# Electrostatic and magnetostatic fields of point dipoles revisited

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After reviewing how the Dirac delta contributions to the electrostatic and magnetostatic fields of a point electric dipole and a point magnetic dipole are usually introduced, we present an alternative procedure for obtaining these terms based on a regularization prescription similar to that used in the computation of the transverse and longitudinal delta functions. We think this method may be useful for the students in other analogous calculations.

*Keywords:* Point dipole fields; Dirac delta; regularization

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## 1. Introduction

A typical problem in electromagnetism is to determine the electromagnetic fields created by localized and prescribed charge and current distributions in vacuum. This task is accomplished by solving Maxwell's equations with these prescribed sources taking into account the appropriate boundary conditions, whenever they exist. For the particular cases of a static charge distribution and a stationary current distribution, the electric and magnetic fields decouple from each other in Maxwell's equations, which now take the form $\nabla \cdot \mathbf{E} = \rho/\epsilon_0,\ \nabla \times \mathbf{E} = \mathbf{0},\ \nabla \cdot \mathbf{B} = 0,\ \nabla \times \mathbf{B} = \mu_0 \mathbf{j}$. Since the divergence and rotational of both $\mathbf{E}$ and $\mathbf{B}$ are known, and the sources are localized in space, Helmholtz theorem allows us to write immediately the solution for the fields, namely, [1,2]

$$\mathbf{E} = -\nabla V \quad \text{with} \quad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{R}} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3\mathbf{r}' \,; \tag{1}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{with} \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{R}} \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3\mathbf{r}' \,, \tag{2}$$

where $\mathcal{R}$ is the region of space where the sources are different from zero. If we operate the gradient in Eq. (1) and the rotational in Eq. (2), we find the usual expressions for the electric field of an arbitrary (static) charge distribution and the magnetic field of an arbitrary (stationary) current distribution, namely, the Coulomb's law (after using the superposition principle) and the Biot-Savart law, respectively.

However, in many situations we are interested only in approximate expressions for the electrostatic and magnetostatic fields at points very far from the sources. In these cases, we do not need to evaluate exactly the previous integrals for $V(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$, and we are lead naturally to the so called multipole expansion which consists, basically, in doing a Taylor expansion of $1/|\mathbf{r} - \mathbf{r}'|$ around $\mathbf{r}' = \mathbf{0}$. Being $r'_{\max}$ the maximum value of $r' := |\mathbf{r}'|$, the condition $r'_{\max} < r$, with $r := |\mathbf{r}|$, allows us to write the power series expansion $1/|\mathbf{r} - \mathbf{r}'| = (1/r) \sum_n (r'/r)^n P_n(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')$, where $\{P_n; n = 0, 1, 2, ...\}$ are the usual Legendre polynomials and we defined $\hat{\mathbf{r}} := \mathbf{r}/r$ and $\hat{\mathbf{r}}' = \mathbf{r}'/r'$. The multipole expansion is of extreme importance as it allows us to systematically obtain a good approximation for the electric and magnetic fields far from the sources, avoiding in this way the calculation of the exact field which, sometimes, is an extremely difficult task. Moreover, for points very far from the sources, only the first terms in the multipole expansion are sufficient to get a good approximation of the field. In many situations, the first non-zero term is already a good approximation to the field. It is straightforward to show that, up to the dipole term, the multipole expansions of the electromagnetic potentials written in Eqs. (1) and (2), are given by

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{r} + \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \ + \ \cdots \,, \tag{3}$$

where

$$q = \int_{\mathcal{R}} \rho(\mathbf{r}')\, d^3\mathbf{r}'$$

is the total charge of the distribution and

$$\mathbf{p} = \int_{\mathcal{R}} \rho(\mathbf{r}')\mathbf{r}'\, d^3\mathbf{r}'$$

is the electric dipole moment of the distribution and

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3} \ + \ \cdots \,, \tag{4}$$

where

$$\mathbf{m} = \frac{1}{2} \int_{\mathcal{R}} \mathbf{r}' \times \mathbf{j}(\mathbf{r}')\, d^3\mathbf{r}'$$

is the magnetic dipole moment of the current distribution. A few comments are in order here. Although the first term in the multipole expansion of the scalar potencial $V$ is the monopole term, in many important situations the net charge of the system is zero and the dominant term becomes the dipole term, as it occurs in polar molecules, like water molecules. Dipole-Dipole interactions, if appropriately used, can explain not only the classical van der Waals forces between two polar molecules or between a polar molecule and a non-polar one, but also the quantum dispersive van der Waals force between two non-polar molecules (or two atoms like two hydrogen atoms). For an introductory discussion on these three kinds of van der Waals interactions see Taddei *et al* [3]. For the vector potential $\mathbf{A}$, note that it already starts with the dipole term. This is a direct consequence of the fact that there is no magnetic monopole in nature (up to now). Hence, the magnetic field of a current loop may be regarded at points very far from the current loop as that of a magnetic dipole. From now on, we shall be concerned only with the dipole fields.

Using relations $\mathbf{E} = -\nabla V$ and $\mathbf{B} = \nabla \times \mathbf{A}$, and the previous expressions for the dipole terms of the potentials $V$ and $\mathbf{A}$, it is not difficult to show that, for $r > r'_{\rm max}$, the electric and magnetic dipole contributions to the fields can be written, respectively, as

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_o}\left[\frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{r^3}\right]$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_o}{4\pi}\left[\frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3}\right]. \qquad (5)$$

The previous expressions become better approximations to the exact $\mathbf{E}$ and $\mathbf{B}$ fields as the distance from the point of interest to the sources increases. For the idealized situation of point dipoles, the previous expressions are exact, for $r \neq 0$, since point dipoles do not have any dimension so that any finite distance, as small as it may be, is still infinitely large compared with the size of an idealized point dipole. It is worth mentioning that the previous expressions for the dipole fields can also be obtained by making a Taylor expansion directly in the expressions for the fields, instead of in the expressions for the potentials [4].

However, for reasons of self-consistency that we shall review in the next section, the exact fields of idealized point dipoles (at origin) contain extra terms proportional to the Dirac delta function $\delta(\mathbf{r})$, as follows, [1,2]

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_o}\left[\frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{r^3} - \frac{4\pi}{3}\mathbf{p}\delta(\mathbf{r})\right];$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_o}{4\pi}\left[\frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3} + \frac{8\pi}{3}\mathbf{m}\delta(\mathbf{r})\right]. \qquad (6)$$

As the Dirac delta function vanishes outside the origin and the classical interaction is always between sources spatially separated, in pure classical electromagnetism these extra terms play no important role. Nonetheless, in quantum mechanics the Dirac delta function term finds its relevance in an extremely important phenomenon, namely, the hyperfine splitting of $s$ states of the hydrogen atom [5,6]. In atomic hydrogen, the interaction between the spin of the electron with the spin of the nucleus gives rise to a transition whose wavelength is approximately 21 cm. Indeed, for spherically symmetrical states, the only contribution to the splitting is due to the delta function term in the second equation of (6), because in these states the wave-function does not vanish at the proton position. The discovery of the 21 cm line, first observed in 1951 by Ewen and Purcell at Harvard [7], was a landmark in the history of the radio astronomy. In fact, we can say that it marked the birth of the spectral line radio astronomy widely used for spectroscopic velocity measurements. And the main reason for that is the following: this hyperfine spin-flip transition is a highly forbidden process in the sense that its mean life is approximately $10^7$ years! Therefore, invoking the Heisenberg uncertainty principle, this transition leads to a very sharp emission line, with a very small energy dispersion. As a consequence, by using the Doppler effect, this transition allows for extremely precise measurements of velocities of hydrogen atom sources, giving rise to a great variety of applications. To mention one of them, just after the first observation of the 21 cm line, the first maps of atomic hydrogen in the milky way revealed that our galaxy has a spiral structure.

Undoubtedly, it is very important for a student to have a good understanding on how these singular delta terms arise. In graduate textbooks and even in undergraduate ones, different derivations of Eqs. (6) are exposed. In this work our main purpose is to present an alternative procedure to obtain the dipole fields based on a regularization prescription which naturally leads to the expressions containing the contributions of the Dirac delta terms. This paper is organized as follows: in Sec. 2 we briefly review two common procedures found in the literature to obtain the Dirac delta terms that appear in Eqs. (6). In Sec. 3, we present an alternative procedure for obtaining the complete point dipole fields (including the delta terms). Section 4 is left for the final comments.

## 2. Usual demonstrations

When discussing the fields of point electric and magnetic dipoles, many textbooks do not include the delta terms in the field expressions [8–13]. As already mentioned, the reason for that lies in the fact that in classical electromagnetism those terms do not contribute to the interaction between different sources. However, more careful textbooks pay attention to this formal, but important aspect of the theory. In this section, for the sake of completeness, and in order to introduce the reader to some standard approaches, we briefly review the most common procedures to obtain the Dirac delta contributions to the static fields of an electric and a magnetic point dipoles. The reader who is familiarized with these approaches may skip this section without compromising the understanding of the next one.

### 2.1. A formal procedure

This first method is based on the consistency of Maxwell's equations. Very general relations involving the electric and magnetic fields are established and, as a consequence, one is forced to include the above mentioned Dirac delta contributions in the expressions for the electric and magnetic fields of point dipoles. Following Jackson's textbook, [2] let us start with the electric case and evaluate the volume integral of the electrostatic field in a spherical region $\mathcal{R}$, of radius $R$, centered at the origin and containing all charges inside. Recalling that the electrostatic field created by an arbitrary charge distribution is given by Coulomb's law (plus the superposition principle) we may write

$$\int_{\mathcal{R}} \mathbf{E}(\mathbf{r})\, d^3\mathbf{r} = \int_{\mathcal{R}} \frac{1}{4\pi\epsilon_0} \left[ \int_{\mathcal{R}} \frac{\rho(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\, d^3\mathbf{r}' \right] d^3\mathbf{r}$$

$$= -\int_{\mathcal{R}} \rho(\mathbf{r}') \left[ \int_{\mathcal{R}} \frac{(\mathbf{r}'-\mathbf{r})}{4\pi\epsilon_0|\mathbf{r}'-\mathbf{r}|^3}\, d^3\mathbf{r} \right] d^3\mathbf{r}'\,, \quad (7)$$

where in the last step we have just interchanged the order of integration and used that $\mathbf{r}-\mathbf{r}' = -(\mathbf{r}'-\mathbf{r})$. Now we notice that the term inside brackets can be identified as the electric field created by an uniformly charged sphere (centered at the origin) of unit charge density evaluated at position $\mathbf{r}'$. The electric field inside an uniformly charged sphere is well known from introductory courses in electrostatics [14] and can be obtained directly from Gauss's law. With this fact in mind, the previous equation can be written as

$$\int_{\mathcal{R}} \mathbf{E}(\mathbf{r})\, d^3\mathbf{r} = -\int_{\mathcal{R}} \rho(\mathbf{r}') \left( \frac{1}{3\epsilon_o}\mathbf{r}' \right) d^3\mathbf{r}' = -\frac{1}{3\epsilon_o}\mathbf{p}\,, \quad (8)$$

where

$$\mathbf{p} = \int_{\mathcal{R}} \rho(\mathbf{r}')\mathbf{r}'\, d^3\mathbf{r}'$$

is the electric dipole moment of the charge distribution inside the spherical region $\mathcal{R}$. Note that this result is a direct consequence of Maxwell equations and hence it must be always satisfied. The problem with the expression for the electric field of a point dipole given by the first equation in (5) is that it does not satisfy the previous equation. This can be easily checked if we write this field explicitly in cartesian coordinates, $\mathbf{E} = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}$, and integrate it in a spherical region $\mathcal{R}$ of finite radius $R$ centered at the point dipole. Choosing the cartesian axis such that $\mathbf{p} = p\hat{\mathbf{z}}$, a direct consequence of the azimuthal symmetry around the $\mathcal{O}z$ axis is that

$$\int_{\mathcal{R}} E_x\, d^3\mathbf{r} = \int_{\mathcal{R}} E_y\, d^3\mathbf{r} = 0.$$

For the $E_z$ component, writing $E_z$ in terms of the spherical coordinates $(r,\theta,\phi)$, we have

$$\int_{\mathcal{R}} d^3\mathbf{r}\, E_z = \lim_{r_0\to 0} \int_{r_0}^{R} r^2 dr \int_0^{\pi} \sin\theta d\theta \times \int_0^{2\pi} d\phi\, \frac{p}{4\pi\epsilon_0} \left( \frac{3\cos^2\theta - 1}{r^3} \right)$$

$$= \lim_{r_0\to 0} \frac{p}{2\epsilon_0} \int_{r_0}^{R} \frac{dr}{r} \int_0^{\pi} (3\cos^2\theta - 1)\sin\theta d\theta$$

$$= \lim_{r_0\to 0} \frac{p}{2\epsilon_0} \int_{r_0}^{R} \frac{dr}{r} \left( \cos^3\theta\Big|_0^{\pi} - \cos\theta\Big|_0^{\pi} \right) = 0, \quad (9)$$

where, due to the singularity at the origin, we adopted the prescription of integrating in a spherical region that excluded the origin and only after making the angular integrations we took the limit $r_0 \to 0$. Therefore, in order to recover consistency between the electric field of a point dipole with Eq. (8), we must include a term which is zero outside the source ($\mathbf{r} \neq 0$) but is such that its integral over any spherical region centered at the point dipole yields the result $(-1/3\epsilon_o)\mathbf{p}$. This is accomplished by adding the delta term $(-\mathbf{p}/3\epsilon_0)\delta(\mathbf{r})$ to the right hand side of the first equation given by (5).

A similar argument can be made for the magnetic field generated by a point magnetic dipole, but we now evaluate the volume integral of the magnetostatic field in a spherical region $\mathcal{R}$, of radius $R$, centered at the origin and containing all currents inside. Recalling that the magnetostatic field created by an arbitrary stationary current distribution is given by Biot-Savart law, we may write

$$\int_{\mathcal{R}} \mathbf{B}(\mathbf{r})\, d^3\mathbf{r} = \int_{\mathcal{R}} \frac{\mu_o}{4\pi} \left[ \int \frac{\mathbf{j}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\, d^3\mathbf{r}' \right] d^3\mathbf{r},$$

$$= -\int \mathbf{j}(\mathbf{r}') \left[ \int_{\mathcal{R}} \frac{\mu_0(\mathbf{r}'-\mathbf{r})}{4\pi|\mathbf{r}'-\mathbf{r}|^3}\, d^3\mathbf{r} \right] d^3\mathbf{r}' \quad (10)$$

where, as in the previous case, we have interchanged the order of integration and used that $\mathbf{r}-\mathbf{r}' = -(\mathbf{r}'-\mathbf{r})$. Analogously to the electric case, note that the term inside brackets can be identified with the electric field created by an uniformly charged sphere (centered at the origin) of unit charge density evaluated at position $\mathbf{r}'$ multiplied by $\mu_0\epsilon_0$. Proceeding as in the previous case, we obtain

$$\int_{\mathcal{R}} \mathbf{B}(\mathbf{r})\, d^3\mathbf{r} = -\int \mathbf{j}(\mathbf{r}') \times \left( \frac{\mu_o}{3}\mathbf{r}' \right) d^3\mathbf{r}' = \frac{2\mu_o}{3}\mathbf{m}\,, \quad (11)$$

where

$$\mathbf{m} = \frac{1}{2}\int_{\mathcal{R}} \mathbf{r}' \times \mathbf{j}(\mathbf{r}')\ d^3\mathbf{r}'$$

is the magnetic dipole moment of the current distribution inside the spherical region $\mathcal{R}$. Using arguments completely analogous to those employed in the previous case, it can be shown that if we use the magnetic field of a point dipole given by the second equation in (5), we will get

$$\int_{\mathcal{R}} \mathbf{B}(\mathbf{r})\ d^3\mathbf{r} = \mathbf{0},$$

in contradiction to the previous result. In order to recover consistency with Eq. (11), we must add to the magnetic dipole field given by (5) the term $(2\mu_o\mathbf{m}/3)\delta(\mathbf{r})$.

### 2.2. A real dipole model

Another way of introducing the Dirac delta terms is by considering a model for a real dipole, that is, to choose a finite source which creates outside the source exactly the field expressions written in Eq. (5). Then we shrink the source appropriately by taking the limit where its size tends to zero, but keeping fixed the total dipole moment of the source [1,6]. For the electric case, one of the simplest models of a real electric dipole is to consider a uniformly polarized sphere of radius $a$ and polarization $\mathbf{P}$. It is a standard exercise in electrostatic to show that the electric field outside the sphere is exactly that given by the first expression of (5), while the electric field inside the sphere is given by

$$\mathbf{E}_{in}(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{a^3}, \qquad (r < a)\,, \tag{12}$$

where $\mathbf{p} = (4/3)\pi a^3\mathbf{P}$ is the electric dipole moment of the sphere. In order to describe a point dipole with electric dipole moment $\mathbf{p}$ we must take the limit where $a \to 0$, but keeping the product $\left((4/3)\pi a^3\right)(\mathbf{P})$ constant and equal to $\mathbf{p}$. Note that, adopting this procedure, the electric field inside the sphere tends to infinity. However, if we calculate the volume integral of the electric field over the region $\mathcal{R}_a$ occupied by the sphere we obtain the finite desired value,

$$\int_{\mathcal{R}_a} \mathbf{E}_{in}(\mathbf{r})\ d^3\mathbf{r} = -\frac{\mathbf{p}}{3\epsilon_0}, \tag{13}$$

in agreement with Eq. (8). As a consequence, we can write the exact expression for the electric field in all space by simply adding the delta term $-(\mathbf{p}/3\epsilon_0)\delta(\mathbf{r})$ to the first equation written in (5).

For the delta term appearing in the expression of the magnetic field of a point magnetic dipole, as written in the second equation of (5), one can follow a complete analogous procedure, but this time considering as a model of a real magnetic dipole a sphere with a uniform magnetization. For a detailed discussion of how to obtain the delta terms for the point dipole electric and magnetic fields using real dipoles described in this section, see the nice paper by Griffiths [6].

## 3. Alternative demonstration

It is well known that in many formal mathematical manipulations the result of an operation is, in fact, a distribution instead of a usual function. One of the most frequent examples of such a thing occurs when we attribute to $\nabla^2(1/r)$ the Dirac delta term $-4\pi\delta(\mathbf{r})$. This result can be demonstrated in many different ways [16, 17]. The most common, but probably the least rigorous one, consists in two steps, namely: first one shows explicitly that for $r \neq 0$, $\nabla^2(1/r) = 0$ and then, using the fact that $\nabla^2(1/r) = \nabla\cdot\nabla(1/r)$, as well as Gauss theorem, one shows that $\int \nabla^2(1/r)\ d^3\mathbf{r} = -4\pi$. A more rigorous demonstration can be accomplished by adopting a regularization prescription. Instead of working with $1/r$, which is singular at the origin, one adopts a regularization prescription and works with the regularized expression $1/\sqrt{r^2+\epsilon^2}$ which is well behaved at the origin, for $\epsilon > 0$. At the end of calculations we must remove the regularization parameter $\epsilon$, so that we shall take the limit $\epsilon \to 0$, as it is usually done in this kind of procedure. In fact, it is straightforward to show that $\nabla^2(1/\sqrt{r^2+\epsilon^2})$ is, in fact, a delta sequence. As $1/r$ is the limit of our regularized function as $\epsilon \to 0$, the demonstration is complete. The reader can find generalizations for regularizations without spherical symmetry for the second-order partial derivatives of $1/r$ in paper by Hnizdo [16].

Motivated by the previous discussion, we shall employ the same kind of approach in order to obtain the complete fields (including the delta terms) of point electric and magnetic dipoles. With this goal, we start by writing the scalar potential (3) and the vector potential (4) for point dipoles in the convenient forms

$$V(\mathbf{r}) = -\frac{1}{4\pi\epsilon_o}\mathbf{p}\cdot\nabla\left(\frac{1}{r}\right), \tag{14}$$

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_o}{4\pi}\mathbf{m}\times\nabla\left(\frac{1}{r}\right). \tag{15}$$

For the sake of clarity, we shall discuss the two cases separately in the following subsections.

### 3.1. Point electric dipole

Adopting the previous regularization prescription means to define the regularized scalar potential, $V_{\rm reg}(\mathbf{r},\epsilon)$, through the transformation

$$V(\mathbf{r}) = -\frac{1}{4\pi\epsilon_o}\mathbf{p}\cdot\nabla\left(\frac{1}{r}\right) \longrightarrow \quad V_{\rm reg}(\mathbf{r},\epsilon) = -\frac{1}{4\pi\epsilon_o}\mathbf{p}\cdot\nabla\left(\frac{1}{\sqrt{r^2+\epsilon^2}}\right). \tag{16}$$

Recalling the gradient of a spherical symmetric function is simply given by $\nabla f(r) = (df/dr)\hat{\mathbf{r}}$, the regularized scalar potential takes the form

$$V_{\rm reg}(\mathbf{r},\epsilon) = \frac{1}{4\pi\epsilon_o}\left[\frac{\mathbf{p}\cdot\mathbf{r}}{(r^2+\epsilon^2)^{\frac{3}{2}}}\right]. \tag{17}$$

The regularized electric field is then given by

$$\begin{aligned}\mathbf{E}_{\text{reg}}(\mathbf{r},\epsilon) &= -\frac{1}{4\pi\epsilon_o}\nabla\left[\frac{\mathbf{p}\cdot\mathbf{r}}{(r^2+\epsilon^2)^{\frac{3}{2}}}\right] \\ &= -\frac{\hat{\mathbf{e}}_{\mathbf{i}}\, p_j}{4\pi\epsilon_o}\,\frac{\partial}{\partial x_i}\left[\frac{x_j}{(r^2+\epsilon^2)^{\frac{3}{2}}}\right] \\ &= \frac{1}{4\pi\epsilon_o}\left[\frac{3(\mathbf{p}\cdot\mathbf{r})\mathbf{r}}{(r^2+\epsilon^2)^{\frac{5}{2}}} - \frac{\mathbf{p}}{(r^2+\epsilon^2)^{\frac{3}{2}}}\right] \\ &= \frac{1}{4\pi\epsilon_o}\left[\frac{3(\mathbf{p}\cdot\mathbf{r})\mathbf{r}-r^2\mathbf{p}}{(r^2+\epsilon^2)^{\frac{5}{2}}} - \frac{\epsilon^2\mathbf{p}}{(r^2+\epsilon^2)^{\frac{5}{2}}}\right], \qquad (18)\end{aligned}$$

where in the second line of the previous equation we wrote $\mathbf{r}$ in a cartesian basis and used Einstein convention of implicit sum over repeated indexes. We now need to remove the regularization parameter, namely, we must take $\epsilon \to 0$. Doing that in the first term on the right hand side of the previous equation leads to the usual term for the electric dipole field of a point dipole, given by the first equation of (5). Regarding the last term on the r.h.s. of the previous equation, it is shown in the Appendix that

$$\frac{\epsilon^2}{(r^2+\epsilon^2)^{\frac{5}{2}}} = \frac{4\pi}{3}\delta_\epsilon(\mathbf{r})\,, \qquad (19)$$

where $\delta_\epsilon(\mathbf{r})$ is a delta sequence. Hence, from Eqs. (18) and (19) we obtain for the regularized electric dipole field

$$\mathbf{E}_{\text{reg}}(\mathbf{r},\epsilon) = \frac{1}{4\pi\epsilon_o}\left[\frac{3(\mathbf{p}\cdot\mathbf{r})\mathbf{r}-r^2\mathbf{p}}{(r^2+\epsilon^2)^{\frac{5}{2}}} - \frac{4\pi}{3}\mathbf{p}\delta_\epsilon(\mathbf{r})\right]. \qquad (20)$$

Therefore, taking $\epsilon \to 0$ in the previous equation and keeping in mind that $\lim_{\epsilon\to 0}\delta_\epsilon(\mathbf{r}) = \delta(\mathbf{r})$, we finally obtain

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_o}\left[\frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}}{r^3} - \frac{4\pi}{3}\mathbf{p}\delta(\mathbf{r})\right], \qquad (21)$$

in agreement with the literature [1, 2, 6, 15].

### 3.2. Point magnetic dipole

Adopting the same regularization prescription as before, we define the regularized vector potential, $\mathbf{A}_{\text{reg}}(\mathbf{r},\epsilon)$, through the transformation

$$\begin{aligned}\mathbf{A}(\mathbf{r}) = -\frac{\mu_o}{4\pi}\mathbf{m}\times\nabla\left(\frac{1}{r}\right) \quad &\longrightarrow \quad \mathbf{A}_{\text{reg}}(\mathbf{r},\epsilon) \\ &= -\frac{\mu_o}{4\pi}\mathbf{m}\times\nabla\left(\frac{1}{\sqrt{r^2+\epsilon^2}}\right). \qquad (22)\end{aligned}$$

Operating the gradient in the previous equation and recalling that the regularized magnetic field is given by $\mathbf{B}_{\text{reg}}(\mathbf{r},\epsilon) = \nabla\times\mathbf{A}_{\text{reg}}(\mathbf{r},\epsilon)$, we obtain

$$\begin{aligned}\mathbf{B}_{\text{reg}}(\mathbf{r},\epsilon) &= \frac{\mu_o}{4\pi}\nabla\times\left[\frac{\mathbf{m}\times\mathbf{r}}{(r^2+\epsilon^2)^{\frac{3}{2}}}\right] \\ &= \frac{\mu_o m_\ell\hat{\mathbf{e}}_k}{4\pi}\epsilon_{ijk}\epsilon_{\ell nj}\frac{\partial}{\partial x_i}\left[\frac{x_n}{(r^2+\epsilon^2)^{\frac{3}{2}}}\right] \\ &= \frac{\mu_o}{4\pi}\left[\frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r}-r^2\mathbf{m}}{(r^2+\epsilon^2)^{5/2}} + \frac{2\epsilon^2\mathbf{m}}{(r^2+\epsilon^2)^{5/2}}\right], \qquad (23)\end{aligned}$$

where Einstein convention was also employed, $\epsilon_{ijk}$ is the Levi-Civita symbol and we used the identity $\epsilon_{ijk}\epsilon_{\ell nj} = \delta_{k\ell}\delta_{in} - \delta_{kn}\delta_{i\ell}$. Identifying again the delta sequence written in (19) and removing the regularization parameter as before, we get

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_o}{4\pi}\left[\frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r}-r^2\mathbf{m}}{(r^2+\epsilon^2)^{5/2}} + \frac{8\pi}{3}\mathbf{m}\delta(\mathbf{r})\right], \qquad (24)$$

in agreement with the literature [1, 2, 6, 15].

## 4. Final comments

In this work we presented an alternative way of obtaining the complete expressions for the electrostatic and magnetostatic fields of point dipoles, including the contributions of the Dirac delta terms. It should be mentioned that the same expressions were obtained by Frahm [17] using an approach that did not resort to the use of a regularization procedure, however longer and involving non-trivial arguments for a reader less familiar with the subject.

Our method consisted, basically, in adopting a regularization prescription to remove the singularity at $r = 0$ of $1/r$. More specifically, we changed $1/r$ to $1/\sqrt{r^2+\epsilon^2}$ in the expressions for the scalar and vector potentials and after calculating the electric and magnetic fields we removed the regularization parameter ($\epsilon \to 0$).

This kind of procedure may be useful in other calculations like, for instance, in the computation of the so called transverse and longitudinal delta functions, denoted by $\delta^{\perp}_{ij}(\mathbf{r})$ and $\delta^{||}_{ij}(\mathbf{r})$ respectively, which appear naturally in the discussion of the Helmholtz theorem. This theorem states, essentially, that any field vector $\mathbf{F}(\mathbf{r})$ in three space dimensions under appropriate boundary conditions at infinity can be written as the sum of a divergenceless part $\mathbf{F}^{\perp}(\mathbf{r})$ plus an irrotational part $\mathbf{F}^{||}(\mathbf{r})$, namely, $\mathbf{F}(\mathbf{r}) = \mathbf{F}^{\perp}(\mathbf{r}) + \mathbf{F}^{||}(\mathbf{r})$. The above delta functions are defined such that $F^{\perp}_i(\mathbf{r}) = \int d^3\mathbf{r}\;\delta^{\perp}_{ij}(\mathbf{r}-\mathbf{r}')F_j(\mathbf{r}')$ and $F^{||}_i(\mathbf{r}) = \int d^3\mathbf{r}\;\delta^{||}_{ij}(\mathbf{r}-\mathbf{r}')F_j(\mathbf{r}')$ (for a rapid introduction to these delta functions see Milonni's book [18] and for a more detailed discussion, see the book by Cohen and collaborators [19]). In order to obtain the explicit expressions for $\delta^{\perp}_{ij}(\mathbf{r})$ and $\delta^{||}_{ij}(\mathbf{r})$ one is faced with the calculation of the singular term $\partial_i\partial_j(1/r)$ [18, 19] and a possible way to circumvent this problem is to adopt precisely the same regularization prescription as the one used in this work, namely, $\partial_i\partial_j(1/r) \to \partial_i\partial_j(1/\sqrt{r^2+\epsilon^2})$.

Let us finish this final section by mentioning briefly another possible application of the above regularization prescription. It is common in the literature to use the vector potential $\mathbf{A}(\mathbf{r}) = (\Phi/2\pi)(\hat{\mathbf{z}}\times\mathbf{r}/|\hat{\mathbf{z}}\times\mathbf{r}|^2)$ to describe a magnetic field given by $\mathbf{B}(x,y,z) = \Phi\delta(x)\delta(y)\hat{\mathbf{z}}$. The usual demonstration consists in two steps: first, one shows explicitly that $\nabla \times \mathbf{A} = \mathbf{0}$, for $s \neq 0$, where $(s, \phi, z)$ are the cylindrical coordinates. Then, one computes the magnetic flux through a small disc enclosing the $\mathcal{O}z$ axis with the aid of Stokes theorem, namely, $\int \mathbf{B} \cdot \hat{\mathbf{n}}\, dS = \oint \mathbf{A} \cdot d\mathbf{l}$. However, it is not allowed to employ Stokes theorem here because $\mathbf{A}(\mathbf{r})$ is not defined along the $\mathcal{O}z$ axis. Again, in order to avoid this problem, we can use a regularization prescription similar to that used in this work. We just make

$$\mathbf{A}(\mathbf{r}) \longrightarrow \mathbf{A}^{(\mathrm{reg})}(\mathbf{r}, \epsilon) = A_\phi^{(\mathrm{reg})}(s, \epsilon)\hat{\phi},$$

where

$$A_\phi^{(\mathrm{reg})}(s, \epsilon) = \frac{\Phi}{2\pi}\frac{1}{\sqrt{s^2+\epsilon^2}}.$$

Calculating explicitly $\mathbf{B}^{(\mathrm{reg})}(\mathbf{r}, \epsilon) = \nabla \times \mathbf{A}^{(\mathrm{reg})}(\mathbf{r}, \epsilon)$, identifying appropriately a delta sequence and removing at the end the regularization parameter ($\epsilon \to 0$), one obtains the desired result, $\mathbf{B}(x, y, z) = \Phi\delta(x)\delta(y)\hat{\mathbf{z}}$. We leave for the interested reader to do this demonstration.

## Appendix

### A. Demonstration of $(\epsilon^2/(r^2+\epsilon^2)^{5/2}) = (4\pi/3)\delta_\epsilon(\mathbf{r})$

A delta sequence $\delta_a(\mathbf{r})$ is defined by [20]

$$\lim_{a\to 0} \int_{\mathcal{R}} \delta_a(\mathbf{r})\, d^3\mathbf{r} = 1 \qquad \text{and}$$

$$\lim_{a\to 0} \delta_a(\mathbf{r}) = 0 \qquad \text{if} \qquad \mathbf{r} \neq 0, \tag{A.1}$$

where $\mathcal{R}$ is all space or, at least, a finite region containing the origin. Let us then show that

$$\delta_\epsilon(\mathbf{r}) = \frac{1}{4\pi}\frac{3\epsilon^2}{(r^2+\epsilon^2)^{5/2}}$$

is, indeed, a delta sequence. First, note that, for $r \neq 0$, we have

$$\lim_{\epsilon\to 0} \delta_\epsilon(\mathbf{r}) = \lim_{\epsilon\to 0} \frac{3}{4\pi}\frac{\epsilon^2}{(r^2+\epsilon^2)^{\frac{5}{2}}} = 0\,.$$

In order to check the second property of a delta sequence, let us integrate $\delta_\epsilon(\mathbf{r})$ in all space and after that take the limit $\epsilon \to 0$,

$$\lim_{\epsilon\to 0} \int_{\mathcal{R}} \delta_\epsilon(\mathbf{r})\, d^3\mathbf{r} = \lim_{\epsilon\to 0} \int_0^\infty \frac{1}{4\pi}\frac{3\epsilon^2}{(r^2+\epsilon^2)^{\frac{5}{2}}}\; r^2 dr \oint d\Omega$$

$$= \lim_{\epsilon\to 0} 3\epsilon^2 \int_0^\infty \frac{r^2}{(r^2+\epsilon^2)^{5/2}}\; dr$$

$$= \lim_{\epsilon\to 0} \frac{r^3}{(r^2+\epsilon^2)^{3/2}}\bigg|_0^\infty = 1.$$

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