Green function for the Grad-Shafranov operator

J. Julio E. Herrera-Velázquez

Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México,
Ciudad Universitaria, Alcaldía Coyoacán Apartado Postal 70-543, 04511 Ciudad de México, México.
e-mail: herrera@nucleares.unam.mx

Received 13 September 2021; accepted 12 October 2021

The Grad-Shafranov equation, often written in cylindrical coordinates, is an elliptic partial differential equation in two dimensions. It describes magnetohydrodynamic equilibria in axisymmetric toroidal plasmas, such as tokamaks, and yields the poloidal magnetic flux function, which is related to the azimuthal component of the vector potential for the magnetic field produced by a circular (toroidal) current density. The Green function for the differential operator can be obtained from the vector potential for the magnetic field of a circular current loop, which is a typical problem in magnetostatics. The purpose of the paper is to collect results scattered in electrodynamics and plasma physics textbooks for the benefit of students in the field, as well as attracting the attention of a wider audience, in the context of electrodynamics and partial differential equations.

Keywords: Magnetohydrodynamic equilibrium; Grad-Shafranov equation; tokamaks; magnetostatics.

DOI: https://doi.org/10.31349/RevMexFisE.19.010211

1. Introduction

The problem of finding the magnetic field of a circular current loop is presented in electrodynamics textbooks such as those by Landau and Lifschitz [1], Jackson [2], and Greiner [3], as an example of the application of the vector potential in magnetostatics. Due to the symmetry of the problem, only the azimuthal component is necessary. This component, on the other hand, is related to the magnetic field flux through the circular surface, which defines magnetic field surfaces around the loop, as explained in plasma physics textbooks, like those of Bellan [4], and Freidberg [5]. On the other hand, the Grad-Shafranov equation arises from finding axisymmetric magnetohydrodynamic equilibria, like those found in Tokamaks, and other toroidal confinement devices, such as the Reversed Field Pinch, the Reversed Field Configuration, the Spheromak, and the Levitated Dipole [6]. It describes the balance of the magnetostatic force $\mathbf{j}\times\mathbf{B}$ and the plasma pressure force ∇p, where $\mathbf{j}$ is the current density within the plasma and $\mathbf{B}$ is the magnetic induction field. Finding these equilibria can be reduced to the solution of a scalar elliptic partial differential equation in two dimensions known as the Grad-Shafranov equation [4,5]. The solution of its free boundary problem requires the knowledge of the Green's function related to the differential operator of the equation, which will be called the Grad-Shafranov operator. While it is often given, it is not derived in fusion books, such as those by Ariola and Pironti [7], and Jardin [8]. The purpose of the paper is to collect these scattered results for the benefit of plasma physics and magnetized fusion students, as well as to bring them to the attention of a wider audience, in the context of magnetostatics and partial differential equations. The paper is organized as follows: The derivation of the vector potential for the circular loop current, in terms of complete elliptic integrals, and its relation with the magnetic flux function is reviewed in Sec. 2. The Grad-Shafranov differential operator is defined in Sec. 3, and its Green's function is obtained from the result in the previous section. In order to understand the context in which such results are relevant for fusion research, the derivation of the Grad-Shafranov equation is completed in Sec. 4, and some concluding remarks are made in in Sec. 5.

2. Magnetic field for the circular current loop

We start by reviewing the derivation of the vector potential for the circular loop current, following Refs. [1–3]. Then we establish the relationship between the vector potential and the magnetic flux across the surface defined by the loop, following Ref. [7].

The equations for a time independent magnetic induction field $\mathbf{B}$ are given by the Gauss' law for its divergence and Ampère's law, for its curl as

$$\nabla\cdot\mathbf{B}=0 \quad , \tag{1a}$$

$$\nabla\times\mathbf{B}=\mu_0\mathbf{j} \quad , \tag{1b}$$

where μ_0 is the vacuum permeability. From Eq. (1a), we have that the magnetic induction field can be written in terms of the rotational of a vector potential $\mathbf{A}$ as $\mathbf{B}=\nabla\times\mathbf{A}$, where following Helmholtz theorem [9]

$$\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV' \,, \tag{2}$$

in which $\mathbf{r}$ is the vector from the origin to the observation point and $\mathbf{r}'$ the vector from the origin to a given source element $\mathbf{j}(\mathbf{r}')$. The relationship between the vector potential and the the magnetic flux ψ across the surface, defined by a circular current loop is established in this section, and a representation for the magnetic field in terms of it flux will be given.

The circular loop current can be expressed in terms of cylindrical coordinates (ρ, φ, z), as shown in Fig. (1), such that the direction along the φ angle, usually called azimuthal, will be called toroidal, and that perpendicular to it, with constant φ, poloidal. Due to the axisymmetry of the problem $\partial/\partial\varphi = 0$, and $\mathbf{j} = j_\varphi \hat{\mathbf{e}}_\varphi$. The vector potential is therefore given from Ampère's law, Eq. (1b), as

$$\nabla^2 \mathbf{A} = -\mu_0 j_\varphi \hat{\mathbf{e}_\varphi}, \tag{3}$$

using Coulomb's gauge $\nabla \cdot \mathbf{A} = 0$. Thus, $\mathbf{A} = A_\varphi \hat{\mathbf{e}}_\varphi$. The toroidal (azimuthal) symmetry implies that $\partial/\partial\varphi = 0$, so the components of the induction field are

$$B_\rho = -\frac{\partial A_\phi}{\partial z}, \quad B_\varphi = 0, \quad B_z = \frac{1}{\rho}\frac{\partial(\rho A_\varphi)}{\partial \rho}. \tag{4}$$

2.1. The vector potential for the circular current loop in terms of elliptic integrals

The circle defining the current loop can be described in the rectangular coordinates by

$$x = a\cos\varphi, \quad y = a\sin\varphi, \tag{5}$$

so the components of the current density j_φ along the circumference are

$$j_x = -j_\varphi \sin\varphi, \quad j_y = j_\varphi \cos\varphi\,, \tag{6}$$

Due to the azimuthal symmetry, without loss of generality, one can take the vector potential at $\varphi = 0$, where $j_x = 0$, $j_y = j_\varphi \cos\varphi$. The objective is to find $A_\varphi(\rho, z)$ on the plane (x, z). Therefore the current density for a loop of radius a, lying on the X, Y plane, with a current I can be expressed as

$$\mathbf{j}(\mathbf{r}') = j_\varphi \hat{\mathbf{e}_\varphi} = I\delta(\rho' - a)\delta(z')\cos\varphi' \hat{\mathbf{e}_\varphi}\,. \tag{7}$$

In cylindrical coordinates

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{\rho^2 + z^2 + \rho'^2 + z'^2 - 2\rho\rho'\cos(\varphi - \varphi')}\,. \tag{8}$$

Substituting Eqs. (7) and (8) in (2), with $\varphi = 0$,

$$A_\varphi = \frac{\mu_0 I}{4\pi}\int \frac{\rho' d\rho' d\varphi' dz' \delta(\rho' - a)\delta(z')\cos\varphi'}{\sqrt{\rho^2 + z^2 + \rho'^2 + z'^2 - 2\rho\rho'\cos\varphi'}}\,, \tag{9}$$

which reduces to

$$A_\varphi = \frac{\mu_0 I a}{4\pi}\int_0^{2\pi} \frac{d\varphi' \cos\varphi'}{\sqrt{a^2 + \rho^2 + z^2 - 2a\rho\cos\varphi'}}\,. \tag{10}$$

Using the change of variable $\psi = (\pi - \varphi)/2, \quad \cos\varphi' = 2\sin^2\psi - 1$, and A_φ can be written in terms of complete elliptic integrals. First, as

$$A_\varphi = \frac{\mu_0 I a}{4\pi}\frac{4}{\sqrt{(a+\rho)^2 + z^2}}\int_0^{\pi/2} \frac{2\sin^2\psi - 1}{\sqrt{1 - k^2\sin^2\psi}}d\psi\,, \tag{11}$$

where

$$k^2 \equiv \frac{4a\rho}{(a+\rho)^2 + z^2}, \quad 0 < k^2 < 1\,, \tag{12}$$

and finally as

$$A_\varphi = \frac{\mu_0 I a}{2\pi}\sqrt{\frac{a}{\rho}}\left[\frac{(2 - k^2)K(k) - 2E(k)}{k}\right]\,, \tag{13}$$

where $K(k)$ and $E(k)$ are the elliptic integrals of first and second type respectively, defined by

$$K(k) = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - k^2\sin^2\psi}},$$

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2\sin^2\psi}\,d\psi \quad . \tag{14}$$

The components of the magnetic field can be obtained from Eq. (4) using the relations

$$\frac{\partial K}{\partial k} = \frac{E}{k(1-k^2)} - \frac{K}{k}, \qquad \frac{\partial E}{\partial k} = \frac{E - K}{k}, \tag{15}$$

or directly form Amperè's law

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\oint \frac{d\mathbf{l} \times \mathbf{r}}{r^3}\,, \tag{16}$$

to obtain

$$B_\rho = \frac{\mu_0 I z}{2\pi\rho}\sqrt{\frac{k^2}{4a\rho}}\left[\frac{2 - k^2}{2 - 2k^2}E - K\right]\,, \tag{17a}$$

$$B_\varphi = 0\,, \tag{17b}$$

$$B_z = \frac{\mu_0 I}{2\pi\rho}\sqrt{\frac{k^2}{4a\rho}}\left[\rho K + \frac{ak^2 - (2 - k^2)\rho}{2 - 2k^2}E\right]\,, \tag{17c}$$

as shown in Ref. [10].

2.2. Relation between the vector potential and the magnetic flux

Let us consider the circular current loop of radius ρ as described in the previous section, and shown in Fig. 1.

The poloidal flux, ψ_p is defined by the integral of the z component of the magnetic field that crosses the circular surface S, and is given by

$$\psi_p(\mathbf{r}) = \int_0^{\rho}\int_0^{2\pi} B_z(s, z)\,s\,ds\,d\varphi$$

$$= 2\pi\int_0^{\rho} B_z(s, z)\,s\,ds = 2\pi\psi\,, \tag{18}$$

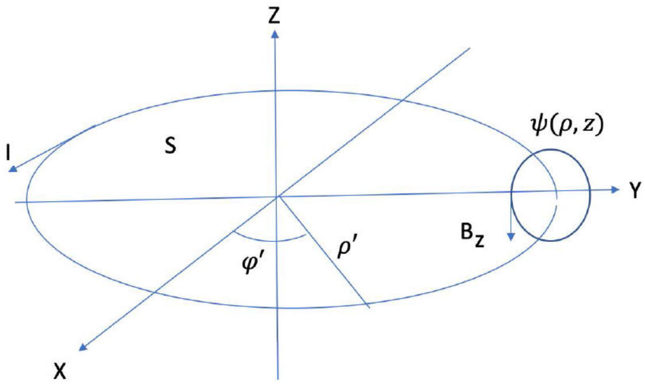

FIGURE 1. A circular current loop lying on the X, Y plane ($Z = 0$) enclosing the surface S within the circle. In section II the contour has a radius $\rho' = a$. The current I induces a magnetic field $\mathbf{B}$, whose component B_z crosses the surface S. The magnetic flux function $\psi(\rho, z)$ describes the magnetic surfaces around the current loop.

On the other hand, from Gauss law for the magnetic field, in the case of azimuthal symmetry (axisymmetry), which means $\partial/\partial\varphi = 0$,

$$\nabla \cdot \mathbf{B} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho) + \frac{\partial B_z}{\partial z} = 0\,, \tag{19}$$

which yields

$$\rho\frac{\partial B_z}{\partial z} = -\frac{\partial}{\partial\rho}(\rho B_\rho)\,. \tag{20}$$

Therefore, from Eqs. (18) and (20),

$$\frac{\partial\psi}{\partial\rho} = \rho B_z, \quad \frac{\partial\psi}{\partial z} = -\rho B_\rho\,. \tag{21}$$

Comparing Eqs. (4) and (21) it can be seen that

$$B_\rho = -\frac{1}{\rho}\frac{\partial\psi}{\partial z}, \quad B_z = \frac{1}{\rho}\frac{\partial\psi}{\partial\rho}, \quad \psi = \rho A_\varphi\,. \tag{22}$$

Thus, the vector potential can be written in terms of ψ, which is proportional to the poloidal flux function ψ_p, modulus a 2π factor, as $A_\varphi = \psi/\rho$. Therefore, the poloidal magnetic field can be written as

$$\mathbf{B}_p = -\frac{\partial\psi}{\partial z}\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho}\hat{\mathbf{e}}_z = \nabla\times(\psi\nabla\varphi) = \nabla\psi\times\nabla\varphi\,. \tag{23}$$

This equation shows that ψ defines a stream function which describes nested surfaces around the circular current loop. The orthogonal coordinate system $(\nabla\psi, \nabla\varphi, \nabla\psi\times\nabla\varphi)$ can be defined, where $\nabla\varphi = (1/\rho)\hat{\mathbf{e}}_\varphi$. In this sense, the stream function ψ plays the role of a "radial coordinate" from the magnetic axis, defined by the circular current loop, and the magnetic surface.

3. The Grad-Shafranov operator and its Green's function

Ampère's law, Eq. (1b), can be expressed in terms of ψ for the circular current density loop as

$$\mu_0 j_\varphi \hat{\mathbf{e}}_\varphi = \nabla\times\nabla\times(\psi\nabla\varphi) = -\triangle^*\psi\nabla\varphi\,, \tag{24}$$

where the Grad-Shafranov elliptic operator $\triangle^*\psi$ is defined as

$$\triangle^*\psi(\mathbf{r}) \equiv \rho^2\nabla\cdot\left[\rho^{-2}\nabla\psi\right] = \rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}\right) + \frac{\partial^2\psi}{\partial z^2} = -\mu_0\rho j_\varphi\,, \tag{25}$$

in which $\mathbf{r} = (\rho, z)$. (Observe this expression is equivalent to Eq. (3), and can be derived from it when $\mathbf{A} = (\psi/\rho)\hat{\mathbf{e}}_\varphi$. It must be remembered for that purpose that $(\triangle^2\mathbf{A})_\varphi = \triangle^2 A_\varphi - A_\varphi/\rho^2$.)

Since j_φ may be in general an extended source, Eq. (25) can be solved by means of the Green's function $G(\mathbf{r};\mathbf{r}')$ of the $\triangle^*$ operator, which is simply the solution for a circular filament loop of radius ρ_0:

$$\triangle^* G(\mathbf{r};\mathbf{r}') = \rho^2\nabla\cdot\left[\rho^{-2}\nabla G(\mathbf{r};\mathbf{r}')\right] = \mu_0\rho\delta(\mathbf{r}-\mathbf{r}')\,. \tag{26}$$

From Eqs. (25) and (26) the following Green's identity can be derived

$$\frac{1}{\rho^2}\psi\triangle^* G(\mathbf{r};\mathbf{r}') - \frac{1}{\rho^2}G(\mathbf{r};\mathbf{r}')\triangle^*\psi = \nabla\cdot\left[\psi\frac{1}{\rho^2}\nabla G(\mathbf{r};\mathbf{r}') - G(\mathbf{r};\mathbf{r}')\frac{1}{\rho^2}\nabla\psi\right]. \tag{27}$$

Substituting Eqs. (25) and (26) in the left hand side of Eq. (27), integrating over the cross section of the current in the (ρ, z) plane, and using the divergence theorem to turn the integral on the right hand side into a contour integral, the solution for the function $\psi(\mathbf{r})$ can be written as

$$\psi(\rho, z) = \int G(\mathbf{r};\mathbf{r}')j_\varphi(\mathbf{r}')dS - \frac{1}{\mu_0}\oint \times\left[\frac{\psi(\mathbf{r}')}{\rho'^2}\frac{\partial}{\partial n'}G(\mathbf{r};\mathbf{r}') - \frac{G(\mathbf{r};\mathbf{r}')}{\rho'^2}\frac{\partial}{\partial n'}\psi(\mathbf{r}')\right]dl\,, \tag{28}$$

where the contour integral includes the boundary conditions, and in the absence of them can be set to naught.

Since $\psi = A_\varphi\rho$, it's now clear that the Green's function for Eq. (26) is given by the toroidal (azimuthal) component of the vector potential A_φ of the circular current loop. From Eq. (13) it can be expressed as

$$G(\mathbf{r};\mathbf{r}') = \frac{\mu_0 I}{2\pi}\sqrt{\rho\rho'}\left[\frac{(2-k^2)K(k) - 2E(k)}{k}\right], \tag{29}$$

where a in Eq. (13) has been replaced by ρ'. This is the expression for $G(\mathbf{r};\mathbf{r}')$ given by Refs. [7,8].

In practice, the boundary conditions in Eq. (28) may be replaced by the contributions of external coils around the main current density $\mathbf{j}$, which in the case of magnetic confinement is that of the plasma. The role of such external coils would be to control the shape and position of the plasma column.

4. The Grad-Shafranov equation

While in devices like the Reversed Field Configuration and the Levitated Dipole, the poloidal magnetic field is sufficient for confinement, in most cases, such as the Tokamak, Reversed Field Pinch and Spheromak, a toroidal (azimuthal) magnetic field is also required. From Ampère's law $\nabla \cdot \mathbf{j} = 0$, so it is possible to treat the current density in an analogous way as the magnetic field in Sec. 2. From this fact, and the assumption of axisymmetry it follows from Ampère's law, in analogy to Eqs. (22), that there exists a stream function χ such that

$$j_\rho = -\frac{1}{\rho}\frac{\partial \chi}{\partial z} = -\frac{1}{\mu_0}\frac{\partial B_\varphi}{\partial z},$$
$$j_z = \frac{1}{\rho}\frac{\partial \chi}{\partial \rho} = \frac{1}{\mu_0 \rho}\frac{\partial}{\partial \rho}(\rho B_\varphi). \quad (30)$$

Additionally,

$$j_\varphi = \frac{1}{\mu_0}\left(\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial \rho}\right) = -\mu_0 \rho \triangle^* \psi\,, \quad (31)$$

as shown earlier. From Eqs. (30) it follows that $B_\varphi = \mu_0 \chi/\rho$, so defining $F(\rho, z) \equiv \mu_0 \chi(\rho, z)$ the toroidal magnetic field can be expressed as $\mathbf{B}_\varphi = (F/\rho)\hat{\mathbf{e}}_\varphi$. Combining this result with Eq. (23), it is found that the total magnetic field for the axisymmetric case can be written as

$$\mathbf{B} = \mathbf{B}_p + \mathbf{B}_\varphi = \nabla\psi \times \nabla\varphi + F\nabla\varphi\,. \quad (32)$$

Thus, from Ampére's law Eq. (1b), $\mu_0 \mathbf{j}_p = \nabla F \times \nabla\varphi = \nabla \times F\nabla\varphi$.

The Grad-Shafranov equation is derived from the balance of magnetostatic force and the plasma pressure force:

$$\nabla p = \mathbf{j} \times \mathbf{B} = \mathbf{j}_\varphi \times \mathbf{B}_p + \mathbf{j}_p \times \mathbf{B}_\varphi\,. \quad (33)$$

from where it is found that

$$\nabla p = -\frac{1}{\mu_0 \rho^2}\left[\triangle^*\psi\nabla\psi - F\nabla F\right]\,. \quad (34)$$

As can be seen from Eq. (33),

$$\nabla p \cdot \mathbf{B} = 0 = \nabla p \cdot \mathbf{j}\,, \quad (35)$$

which means that the pressure is constant on surfaces where both the magnetic field and the current density lie. Since the magnetic surfaces are defined by the magnetic flux stream function, then $\nabla p(\psi) = p'(\psi)\nabla\psi$ and $\nabla F(\psi) = F'(\psi)\nabla\psi$, where the primes denote derivative with respect to the argument, so the pressure balance equation can finally be written as a scalar partial differential equation known as the Grad-Shafranov equation:

$$\triangle^*\psi = -\mu_0\rho^2 p'(\psi) - F(\psi)F'(\psi) = -\mu_0 \rho j_\varphi\,. \quad (36)$$

Since it is in general a nonlinear equation it is usually solved numerically, as explained in Ref. [8], for instance. Additionally, the toroidal plasma column may be surrounded by circular coils for control of its shape and position. In that case, finding the equilibrium is posed as a free-boundary problem, in which the poloidal flux function has a contribution by each of the i coils given by

$$\psi_i(\mathbf{r}) = \int j_\varphi(\mathbf{r}')G(\mathbf{r};\mathbf{r}')dS\,. \quad (37)$$

5. Conclusions

The Grad-Shafranov equation describes the magnetohydrodynamic equilibrium for toroidal axisymmetric plasma confinement devices. Its solutions define nested magnetic surfaces around a magnetic axis, in terms of a poloidal flux function ψ. On the other hand, this flux function is related to the azimuthal (toroidal) component of the vector potential by, $A_\varphi = \psi/\rho$. Since the solution for A_φ is that of the circular current loop, it is possible to obtain through this relationship the Green function for the elliptic differential operator $\triangle^*$ in terms of complete elliptic integrals, as shown in Sec. 3. The solution for ψ in terms of the Green's function, including the line integral, which includes the boundary conditions was shown.

1. L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* Chap. IV, Problem 2 (Addison-Wesley, 1960), p. 124–125.
2. J. D. Jackson, *Classical Electrodynamics* 3rd edition (Wiley, 1998), Chap. 5, Sec. 5, p. 181–182.
3. W. Greiner, *Classical Electrodynamics* (Springer, 1998), Chap. 9, Example 9.1, p. 206–209.
4. P. Bellan, *Fundamentals of Plasma Physics*, 2nd edition (Cambridge, 2006), Chap. 9, Sec. 8, p. 320–328.
5. J. Freidberg *Ideal MHD*, Chap. 9, Sec. 2 (Cambridge, 2014), Chap. 9, Sec. 2, p. 124–128.
6. F. F. Chen *Introduction to Plasma Physics and Controlled Fusion*, 3rd edition (Springer, 2016), Chap. 10, Sec. 2, p. 357-386.
7. M. Ariola and A. Pironti, *Magnetic Control of Tokamak plasmas* Appendix A.1, (Springer, 2016), p. 181-183.
8. S. Jardin, *Computational Methods in Plasma Physics* Chap. 4, (CRC Press, 1998), p. 109-113.
9. D. J. Griffiths, *Introduction to Electrodynamics* (Cambridge University Press, 2017), 4th edition, Appendix B, p. 582-584.
10. R.H. Good, Elliptic Integrals, the forgotten functions, Eur. J. Phys. **22**, 119–126. DOI: `10.1088/0143-0807/22/2/303`.