. Notice that $\Pi^R(k)$ contains the contribution of virtual massive pairs whose masses take values from $4m^2 \leqslant z^2 \leqslant \infty$.

For the solution of Eq. (139), whose second term is $\Pi^R(k)$ and it is proportional to k^4, one can write $k^2(1 - \Pi^R(k)/k^2) = 0$, leading to the physical solution of the dispersion equation as $k^2 = 0$, that is, the light cone equation $\mathbf{k}^2 = \omega^2$.

In a medium, at finite temperature, there are two basic vectors, the photon four momentum k_μ and the system four-velocity u_μ. The latter gives rise to another four dimensional transverse tensor,

$$U_{\mu\nu} = \frac{k_\mu k_\nu}{k^2} - \frac{k_\mu u_\nu}{(ku)} - \frac{k_\nu u_\mu}{(ku)} + \frac{u_\mu u_\nu k^2}{(ku)^2}, \quad (142)$$

where $(uk) = u_\mu k_\mu$. Finally, the polarization operator tensor $\Pi_{\mu\nu}$ (see Fig.1) can be written in a medium (in a moving coordinate system) as

$$\Pi_{\mu\nu} = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) A + \left(\frac{k_\mu k_\nu}{k^2} - \frac{k_\mu u_\nu}{(ku)} - \frac{k_\nu u_\mu}{(ku)} + \frac{u_\mu u_\nu k^2}{(ku)^2}\right) B. \quad (143)$$

In the rest system, where the four-velocity is $u_\mu = (0,0,0,u_4)$, results

$$\Pi_{ij} = \left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right) A(\mathbf{k}^2, k_4^2) + \Pi_{44} \frac{k_i k_j}{\mathbf{k}^2} \frac{k_4^2}{\mathbf{k}}^2, \quad i,j = 1,2,3, \quad (144)$$

where $\Pi_{i4} = \Pi_{4i} = -\Pi_{44}(k_i k_4/\mathbf{k}^2)$. The expressions for A and Π_{44} are found below [11]. According to Eq. (112), Eq. (129) and Eq. (136), we have

$$\Pi_{44}(k_4=0,k\to 0)=\Pi_{44}(0)=\frac{e^2}{(2\pi)^3\beta}\mathrm{Tr}$$
$$\times\int d^3s\sum\gamma_4 G(s)\Gamma_4(s,s,0)G(s)$$
$$=-\frac{e^2}{(2\pi)^3\beta}\mathrm{Tr}\int d^3s\sum\gamma_4 G(s)\frac{\partial G^{-1}(s)}{\partial\mu}G(s)$$
$$=e^2\frac{\partial}{\partial\mu}[\frac{1}{(2\pi)^3\beta}\mathrm{Tr}\sum\int d^3s\gamma_4,G(s)]$$
$$=-e^2\frac{\partial\rho_e(0)}{\partial\mu}. \tag{145}$$

where ρ_e is the charge density. From here we can obtain an exact expression for the Debye radius λ

$$\Pi_{44}(0)=-e^2\frac{\partial\rho_e(0)}{\partial\mu}=-\lambda^{-2}. \tag{146}$$

It is interesting the limit $k_4<\mathbf{k}\to 0$ for which in the one-loop approximation (see below)

$$-\Pi_{44}=\lambda^{-2}=\frac{e^2}{3\beta^2}, \tag{147}$$

which means that the Debye radius decrease with increasing temperature.

If $T\sim 2mc^2=O(10^9)\ K$ increases, it leads also to pair creation (for instance, in neutron stars).

In the limit $k_4=0,\mathbf{k}\to 0$,

$$A=-\frac{e^2\mathbf{k}^2}{6\pi^2}\ln(m\beta). \tag{148}$$

It must be stressed that in QED at finite temperature the velocity of electromagnetic waves is smaller than in vacuum and depends from the properties of the medium.

Let us consider Eq. (121) in the one loop approximation

$$\Pi_{\mu\eta}(x,x')=e^2\int d^4y\mathrm{Tr}\,\gamma_\mu G(x,y)\gamma_\eta G(y,x'), \tag{149}$$

where y_4 is integrated in the interval $[0,\beta]$. Taking Eq. (83) into account, one gets

$$\Pi_{\mu\eta}(k)=\frac{e^2}{(2\pi)^3\beta}\sum_{p_4}\int d^3\mathbf{p}\mathrm{Tr}$$
$$\times\frac{\gamma_\mu(-i\gamma_\rho p^*_\rho+m)\gamma_\eta(-i\gamma_\rho[p^*_\rho+k_\rho]+m)}{([p^*+k]^2+m^2)(p^{*2}+m^2)}, \tag{150}$$

where $p^*_\mu=p_i$, for $i=1,2,3$ and $p^*_4=p_4+i\mu$. After taking the trace we have

$$\Pi_{\mu\eta}(k)=\frac{4e^2}{(2\pi)^3\beta}\sum_{p_4}\int d^3\mathbf{p}$$
$$\times\frac{(p^{*2}+[p^*k]+m^2)\delta_{\mu\eta}-(2p^*_\mu p^*_\eta+p^*_\eta k_\mu+k_\eta p^*_\mu)}{([p^*+k]^2+m^2)(p^{*2}+m^2)}. \tag{151}$$

After calculating the sum in p_4, we obtain

$$\Pi_{\mu\eta}(k)=\Pi^s_{\mu\eta}(k)+\Pi^0_{\mu\eta}(k). \tag{152}$$

The first term depends on the temperature T and chemical potential μ (statistical part, see Eq. (144)), where

$$\Pi^s_{44}(k)=-\frac{e^2}{\pi^2}\int_0^\infty\frac{p^2dp}{\varepsilon_p}\left(\frac{1}{1+e^{(\varepsilon_p+\mu)\beta}}+\frac{1}{1+e^{(\varepsilon_p-\mu)\beta}}\right)$$
$$\times\left(1+\frac{k^2+k_4^2-4\varepsilon_p^2}{8pk}\ln a-i\frac{k_4\varepsilon_p}{2pk}\ln b\right), \tag{153}$$

$$A(k)=-\frac{e^2}{2\pi^2}\int_0^\infty\frac{p^2dp}{\varepsilon_p}\left(\frac{1}{1+e^{(\varepsilon_p+\mu)\beta}}+\frac{1}{1+e^{(\varepsilon_p-\mu)\beta}}\right)$$
$$\times\left(1-\frac{k_4^2}{k^2}+\frac{k^4-k_4^4+4\varepsilon_p^2k_4^2+4k^2p^2}{8pk^3}\ln a\right.$$
$$\left.+i\frac{k_4\varepsilon_p}{2pk^3}(k^2+k_4^2)\ln b\right), \tag{154}$$

where $\varepsilon_p=\sqrt{p^2+m^2}$, p, k are the modulus of the spatial momenta vectors $\mathbf{p}$, $\mathbf{k}$, and

$$a=\frac{(k^2-2pk+k_4^2)^2+4\varepsilon_p^2k_4^2}{(k^2+2pk+k_4^2)^2+4\varepsilon_p^2k_4^2}, \tag{155}$$

and

$$b=\frac{(k^2+k_4^2)^2-4(pk+i\varepsilon_pk_4)^2}{(k^2+k_4^2)^2-4(pk-i\varepsilon_pk_4)^2}. \tag{156}$$

The second term in Eq. (152) is divergent, and is the term obtained in QED, when the photon frequency term $i\omega$ replaces the Euclidean variable k_4. After subtracting the divergence, we obtain the renormalized term $(\Pi^{0R}_{\mu\eta})=\Pi^RT_{\mu\eta}$ (see Eq. (139)). In a hot medium where we are close to thermodynamic equilibrium among photons and electrons plus positrons, it is valid the dispersion equation for photons propagating in the system in the form

$$D^{-1}_{\mu\nu}=\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\left(k^2-\Pi^{0R}(k)\right)$$
$$-\Pi^s_{\mu\nu}(\mathbf{k},k_4)+Ck_\mu k_\nu=0, \tag{157}$$

where C is a gauge fixing parameter, and by taking its Fourier transform, we would get a time-dependent equation. The Eq. (157) differs from the dispersion equation in vacuum essentially in the term $\Pi^s_{\mu\nu}(\mathbf{k},k_4)$. From its solution one gets three modes of propagation, two of them having transverse polarizations (which in absence of the medium correspond to the two transverse modes propagating in vacuum). In the medium their polarizations are respectively $\mathbf{p}_1$, $\mathbf{p}_2$, and their wave vectors are parallel to $\mathbf{k}_3$, it is $\mathbf{p}_1\cdot\mathbf{k}_3=\mathbf{p}_2\cdot\mathbf{k}_3=0$. The third mode, which is longitudinal (and does not exist in vacuum), has its polarization $\mathbf{p}_3$ parallel to the wave vector $\mathbf{k}_3$.

We have $\lim_{\mathbf{k}\to 0}\omega(\mathbf{k},T) \sim eT/3$. The photon in that medium is a quasi-particle, and at high temperatures it has a nonzero effective mass. It is a quasi-particle mass, and can be named plasmon mass, to which contribute both processes represented by Fig. 1. For the limit $\omega = 0$, the transverse modes are proportional to $\mathbf{k}$ as $\mathbf{k} \to 0$. The longitudinal mode leads to a nonzero but purely imaginary solution for $\mathbf{k}$, interpreted as the Debye length $\lambda = 3\hbar c/\alpha^{1/2}kT$, where α is the fine structure constant (expressed it in CGS units, remind that in most of all other formulae written in the present paper, we shall use natural units $\hbar = c = 1$ in which $e^2 = \alpha$). The Debye length accounts for the screening of longitudinal waves in a medium produced by the overlapping of fields created by the presence of particles of opposite sign. We remark here that the photon modes propagate at speeds smaller than c, which is consistent with the arising of effective masses. For the reader interested in more details about the propagation of the photon in an electron-positron background, we suggest to find it in the paper [27], by one of the present authors (H.P.R.) and L. Villegas Lelovsky, where the solutions of the dispersion equation Eq. (157) for the photon-electron-positron system are discussed in detail.

7. The effective action

Green functions are given by second functional differentiation of the functional Z with regard to the sources of the fields. One can also use the functional $W = \ln Z$. Starting from this functional W it can be defined by a Legendre transformation, the so-called *effective action* as,

$$\Gamma(\bar{\psi},\psi, A_\mu) = W(J_\mu, \bar{\eta}, \eta) - \int d^4x (J_\mu(x)A_\mu(x) + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)), \tag{158}$$

where the $\bar{\psi}, \psi, A_\mu$ fields are the quantum averages of the initial fields. The effective action is such that the functional differentiation $(\delta\Gamma/\delta A_\mu(x)) = J_\mu(x)$, gives Maxwell's equations in the presence of the current $J_\mu(x)$. Likewise, one can get the equations for the electron-positron field. Of special interest are the quantities obtained by a second functional differentiation,

$$D^{-1}_{\mu\nu}(x,x') = \frac{\delta^2\Gamma}{\delta A_\mu(x)\delta A_\nu(x')} = D^{-1}_{0\mu\nu}(x,x') - \Pi_{\mu\nu}(x,x'), \tag{159}$$

$$G^{-1}(x,x') = \frac{\delta^2\Gamma}{\delta\psi(x)\delta\bar{\psi}(x')} = G_0^{-1}(x,x') + \Sigma(x,x'), \tag{160}$$

which give the inverse Green functions, which contain the contributions from free Green functions or propagators $D^{-1}_{0\mu\nu}$, G_0^{-1} plus the vacuum polarization tensor $\Pi_{\mu\nu}$ and the electron self-energy Σ respectively. From the equations $D^{-1}_{\mu\nu}D_{\nu\lambda} = \delta_{\mu\lambda}$ and $G^{-1}G = 1$, the Schwinger-Dyson equations are obtained.

If their transforms to momentum space are equated to zero $D^{-1}_{\mu\nu}(k,k') = 0$, and $G^{-1}(p,p') = 0$ their solution leads to the spectra of particles, that is, the dispersion laws that relate their energy and momentum, for photons as well as for electrons and positrons. Taking into account the quantum corrections of higher order contained in $\Pi_{\mu\nu}$ and Σ. The equations are

$$D^{-1}_{0\mu\nu} - \Pi_{\mu\nu} = 0, \tag{161}$$

$$G_0^{-1} + \Sigma = 0. \tag{162}$$

In field theory, the first order functional differentiation of the effective action gives us the equations of motion for average fields, taking into account the quantum corrections, and the second differentiation, the dispersion equations of the particles. These equations, or similar ones, are equally valid, in the case of multi-particle systems, at low energy, as it is condensed matter at finite temperature (which a $T = 0$ is called many body theory).

8. Charged particle in a magnetic field in non-relativistic quantum mechanics

We shall study the motion of a charged particle in an external constant magnetic field. The vector momentum must be written as $\mathbf{P} = \mathbf{p} - (e/c)\mathbf{A}$, where $\mathbf{A}$ is the vector potential. For a constant magnetic field $\mathbf{B}$ along the x_3 axis, the vector potential may be taken as $\mathbf{A} = (-Bx_2, 0, 0)$ (the expression for $\mathbf{A}$ is not unique, due to gauge invariance). The Hamiltonian is [28]:

$$H = \frac{1}{2m}\left(\left[-i\hbar\frac{\partial}{\partial x_1} + \frac{e}{c}Bx_2\right]^2 + \left[-i\hbar\frac{\partial}{\partial x_2}\right]^2 + \left[-i\hbar\frac{\partial}{\partial x_3}\right]^2 \right) - \boldsymbol{\mu}\cdot\mathbf{B}, \tag{163}$$

where m is the charged particle mass, and $\boldsymbol{\mu} = g\mu_B\mathbf{S}$ is the particle intrinsec magnetic moment. Here $\mu_B = e\hbar/2mc$ is the Bohr magneton, $\mathbf{S}$ the spin given in terms of the Pauli matrices as $\mathbf{S} = \boldsymbol{\sigma}/2$, and g the Lande g-factor. For bare electrons $g = 2$.

The external magnetic field A_μ appears only in the first term of the expression Eq. (163), as the term Bx_2. By changing the gauge, it would appear also the term Bx_1, since B is taken parallel to the x_3 axis. (For instance, another gauge equivalent expression is $A_\mu = B(x_2, -x_1, 0)/2$). An important fact is the non-conservation of linear momentum orthogonal to the field. That is $[H, p_1] \neq 0$ and $[H, p_2] \neq 0$, but $[H, p_3] = 0$. The angular momentum $\mathbf{L}$ is conserved also parallel to x_3. This is due to the fact that the isotropy of space in the zero field case is broken by the external magnetic field. Only two symmetries remain, parallel to **B** and rotational around **B**. This is seen in the wavefunction.

There are several physical motivations to study the non-relativistic problem of an electron in a magnetic field. To solve the Schrödinger equation $i\partial\Psi/\partial t = H\Psi$, we must use separation of variables, $\Psi = \psi(\mathbf{x})\chi(t)$, where $\chi(t) = e^{i\varepsilon t/\hbar}$. Notice that the x_1 and x_3 coordinates do not appear in the Hamiltonian, thus, the generalized momenta p_1, p_3 are conserved. This leads us to assume a wave function of form $\psi(\mathbf{x}) = Ae^{\frac{i}{\hbar}(p_1x_1+p_3x_3)}\phi(x_2)$, where A will be chosen appropriate for the normalization of ψ. After substituting ψ in Eq. (163), we get an equation which can be written as:

$$\frac{d^2\psi}{dx_2^2} + \frac{2m}{\hbar^2}\Bigg(\varepsilon + \boldsymbol{\mu}\cdot\mathbf{B} - \frac{p_3^2}{2m} - \frac{m}{2}\left[\frac{eB}{mc}\right](x_2 - x_{20})^2\Bigg)\phi = 0, \qquad (164)$$

where $x_{20} = -(cp_1/eB)$. The Eq. (164) is similar to the Schrödinger equation for the oscillator, it can be written as:

$$\frac{d^2\psi}{dx_2^2} + \frac{2m}{\hbar^2}\left(\varepsilon' - \frac{1}{2}m\omega^2(x_2 - x_{20})^2\right)\phi = 0,$$

where

$$\varepsilon' = \varepsilon + \boldsymbol{\mu}\cdot\mathbf{B} - \frac{p_3^2}{2m}.$$

We may now write the energy eigenvalues as:

$$\varepsilon_{p_3,n} = \frac{p_3^2}{2m} + \left(n + \frac{1}{2} \mp \frac{1}{2}\right)\frac{eB\hbar}{mc}. \qquad (165)$$

We observe first of all, that the dynamics along the magnetic field is similar to that of the free particle but this does not mean that the magnetic field does not has influence on the motion of the particle parallel to it. Let us call it longitudinal energy, whereas orthogonal to the external field, the energy behaves like that of the linear oscillator, and depends linearly on the magnetic field B. Its energy eigenvalues are expressed through their quantization through the integers $n = 0, 1, 2...$, called the Landau quantum numbers. The spin contribution is proportional to the eigenvalues of the Pauli matrix σ_3, given by $\sigma_3 = \pm 1$ which we understand as implying spin projections $(+1/2)$ along, and $(-1/2)$ opposite to the field direction. It leads to a non-degenerate ground state $n = 0$, $\sigma_3 = -1$, and all other states are two-fold degenerate. Notice that $\varepsilon_{p_3,n}$ is also independent of p_1, that is, it is also degenerate with regard to $x_{20} = -cp_1/eB$, which is the orbit's center coordinate. The degeneracy term $eB/\hbar c$ leads to a number of states per Landau energy level which grows proportional to B. The wavefunctions are given by the product of the free particle wavefunction along B multiplied by the factor $\phi_n(\xi) = A_n e^{-\xi^2/2}H_n(\xi)$ where

$$\xi = (x_2 - x_{20})\sqrt{eB/\hbar c}, \quad A_n = \left[\sqrt{eB}/\sqrt{\pi\hbar c}2^n n!\right]^{1/2}$$

and H_n are Hermite polynomials. The orthonormality condition

$$\int_{-\infty}^{\infty}\phi_n(\xi)\phi_m(\xi)d\xi = \delta_{mn}$$

is satisfied. Notice that for the particle in the ground state, the wavefunction is reduced to a Gaussian whose dispersion decreases as $(eB)^{-1}$. The larger the magnetic field intensity, the smaller the dispersion of the wave function, which adopt a peaked form.

The previous problem becomes interesting in several applications, for instance, the study of the quantum Hall effect. (Hall effect is produced by an electric current propagating orthogonal to a magnetic field: another current orthogonal to the magnetic field ia created. In this case, the dimensions must be reduced to two, which are those orthogonal to the field **B**. It keeps the degeneracy term $eB/\hbar c$. Usually the orbital effective mass m^* is much smaller than the electron mass m, and this fact makes unimportant the spin degeneracy. Also, the non-relativistic case is useful for the case of small energy of the particles involved (electrons and positrons) in absence of the external field (in other words, it is $E \ll mc^2$), and small magnetic fields, that is $B \ll B_c$, where B_c is the so-called Schwinger field intensity of order 4×10^{13} Gauss. For instance, it may become useful in the study of electrons and protons coming in the Solar wind, and trapped by the Earth magnetic field. The relativistic case involve high energies of the particles and fields and it is especially important in astroparticle physics.

9. Relativistic charged particle. Dirac equation in a magnetic field

In the relativistic case, [29] we have to solve the Dirac equation in the constant magnetic field

$$[i\gamma^\mu(\partial_\mu - ieA_\mu) - m]\Psi = 0. \qquad (166)$$

The energy eigenvalues (in CGS units) are given by

$$\varepsilon_{p_3,n} = \sqrt{p_3^2c^2 + m^2c^4 + |e|B\hbar c(2n+1) - |e|sgn(e)B\hbar c\sigma_3}, \qquad (167)$$

where p_3 is the momentum along **B**, $sgn(e) = (\pm)$, with $(-)$ for electrons and $(+)$ for positrons, m is the electron mass, σ_3 are spin eigenvalues along x_3 and $n = 0, 1, ...$ are the Landau quantum numbers. These are two-fold spin degenerate, except the ground state in which $n = 0$, and for electrons it is $\sigma_3 = -1$ whereas for positrons $\sigma_3 = 1$. This means that in the ground state the spin of electrons and positrons must have also opposite directions. The expressions for the spinor wavefunctions are:

$$\psi^{\pm}_{n,p_2,p_3,\sigma_3}(\mathbf{x}) = \left(\frac{\varepsilon_{p_3,n} \pm m}{2\varepsilon_{p_3,n}}\right)^{1/2} \frac{e^{ip_2x_2+ip_3x_3}}{2\pi} \begin{pmatrix} \phi_{n-1}(\xi) \\ 0 \\ \dfrac{p_3\phi_{n-1}(\xi)}{m \pm \varepsilon_{p_3,n}} \\ \dfrac{i(2neB)^{1/2}\phi_n(\xi)}{m \pm \varepsilon_{p_3,n}} \end{pmatrix}, \tag{168}$$

$$\psi^{\pm}_{n,p_2,p_3,-\sigma_3}(\mathbf{x}) = \left(\frac{\varepsilon_{p_3,n} \pm m}{2\varepsilon_{p_3,n}}\right)^{1/2} \frac{e^{ip_2x_2+ip_3x_3}}{2\pi} \begin{pmatrix} 0 \\ \phi_n(\xi) \\ -\dfrac{i(2neB)^{1/2}\phi_{n-1}(\xi)}{m \pm \varepsilon_{p_3,n}} \\ -\dfrac{p_3\phi_n(\xi)}{m \pm \varepsilon_{p_3,n}} \end{pmatrix}. \tag{169}$$

Where upper index means particle and antiparticle, whereas the lower ones are n Landau number, p_2, p_3 are momentum components and the last is spin projections $\sigma_3 = \pm 1$. Notice that the energy of the ground state in the nonrelativistic case is $\varepsilon_{p_3,0} = p_3^2/2m$ and in the relativistic case it is $\varepsilon_{p_3,0} = \sqrt{p_3^2c^2 + m^2c^4}$. The orbital and spin contributions cancel each other, and the motion is like that of a free particle moving along a straight line parallel to **B**. This happens for electrons with $\sigma_3 = -1$, since the negative charge determines the cancelation of orbital and spin terms. On the opposite, for positrons $\sigma_3 = +1$.

10. Electron-positron temperature-dependent Green function

The time ($t = x_4$)-dependent Green function, in the Furry picture, (for instance, when strong external fields are present) in the magnetic field case is [22, 30]:

$$G(\mathbf{x}, x_4, \mathbf{x}', x_4') = \begin{cases} -i\sum_q e^{-i\varepsilon_q(x_4-x_4')}\mathfrak{G}^+(\mathbf{x}, \mathbf{x}'), & \text{for} \quad x_4 > x_4' \\ i\sum_q e^{i\varepsilon_q(x_4-x_4')}\mathfrak{G}^-(\mathbf{x}, \mathbf{x}'), & \text{for} \quad x_4 < x_4' \end{cases}, \tag{170}$$

where q denotes the set of quantum numbers (p_2, p_3, n) and $\sum_q$ indicates integration on p_2, p_3 and sum over $n = 0, 1....$ The expression for the spatial dependent electron-positron temperature Green function is:

$$\mathfrak{G}^{\pm}{}_q(\mathbf{x}, \mathbf{x}') = \sum_{\sigma_3} \psi^{\pm}_{q,\sigma_3}(\mathbf{x})\bar{\psi}^{\pm}_{q,\sigma_3}(\mathbf{x}') = \frac{e^{ip_2(x_2-x_2')+ip_3(x_3-x_3')}}{8\pi^2\varepsilon_q} M, \tag{171}$$

where

$$M = \begin{bmatrix} C_{n-1,n-1}(\varepsilon_q) & 0 & -D_{n-1,n-1} & -E_{n-1,n} \\ 0 & C_{n,n}(\varepsilon_q) & E_{n,n-1} & D_{n,n} \\ D_{n-1,n-1} & E_{n-1,n}, & C_{n-1,n-1}(-\varepsilon_q) & 0 \\ -E_{n,n-1} & D_{n,n} & 0 & C_{n,n}(-\varepsilon_q) \end{bmatrix}, \tag{172}$$

and $C_{k,k'}(\varepsilon_q) = (\varepsilon_q \pm m)\phi_k(\xi)\phi_{k'}(\xi')$, $D_{k,k'} = \pm p_3\phi_k(\xi)\phi_{k'}(\xi')$, and $E_{k,k'} = \mp i(2eBn)^{1/2}\phi_k(\xi)\phi_{k'}(\xi')$, here $\phi_m(\xi)$ are Hermite functions.

It must be understood above that $\phi_{-1}(\xi) = 0$. Taking the Fourier transform in time of Eq. (170), and making the analytical continuation $p_0 = -ip_4 + \mu$, we get, for the x_4 Fourier transform of the electron-positron Green function, the expression

$$G(-ip_4 + \mu, \mathbf{x}, \mathbf{x}') = \frac{\mathfrak{G}^+{}_q(\mathbf{x}, \mathbf{x}')}{-ip_4 + \mu - \varepsilon_q} + \frac{\mathfrak{G}^-{}_q(\mathbf{x}, \mathbf{x}')}{-ip_4 + \mu + \varepsilon_q}. \tag{173}$$

After multiplying by $e^{ip_4x_4}$ and summation over $p_4 = (2s + 1)\pi/\beta$ (s runs from $-\infty$ to $+\infty$) we have the following expression for temperature-dependent Green function

$$G(\mathbf{x}, x_4, \mathbf{x}', x_4') = \begin{cases} \sum_q [n_e - 1]e^{-i(\varepsilon_q-\mu)(x_4-x_4')}\mathfrak{G}^+{}_q(\mathbf{x}, \mathbf{x}') - n_p e^{i(\varepsilon_q+\mu)(x_4-x_4')}\mathfrak{G}^-{}_q(\mathbf{x}, \mathbf{x}'), & \text{for } x_4 > x_4' \\ \sum_q n_e e^{-i(\varepsilon_q-\mu)(x_4-x_4')}\mathfrak{G}^+{}_q(\mathbf{x}, \mathbf{x}') - [n_p - 1]e^{i(\varepsilon_q+\mu)(x_4-x_4')}\mathfrak{G}^-{}_q(\mathbf{x}, \mathbf{x}'), & \text{for } x_4 < x_4' \end{cases}, \tag{174}$$

where $n_e(\varepsilon_q) = (1 + e^{(\varepsilon_q-\mu)\beta})^{-1}$ and $n_p(\varepsilon_q) = (1 + e^{(\varepsilon_q+\mu)\beta})^{-1}$ are the mean density in momentum space of electrons and positrons respectively. At zero temperature $n_e(\varepsilon_q) = \theta(\mu - \varepsilon_q)$, and the average number of positrons vanish $n_p(\varepsilon_q) = 0$. At high temperatures $kT \simeq mc^2$ they are significant (this happens at very hot stars in regions where $T \gtrsim 10^9$ K).

Notice that the charge density ρ_e of the system electron-positron is

$$\begin{aligned}\rho_e &= e \lim_{V\to\infty} V^{-1} \int d^3\mathbf{x} \mathrm{Tr}\, \gamma_4 G(x,x)\\ &= \frac{e^2 B}{4\pi^2} \sum_0^\infty \alpha_n \int_\infty^\infty dp_3 (n_e(\varepsilon_q) - n_p(\varepsilon_q)),\end{aligned} \tag{175}$$

where $\alpha_n = 2 - \delta_{0n}$.

11. Polarization operator and wave propagation

We can study systems close to thermodynamic equilibrium, by means of time-dependent Green functions if we do an analytic continuation at finite temperature. For instance, by taking $x_4 = it$ (or $x_4 = ict$, for instance in Gaussian units), and $p_4 \to i\varepsilon$, $k_4 \to i\omega$. The general tensor structure is established in terms of the matrix which can be built from the characteristic vectors and tensors that enter in the problem. For this purpose we have used the four dimensional transversality of the tensor $\Pi_{\mu\nu}$ that results from the gauge invariance. Its property of containing the external field and the chemical potential so that the sum of their powers be even follows from the charge (C) and parity (P) symmetries assumed as valid for the underlying interaction. The unitary condition requires that the polarization operator be represented by an Hermitian matrix until the absorption is explicitly taken into account. This determines that the coefficients, with which the symmetric and antisymmetric matrices enter into the decomposition of $\Pi_{\mu\nu}$, are respectively real and imaginary. Also, the explicit introduction of the 4-velocity vector u_μ, into the density matrix allows us to conclude that the parity change of the polarization operator under the reflection of u_μ, in the rest frame, coincides with that under the charge conjugation. This eliminates some tensor structures leading us to the conclusion that antisymmetric structures in $\Pi_{\mu\nu}$ contain only two basic tensors containing odd powers of the chemical potential μ, as well as odd powers of the external field tensor $F_{\mu\nu}$ (this also agrees with the generalized Furry theorem). These antisymmetric structures are physically responsible for the appearance of the Faraday and Hall effects as well as of elliptically polarized eigenmodes. We must remark here that by taking the field **B** along the z axis, its only nonvanishing components of $F_{\mu\nu}$ are $F_{12} = -F_{21} = B$. This is valid for all reference frames moving parallel to **B**. Its dual tensor is $F^*_{\rho\lambda} = -(1/2)\epsilon_{\rho\lambda\mu\nu}F_{\mu\nu}$ where $\epsilon_{\rho\lambda\mu\nu}$ is the antisymmetric unit tensor in four dimensions. The tensor invariants are $\mathfrak{F} = -(1/4)F_{\mu\nu}F_{\mu\nu} > 0$ and $\mathfrak{G} = -(1/4)F_{\lambda\rho}F^*_{\lambda\rho} = 0$. From the structure of $\Pi_{\mu\nu}$ in a magnetized plasma, in the case of non-vanishing temperature T as well as chemical potential μ [22, 30], we can find the polarization properties of three electromagnetic eigenmodes propagating in the system [30, 31]. Under those conditions the polarization tensor may be expanded in terms of six independent transverse tensors [30]

$$\Pi_{\mu\nu} = \sum_{n=1}^{6} \pi^{(i)} \Psi^{(i)}_{\mu\nu}. \tag{176}$$

As is shown in [30], symmetry properties play an important role in quantum statistics. The theory is invariant under the simultaneous inversion of the electromagnetic four vector $A_\mu \to -A_\mu$ and the four-velocity $u_\mu \to -u_\mu$, (CPT symmetry) and the generalized Furry's theorem, reduce the number of the basic tensors from an initial set of 9 to a final set of 6. The basic tensors written in Euclidean variables are

$$\begin{aligned}
\Psi^{(1)}_{\mu\nu} &= k^2\delta_{\mu\nu} - k_\mu k_\nu,\\
\Psi^{(2)}_{\mu\nu} &= F_{\mu\lambda}k_\lambda F_{\nu\eta}k_\eta,\\
\Psi^{(3)}_{\mu\nu} &= -k^2\left(\delta_{\mu\eta} - \frac{k_\mu k_\eta}{k^2}\right) F_{\eta\lambda}F_{\lambda}\rho \left(\delta_{\rho\nu} - \frac{k_\rho k_\nu}{k^2}\right),\\
\Psi^{(4)}_{\mu\nu} &= \left(u_\mu - \frac{(uk)k_\mu}{k^2}\right)\left(u_\nu - \frac{(uk)k_\nu}{k^2}\right),\\
\Psi^{(5)}_{\mu\nu} &= (uk)(k_\mu F_{\nu\eta}k_\eta - k_\nu F_{\mu\eta}k_\eta + k^2 F_{\mu\nu}),\\
\Psi^{(6)}_{\mu\nu} &= u_\mu F_{\nu\eta}k_\eta - u_\nu F_{\mu\eta}k_\eta + (uk)F_{\mu\nu}.
\end{aligned} \tag{177}$$

As a next step, we shall introduce four mutually orthogonal basic vectors in the four dimensional space (we shall use here $x_4 \to x_0$ after the corresponding analytic continuation $k_4 = i\omega$, so that quantities are given in Minkowski space) $c^{(1)}_\mu = k^2 F^2_{\mu\nu}k_\nu - k_\mu(kF^2k)$, $c^{(2)}_\mu = F^*_{\mu\nu}k_\nu$, $c^{(3)}_\mu = F_{\mu\nu}k_\nu$, $c^{(4)}_\mu = k_\mu$ and normalize $c^{(1,3,4)}_\mu$ to unity and $c^{(2)}_\mu$ to minus unity:

$$\begin{aligned}
a^{(1)}_\mu &= c^{(1)}_\mu(-k^2(kF^2k)(kF^{*2}k))^{-1/2},\\
a^{(2)}_\mu &= -c^{(2)}_\mu(kF^{*2}k))^{-1/2},\\
a^{(3)}_\mu &= c^{(3)}_\mu(-kF^2k))^{-1/2},\\
a^{(4)}_\mu &= c^{(4)}_\mu(k^2)^{-1/2}.
\end{aligned} \tag{178}$$

We want to remark that in general $kF^2k = -B^2(k_1^2 + k_2^2) \leq 0$, and denoting $k_1^2 + k_2^2 = k_\perp^2$, we have $B^2k_\perp^2 \geq 0$. We shall also have that $kF^{*2}k$ can be both > 0 or < 0 [22]. Let us now take the vectors $a^{(i)}_\mu$, where $i = 1, 2, 3, 4$ as four potential vectors and let us find the directions of polarization vectors in terms of their components parallel and orthogonal to **B**. If **b**, b_0 are the spatial and time components of the vectors $a_\mu^{(i)}$, let us use the expressions

$$\mathbf{E}^{(i)} = -\frac{\partial \mathbf{b}^{(i)}}{\partial x_0} - \frac{\partial b_0^{(i)}}{\partial \mathbf{x}}, \tag{179}$$

$$\mathbf{H}^{(i)} = \nabla \times \mathbf{b}^{(i)}, \tag{180}$$

where $\mathbf{E}^{(i)}$ and $\mathbf{H}^{(i)}$, with $i = 1, 2, 3$ correspond to electric and magnetic fields respectively. Both expressions are assumed as multiplied by an exponential factor of form $e^{ik_\mu x_\mu}$. From equations Eqs. (178- 180), we have [31]

$$\begin{aligned}
&E^{(1)} = -\frac{\mathbf{k}_\perp}{k_\perp}\omega, \quad H^{(1)} = \left(\frac{\mathbf{k}_\perp}{k_\perp} \times \mathbf{k}_3\right) k^2 \\
&E_\perp^{(2)} = -\mathbf{k}_\perp k_3, \quad E_3^{(2)} = \frac{\mathbf{k}_3}{k_3}(k_3^2 - \omega^2), \\
&H^{(2)} = \left(\mathbf{k}_\perp \times \frac{\mathbf{k}_3}{k_3}\right)\omega \quad E^{(3)} = \left(\frac{\mathbf{k}_\perp}{k_\perp} \times \frac{\mathbf{k}_3}{k_3}\right)\omega, \\
&H_\perp^{(3)} = -\frac{\mathbf{k}_\perp}{k_\perp}k_3, \quad H_3^{(3)} = -\frac{\mathbf{k}_3}{k_3}k_\perp.
\end{aligned} \tag{181}$$

Thus, the magnetic field, breaking the symmetry of space, gives rise not only to a discrete set of basic modes of propagation, but also to a set of allowed polarizations of these modes.

11.1. The charge symmetric case

In this case, the chemical potential $\mu = 0$. This means that $\Pi_{\mu\nu}$ depends only on even in $F_{\mu\nu}$ tensors. By doing the corresponding substitutions one has

$$\begin{aligned}
\Pi_{\mu\nu} a_\nu^{(1)} &= p a_\mu^{(1)} + q a_\mu^{(2)}, \\
\Pi_{\mu\nu} a_\nu^{(2)} &= -q a_\mu^{(1)} + s a_\mu^{(2)}, \\
\Pi_{\mu\nu} a_\nu^{(3)} &= t a_\nu^{(3)}, \\
\Pi_{\mu\nu} a_\nu^{(4)} &= 0,
\end{aligned} \tag{182}$$

where the explicit expressions of p, q, s, t in terms of the coefficients $\pi^{(i)}$ (calculated explicitly in the one-loop approximation) are [22]

$$\begin{aligned}
p &= k^2\pi^{(1)} + (kF^*k)\pi^{(3)} - \frac{(uk)^2(kF^2k)}{(k\tilde{F}^*k)k^2}\pi^{(4)}, \\
q &= \frac{(uk)(uF^*k)}{(kF^*k)}\sqrt{-\frac{(kF^2k)}{k^2}}\pi^{(4)}, \\
s &= k^2\pi^{(1)} - \frac{(uF^*k)^2}{(kF^*k)}\pi^{(4)}, \\
t &= k^2\pi^{(1)} - (kF^2k)\pi^{(2)} + 2\mathfrak{F}k^2\pi^{(3)}.
\end{aligned} \tag{183}$$

It is easy to show that mode $a_\mu^{(3)}$ is such that its polarization vector $\mathbf{E}^{(3)}$ is orthogonal to the plane determined by **B** and **k**. The mode $a_\mu^{(1)}$ (to transversal propagation to **B**) is purely longitudinal and the mode $a_\mu^{(2)}$ is purely transverse. For another particular case of propagation along **B**, the mode $a_\mu^{(1)}$ is purely transversal while that of the second mode $a_\mu^{(2)}$ is longitudinal.

From the expressions Eq. (177) and Eq. (178) it follows that in terms of the basic vectors $a_\mu^{(1,2)}$ the polarization operator is a matrix with diagonal terms p, s and off-diagonal terms $q, -q$. The diagonalization of this matrix gives the following eigenvalues

$$\kappa_{1,2} = \frac{1}{2}\left(p + s \pm \sqrt{(p-s)^2 - 4q^2}\right). \tag{184}$$

According to Eq. (161), the dispersion equations can be found as the solutions of the equations $k^2 = \kappa_{1,2}$, which together to $k^2 = \kappa_3$, from Eq. (182) gives the three non-vanishing eigenvalues as depending from the four scalars p, q, s, t. We stress here that these results are approximation-independent, valid for the polarization operator tensor, which is gauge invariant ($\Pi_{\mu\nu}k_\nu = 0$).

11.2. The charge asymmetric case

To this case, $\mu \neq 0$, the two antisymmetric tensors $\Psi_{\mu\nu}^{(5)}$ and $\Psi_{\mu\nu}^{(6)}$ contribute also to the polarization operator. We have

$$\begin{aligned}
\Pi_{\mu\nu} a_\nu^{(1)} &= p a_\mu^{(1)} + q a_\mu^{(2)} + r a_\mu^{(3)}, \\
\Pi_{\mu\nu} a_\nu^{(2)} &= -q a_\mu^{(1)} + s a_\mu^{(2)} + v a_\mu^{(3)}, \\
\Pi_{\mu\nu} a_\nu^{(3)} &= -r a_\mu^{(1)} + v a_\mu^{(3)} + t a_\mu^{(3)}, \\
\Pi_{\mu\nu} a_\nu^{(4)} &= 0,
\end{aligned} \tag{185}$$

where the scalars p, q, t, s were given in Eq. (183), and the pseudoscalars r, v are:

$$\begin{aligned}
r &= -(uk)\left[\sqrt{k^2(kF^*k)}\pi^{(5)} + 2\mathfrak{F}\sqrt{\frac{k^2}{(kF^*k)}}\pi^{(6)}\right], \\
v &= -(uF^*k)\sqrt{\frac{(kF^2k)}{(kF^*k)}}\pi^{(5)}.
\end{aligned} \tag{186}$$

12. Conclusions and applications: chiral symmetry breaking and Faraday Effect

We have given an introduction to the method of quantum field theory at finite temperature and density, pointing out and discussing basic concepts and tools. We used the methods of functional differentiation and path integrals, with Grassmann and boson variables, as calculation methods. These are not unique ways, and it is instructive to compare, especially if a wide use of Feynman diagrams is done, with other methods. Starting from the density matrix basic equation and an imaginary time variable, we are able to obtain Schwinger-Dyson equations for systems of hot and dense system particles. Once established the used technique, we may consider specific problems, involving several fields and conserved quantities. For instance, in the present paper we have concentrated first on quantum electrodynamics where $U(1)$ gauge and CPT invariances are satisfied. At the end, we concentrated in the problem of an external magnetic field. The reader must understand the fact that once an specific direction of the (assumed constant) magnetic field is chosen, spatial symmetry is broken, and linear and angular momentum are conserved only parallel to the field (rotational invariance exists only around an axis parallel to the field). In the second part of the present work, to be done in next paper, we shall work in the wider scenario of the Standard Model.

In the applications of the theory established in the present paper, we start with an effect created by a small electric field, parallel to a strong magnetic field in an electron-positron medium. It produces a chiral magnetic effect in the current due to the electron-positron pairs moving parallel to the external magnetic field. If there is an imbalance of charge in the electron-positron system, another chiral effect is produced on photons propagating parallel to the external field, leading to the well-known Faraday effect.

12.1. Chiral magnetic effect

Let us remind that in a charged $e^{\pm}$ medium, for propagation along the field **B**, in addition to the two transverse modes (see (11.1.) and (11.2.)), there is a longitudinally polarized mode along **B** given by the pseudovector: $b_\mu^{(2)}(k) = ac_\mu^{(2)}$, (independently of the charge symmetric), where $c_\mu^{(2)} = R_2(F^*k)_\mu$ is a normalized pseudovector, (the normalization parameter is $R_2 = 1/Bz_1^{1/2}$, where from now on we will call $z_1 = k_3^2 - \omega^2$) [32,33]. This pseudovector does not violate the invariance CPT of the underlying theory. In other words, the electromagnetic field A_μ is a four vector, but $B_i = \epsilon_{ijk}F_{jk}$ is a pseudovector in $3D$ space. The parameter a (which has dimension of vector potential) is determined by the applied perturbative electric field. Its electric polarization vector being in the direction along **B** [31]

$$\mathbf{E}_B = E^{(2)}\mathbf{e}_B = a(k_3^2 - \omega^2)^{\frac{1}{2}}\mathbf{e}_B, \tag{187}$$

where $\mathbf{e}_B = \mathbf{B}/B$ is a unit pseudovector. The longitudinal mode is not on the light cone, that is $z_1 \neq 0$ [31]. If one consider a very small electric field acting parallel to **B**, a current is produced along the field **B**. In [33] it is shown that if a perturbative electric field $\mathbf{E} \parallel \mathbf{B}$, is applied to an electron-positron magnetized background in thermodynamic equilibrium, associated to a longitudinal pure electric mode (pseudovector mode, for which $\mathbf{E}\cdot\mathbf{B} \neq 0$), it produces an axial current leading to the breaking of the previously existing statistical chiral balance of the densities of charged particles.

An expansion of the electromagnetic current density (it depends on the two relativistic invariants: $\mathfrak{F}$ and $\mathfrak{G}$, where $\mathfrak{G} = \mathbf{B}\cdot\mathbf{E} \neq 0$ only for the mode $b_\mu^{(2)}$) in functional series of a_ν gives:

$$j_\mu(A_\mu^{ext} + a_\mu) = j_\mu(A_\mu^{ext}) + \frac{\delta j_\mu}{\delta A_\nu^{ext}} a_\nu + \ldots \ , \tag{188}$$

where the total external electromagnetic field is $A_\mu^{ext} + a_\mu$, with a_μ a small perturbative radiation field (its electric field $E \ll B$, **B** is generated by a four-potential A_μ^{ext}). Its linear term in a_ν is [34, 35]:

$$j_i = \Pi_{i\nu}a_\nu = Y_{ij}E_j, \tag{189}$$

where $E_j = i(\omega a_j - k_j a_0)$ is the electric field, with $a_4 = ia_0$ and $k_4 = i\omega$, also $j_\mu(A_\mu^{ext}) = N_0\delta_{\mu 4}$. The term $Y_{ij} = \Pi_{ij}/i\omega$ is the complex conductivity tensor. The third term in Eq. (189) comes from the second one by using the four-dimensional transversality of $\Pi_{\mu\nu}$ due to gauge invariance, $\Pi_{\mu\nu}k_\nu = 0$ [22, 30, 31, 36]. In Eq. (188) a_μ is in general a linear function of the eigenmodes $b_\mu^{(i)}$. Below we particularize to the case in which the eigenvector $a_\mu = b_\mu^{(2)}$, for which the electric field vector is parallel to $\mathbf{B}$ (notice that only terms containing odd number of $b_\mu^{(2)}$ legs in Eq. (188) lead to pseudovector terms).

Charged fermions interacting with the longitudinal mode, exchange energy by the transfer of momentum k_3, while the Landau quantum numbers remain unchanged [34]. Then we may consider the fermion interaction with the longitudinal mode as a problem in $(1 + 1)$ dimensions, which is strictly valid if we consider only the lowest Landau level (LLL). We would like to point out that the two-dimensional Dirac matrices obey the identity [37]:

$$\gamma^\mu\gamma^5 = -\epsilon^{\mu\nu}\gamma_\nu. \tag{190}$$

This implies that the axial $j_{\mu A}$ and vector j_μ currents exchange their $(0, 3)$ components according to the same relation. Thus, in the $(1 + 1)$ case, we can study the properties of the axial vector current by using results already derived for the vector current.

Now we must observe that in the linear approximation of j_i, see the Eq. (189), and taking into account the eigenvalue equation to longitudinally polarized mode $\Pi_{\mu\nu}b_\nu^{(2)} = sb_\mu^{(2)}$, one gets also:

$$j_i = \Pi_{i\nu}a_\nu = sb_i^{(2)}, \tag{191}$$

where we can write the scalar $s = c_\mu^{(2)}\Pi_{\mu\nu}c_\nu^{(2)}$, which is the eigenvalue of the photon self-energy tensor corresponding to the longitudinal mode [22, 30, 36]. The remarkable fact is that, as $b_\nu^{(2)}$ is a pseudovector, for propagation along **B** the current j_ν is also a pseudovector, which is a necessary condition for the breaking of chiral symmetry.

It is easy to find a gauge transformation (in which it is obtained $b_3^{(2)} = (k_4/z_1)E_3$) leading to $j_3 = s(k_4/z_1)E_3$ (from Eq. (191)), where $E_3 = E^{(2)}(\mathbf{e}_B \cdot \mathbf{e}_3)$. This equation is equivalent to

$$j_3 = \frac{\Pi_{33}}{k_4}E_3, \tag{192}$$

which is deduced from relation $\Pi_{33} = s(k_4^2/z_1)$, which is obtained from the expression $s = c_\mu^{(2)}\Pi_{\mu\nu}c_\nu^{(2)}$, and from the two-dimensional tranversality $\Pi_{\mu\nu}k_\nu = 0$, where $\mu,\nu = 3,4$.

We are interested only in the real part of j_3, and to obtain it, we will restrict ourselves to the imaginary part of the photon self-energy tensor, after taking $k_4 = i\omega$. From Eq. (190), Eq. (191) and by using the two-dimensional transversality condition of $\Pi_{\mu\nu}$, it is obtained:

$$k_\mu j_{\mu A} = \frac{z_1}{k_4} j_3 \neq 0, \tag{193}$$

while $k_\mu j_\mu = 0$, which expresses the conservation law for the vector current. Eq. (193) expresses the non-conservation of the two-dimensional axial current, whereas Eq. (192) puts in evidence the role of the electric field, characterizing the longitudinal pseudovector mode, in the breaking of the chiral symmetry in both the C-symmetric and non-symmetric cases, which produces an electric current along **B**. This proves that a chiral magnetic effect is produced in the frame of QED. Notice that the chiral conductivity associated to the longitudinal photons can be obtained calculating the scalar s and subsequently its imaginary part (see Appendix B). Now, as an example, we shall calculate the scalar s, similar procedure can be done to determinate the scalars p, q, t, r, v (the Hall conductivity is calculated by using the scalar r).

The x_4 Fourier transform of the polarization tensor in the one loop approximation Eq. (149) is [22, 30, 31, 35, 36]:

$$\Pi_{\nu\rho}(k_4,\mathbf{x},\mathbf{x}'|A^{ext}) = \frac{e^2}{\beta}\mathrm{Tr}\sum_{p_4}\gamma_\nu G(p_4,\mathbf{x},\mathbf{x}'|A^{ext})\gamma_\rho \times G(p_4+k_4,\mathbf{x},\mathbf{x}'|A^{ext}), \tag{194}$$

where $G(p_4,\mathbf{x},\mathbf{x}'|A^{ext})$ is given by Eq. (173), which can be written as

$$G(p_4,\mathbf{x},\mathbf{x}'|A^{ext}) = -\frac{1}{2\pi^2}\sum_{p_4}\int dp_2dp_3[(p_4+i\mu)^2+\varepsilon_q^2]^{-1} \times M e^{i[p_2(x_2-x_2')+p_3(x_3-x_3')]}, \tag{195}$$

where the matrix $M(p_3,p_4,n,\zeta,\zeta')$ is:

$$\begin{pmatrix} H_{n-1,n-1}(-ip_4+\mu) & 0 & -D_{n-1,n-1} & -E_{n-1,n} \\ 0 & H_{n,n}(-ip_4+\mu) & -E_{n-1,n} & D_{n,n} \\ D_{n-1,n-1} & -E_{n-1,n} & H_{n-1,n-1}(ip_4-\mu) & 0 \\ -E_{n-1,n} & -D_{n,n} & 0 & H_{n,n}(ip_4-\mu) \end{pmatrix} \tag{196}$$

Here

$$H_{k,k'}(ip_4-\mu) = (m+ip_4-\mu)\phi_k(\zeta)\phi_{k'}(\zeta'), \tag{197}$$

while $D_{k,k'}$, $E_{k,k'}$ were defined in Eq. (172).

On the other hand, from Eq. (185), the polarization operator can be represented by the matrix:

$$\Pi_{\mu\nu} \equiv \begin{bmatrix} p & q & r \\ -q & s & v \\ -r & v & t \end{bmatrix}. \tag{198}$$

In the case of propagation along external field **B**, from above equation, we have:

$$\Pi_{\mu\nu} \equiv \begin{bmatrix} t & 0 & r \\ 0 & s & 0 \\ -r & 0 & t \end{bmatrix}, \tag{199}$$

which is equivalent to:

$$\Pi_{\mu\nu} \equiv \begin{bmatrix} t & r & 0 \\ -r & t & 0 \\ 0 & 0 & s \end{bmatrix}. \tag{200}$$

Now, from Eq. (195), Eq. (194) and taking into account Eq. (200), we obtain the following expression to the Fourier transform of the scalar s in the one loop approximation:

$$s(k\mid A,\mu,\beta^{-1}) = \Pi_{33}(k|A,\mu,\beta^{-1}) = \frac{e^3B}{2\pi^2\beta}\sum_{p_4}\sum_{n,n'} \times \int_{-\infty}^{+\infty}\frac{dp_3C_{33,\,44}}{(p_4^{2\prime}+\varepsilon_{n,p_3}^2)[(p_4^{2\prime}+k_4)^2+\varepsilon_{n',p_3}^2]}, \tag{201}$$

where $p_4' = p_4 + i\mu$, $\varepsilon_{n,p_3}, \varepsilon_{n',p_3}$ are given by Eq. (167). The coefficients $C_{33,\,44}$ are:

$$C_{33,\,44} = [\mp p_{4'}(p_4'+k_4) \pm p_3(p_3+k_3)+m^2]F_{n,n'} + 2eB\sqrt{nn'}G_{n,n'}, \tag{202}$$

where the $\pm$ signs are taken for the first and second pairs of subindices, respectively and

$$F_{n,n'}\left(\frac{k_\perp^2}{2eB}\right) = \mid T_{n-1,n'-1}\mid^2 + \mid T_{n,n'}\mid^2, \tag{203}$$

$$G_{n,n'}\left(\frac{k_\perp^2}{2eB}\right) = \mid T_{n,n'}T_{n-1,n'-1} + T_{n,n'}T_{n-1,n'-1}\mid, \tag{204}$$

with

$$T_{n,m}(p,y) = \int e^{ipy}\phi_n(x)\phi_m(x+y)dx = \left(\frac{m!}{n!}\right)^{1/2} \times \left(-\frac{y-ip}{\sqrt{2}}\right)^{n-m} e^{-ipy-\frac{p^2+y^2}{4}} \times L_m^{n-m}\left(\frac{p^2+y^2}{2}\right), \tag{205}$$

where L_m^{n-m} are the generalized Laguerre polynomials. The sum over $\sum_{p_4}$ is done by using the Matsubara formalism [11,22,35], where we have:

$$\int_{-\infty}^{\infty} \frac{dp_4}{2\pi} \to \frac{1}{\beta}\sum_{p_4}, \quad p_4' = \frac{(2s+1)\pi}{\beta}, \quad s = 0, \pm1, \pm2, \ldots, \tag{206}$$

and the sum is done taking into account the Eq. (73) [11]. From Eq. (201) and doing the sum over p_4, we obtain the scalar s in the one loop approximation [22, 30–33, 36]:

$$s = \sum_{n,n'=0}^{\infty} \int_{+\infty}^{-\infty} \frac{dp_3}{\varepsilon_q}\left(\chi_{nn'} - \frac{[2p_3k_3 + J_{nn'}]}{D}\phi_{nn'}\right) \times (n_e(\varepsilon_q) + n_p(\varepsilon_q) - 1), \tag{207}$$

where $n_{e,p}$ are the mean density in momentum space of electrons and positrons respectively, and ε_q, with $q = (n, p_3)$, is given by Eq. (167), with $n, n' = 0, 1, 2, 3...$. Here the term -1 inside the square brackets accounts for the quantum vacuum limit ($\mu = T = 0$), while:

$$\chi_{nn'} = -\frac{e^4B^2}{2\pi^2 z_1}(n-n')F_{nn'}^{(1)}, \tag{208}$$

$$J_{nn'} = z_1 + 2eB(n'-n), \tag{209}$$

$$\phi_{nn'} = \frac{e^3B}{2\pi^2}\left(\left[\frac{2e^2B^2(n-n')^2}{z_1} + \left\{2m^2 + eB(n+n')\right\}\right]F_{nn'} + 2eB\sqrt{nn'}G_{nn'}\right), \tag{210}$$

and

$$D = 4z_1p_3^2 + 4p_3k_3J_{nn'} + J_{nn'}^2 - 4\omega^2\varepsilon_{n,0}^2, \tag{211}$$

with $\varepsilon_{n,0} = \sqrt{m^2 + 8eBn}$. Taking into account Eq. (205), the equations Eq. (203) and Eq. (204) can be written:

$$F_{nn'}(x) = \left([L_{n'-1}^{n-n'}(x)]^2 + \frac{n}{n'}[L_{n'}^{n-n'}(x)]^2\right) \times \frac{(n'-1)!}{(n-1)!}x^{n-n'}e^{-x}, \tag{212}$$

$$G_{nn'}(x) = 2\sqrt{\frac{n}{n'}}\frac{(n'-1)!}{(n-1)!}x^{n-n'} \times L_{n'-1}^{n-n'}(x)L_{n'}^{n-n'}(x)e^{-x}, \tag{213}$$

where $x = z_2/2eB$, with $z_2 = k_\perp^2 = k_1^2 + k_2^2$.

12.2. Faraday effect

Faraday effect is produced by electromagnetic waves moving parallel to a magnetic field in a charged medium, For instance, it may be a medium containing electrons, positrons and heavy ions, so that any disbalance of charge among electrons and positrons be balanced by the ionic background. The effect is a rotation of the polarization vector of the electromagnetic wave (photon), induced by the excess of charge of the electrons or positrons. Mathematically this is determined in the polarization tensor by the scalar r, which contains a term proportional to the difference of electron minus positron densities. As the scalar r is pure imaginary, let us call it $r = iI_r$. It can be written $\kappa_{1,2} = t \pm \sqrt{I_r^2}$. The associated eigenvectors can be written as

$$b_\mu^{(1,3)} = \bar{c}_\mu^{(1)} \pm ic_\mu^{(3)}, \tag{214}$$

where $\bar{c}_\mu^{(1)} = c_\mu^{(1)}/Bk^2$ and $c_\mu^{(1,3)}$ were given previously, in the paragraph after Eq. (177). One can write the equation for the photons propagating along B ($k_1 = k_2 = 0$), and polarized orthogonal to it

$$k_3^2 - \omega^2 = \kappa_{1,3}. \tag{215}$$

This equation leads to two solutions having opposite circular polarizations, and different speeds, for the light propagating parallel to **B** in the magnetized medium, induced by the electric charge imbalance. The effect means a chiral effect of photons, determined by the sign of the chemical potential μ, since I_r is an odd function of μ. This leads to the Faraday effect [35] in the magnetized electron-positron plasma, due to the fact that $\mu \neq 0$. As said earlier, the total net charge carried by electrons and positrons, is assumed as balanced by a positive charged background of ions. The total system is neutral, but it is not invariant under change conjugation. By writing $\kappa_{1,2} = t \pm I_r$, we rewrite the photon dispersion equation in the magnetized medium as

$$z_1 = k_3^2 - \omega^2 = t \pm I_r, \tag{216}$$

where,

$$r = iI_r, \qquad t = -\frac{e^3B}{4\pi^2}I_t,$$

and I_r, I_t are the integrals

$$I_r = \frac{e^3B\omega}{2\pi^2}\int_{-\infty}^{\infty} dp_3 f(p_3, k_3, B, \omega) \times (n_e(\varepsilon_p, \mu) - n_p(\varepsilon_p, \mu)), \tag{217}$$

with

$$f(p_3,z_1,B,T)=\sum_{n,n'}F^{(3)}_{n\,n'}(0)\left(\frac{z_1+2eB(n+n')}{D}\right),$$
$$D=[2p_3k_3+z_1+2eB(n'-n)]^2-4\omega^2\varepsilon_p^2,$$

and

$$I_t=\int_{-\infty}^{\infty}dp_3 g(p_3,z_1,B,T)(n_e(\varepsilon_p,\mu)+n_p(\varepsilon_p,\mu)),\quad (218)$$

with

$$g=\sum_{n,n'}\frac{F^{(2)}_{n\,n'}(0)}{\varepsilon_{n,p}}\left(1-\frac{(2p_3k_3+J_{nn'})(z_1+2eB(n+n'))}{D}\right),$$

where $n_{e,p}(\varepsilon_p)=(1+e^{(\varepsilon_p\mp\mu)\beta})^{-1}$, are the Fermi-Dirac distribution for electrons and positrons, n is the Landau quantum number, the energy levels are given by $\varepsilon_p=\sqrt{p_3^2+m^2+2neB}$ and

$$J_{nn'}=z_1+2eB(n'-n),\quad F^{(2,3)}_{n\,n'}(0)=\delta_{n,n'-1}\pm\delta_{n-1,n'}.$$

Notice that the function $f(\mu)=n_e(\mu)-n_p(\mu)$ has odd parity with regard μ. The term I_r is a scalar, thus, it is even in the electromagnetic field $F_{\mu\nu}$, but it is multiplied by the tensor $\Psi^{(5)}_{\mu\nu}$, odd in $F_{\mu\nu}$. Thus, the Faraday effect is an illustrative example of the Furry theorem.

Appendix

A. About propagators

In quantum field theory is used the term *propagator* to a function giving the probability amplitude for a particle to move from one point to other in space-time. Its Fourier transform describes its motion with some specific energy and momentum. It is also understood as the inverse of the wave operator corresponding to some field or particle, which are called (causal) Green's functions. In quantum electrodynamics it is frequently written the propagator for free fermions as G_F, which is in general a matrix in spinor space. For instance

$$(i\gamma_\mu\partial_\mu-m)\,G_F(x'-x)=I\delta^4(x'-x),\qquad (A.1)$$

where $x=(x_1,x_2,x_3,x_4(=ict))$ are the space-time coordinates in Euclidean variables and I is the unit four matrix in spinor space and

$$G_F(x'-x)=\frac{1}{2\pi^4}\int d^4p e^{ip\cdot(x'-x)}\bar{G}_F(p).\qquad (A.2)$$

We have $\bar{G}_F(p)=(\gamma_\mu\partial_\mu+m)/(p^2+m^2)$ for the propagator in momentum space.

The free photon propagator in momentum space can be written in Euclidean variables $(k_\mu=(k_1,k_2,k_3,k_4))$, as

$$D(k^2)=\frac{\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}}{k^2}.\qquad (A.3)$$

Notice that we are speaking in this subsection about quantum field theory definitions, *i.e.* x_4 is to be interpreted as imaginary time, $x_4=ict$. But as there is a parallelism of methods with relativistic quantum statistics, we extend the quantum field language to be used also in the temperature case, where x_4 is a variable defined in the real interval $[0,\beta]$, whose Fourier counterparts are either $p_4=(2n+1)kT$ for fermions and $k_4=2nkT$ for bosons, where $n=0,\pm1,...\pm\infty$. It leads to the possibility of describing thermodynamical processes. But by means of an analytic continuation of appropriate quantities, one can deal with dynamical processes, like propagation of particles and/or currents, which are not equilibrium processes, but means a small departure from it (small enough to be able to speak of an average temperature). In such cases, for photons $k_4\to i\omega$, and for electrons and positrons $p_4\to i\varepsilon_q$, where ω and ε_q are respectively their energies.

B. Calculation of $Im[s]$

The denominator D of the integral s (Eq. (207), which have singularities due to D) given by:

$$D=4z_1p_3(p_3+k_3)+z_1^2-4\omega^2\varepsilon_{n,0}^2,\qquad (B.1)$$

where $z_1=k_3^2-\omega^2$ and $\varepsilon_{n,0}^2=m^2+2enB$, it can be written in the form symmetric under the exchange $\varepsilon_q\leftrightarrow\varepsilon_{q'}$, $\omega\leftrightarrow-\omega$ [30]

$$D^{-1}=\frac{1}{8\varepsilon_{q'}\varepsilon_q\omega}\left(\frac{1}{\varepsilon_{q'}-\varepsilon_q-\omega+i\epsilon}-\frac{1}{\varepsilon_{q'}-\varepsilon_q+\omega+i\epsilon}-\frac{1}{\varepsilon_{q'}+\varepsilon_q-\omega+i\epsilon}+\frac{1}{\varepsilon_{q'}+\varepsilon_q+\omega+i\epsilon}\right),\qquad (B.2)$$

where $\varepsilon_{q'}=\sqrt{(p_3+k_3)^2+m^2+2enB}$ and $\varepsilon_q=\sqrt{p_3^2+m^2+2enB}$, with $q=(n,p_3)$. The first pair of singularities are related to excitation of particles to higher energies and the second two are connected to the pair creation. We have added an infinitesimal positive imaginary part $i\epsilon$ to ω, and by using the relation

$$\frac{1}{s-\omega-i\epsilon}=P\frac{1}{s-\omega}+i\pi\delta(s-\omega),\qquad (B.3)$$

where P corresponds to the principal value in the expression, we get for the imaginary part of D^{-1} [30]

$$ImD^{-1}=\pm\frac{\pi}{8\varepsilon_q\varepsilon_{q'}\omega}\big(\delta[\varepsilon_{q'}-\varepsilon_q\mp\omega]+\delta[\varepsilon_{q'}-\varepsilon_q\pm\omega]-\delta[\varepsilon_{q'}+\varepsilon_q\mp\omega]\big),\qquad (B.4)$$

where the $\pm$ signs applies respectively to $\omega \gtrless 0$. We can use now Eq. (B.4) to obtain the imaginary part the scalar s (Eq. (207)) according to the relation

$$\int_{-\infty}^{\infty} dp_3 f(p_3)\delta(g(p_3)) = \sum_m \frac{f(p_3^m)}{\mid g'(p_3^m) \mid}, \tag{B.5}$$

where p_3^m, with $m = (1, 2)$ are the roots of $g(p_3) = 0$. It may be easily shown that while p_3 runs within $(-\infty < p_3 < \infty)$, the denominator of the expression Eq. (207) may vanish only for real z_1 [30]. Thus, the integral in Eq. (207) represents an analytic function in the z_1 plane except possible singularities located somewhere on the real axis, which corresponds with the absorption region ($Im[\Pi_{33}]$ is responsible of absorption process for the longitudinal mode), where

$$p_3^{(1,2)} = \frac{-k_3 z_1 \pm \omega\Lambda}{2z_1}, \tag{B.6}$$

are the roots of denominator in Eq. (207) [30] and $\Lambda = \sqrt{z_1(z_1 + 4\varepsilon_{n,0}^2)}$. In our case $g(p_3) = \omega \pm (\varepsilon_{q'} \pm \varepsilon_q)$, thus

$$\omega = \varepsilon_{q'} \pm \varepsilon_q, \quad k_3 = p_3' \pm p_3, \tag{B.7}$$

and the corresponding values of the energies are given by

$$\varepsilon_r = \frac{-\omega z_1 + |k_3|\Lambda}{2z_1}, \tag{B.8}$$

$$\varepsilon_s = \frac{\omega z_1 + |k_3|\Lambda}{2z_1}, \tag{B.9}$$

where $r, s = (n, \omega, k_3)$. The $\pm$ signs in Eq. (B.7) corresponds to the pair creation (ε_s) and excitation cases (ε_r) respectively. By substituting these expressions it is easy to obtain:

$$\left|\frac{d}{dp_3}(g(p_3))\right| = \frac{\Lambda}{2\varepsilon_q^m \varepsilon_{q'}^m}. \tag{B.10}$$

In the evaluation of the integral Eq. (207) containing the second delta Eq. (B.4), the following exchange is made $p_3 + k_3 \leftrightarrow -p_3$, $n' \leftrightarrow n$.

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