Thermodynamic functions of Manning-Rosen plus a class of Yukawa potential using Euler-Maclaurin formula

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By employing the Nikiforov-Uvarov (NU) method, we solved the time-independent Schrödinger equation (SE) with Manning-Rosen plus a class of Yukawa potential (MRCYP) model. The eigensolutions expressions were obtained and thermodynamic function expressions were also obtained using Euler-Maclaurin formula in closed forms. Numerical results of the energy were obtained with respect to different screening parameters and quantum numbers. In addition, the effects of temperature on the thermodynamic functions were discussed for various screening parameters and quantum states. Unique critical temperatures of entropy and specific heat capacity were seen to exist for the selected screening parameters and quantum numbers. Our results are also in sync with the results in literatures and they promise to be relevant in various areas of study, including atomic, particle and nuclear physics.

Keywords: Nikiforov-Uvarov method; Manning-Rosen potential; Yukawa potential; Thermodynamic functions; Euler-Maclaurin formula.

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1. Introduction

Numerous studies have been carried out on the properties of quantum mechanical systems in the nonrelativistic regime [1]. This has been achieved via the solutions of the Schrödinger equation (SE) with various potential models. Different techniques and approximation approaches have been used over the years to obtain the solutions of the SE [2–13]. The most commonly used method is the Nikiforov-Uvarov (NU) method [14], as applicable to our studies. Several potential models have been studied using the NU method. These include Hulthen Potential [15, 16], generalized Hulthen potential [17, 18], Kratzer Potential [6, 19], Wood-Saxon Potential [20], Deng-Fan molecular potentials [21], Manning-Rosen Potential [22–24], Yukawa potential [25–27], just to mention a few.

In addition, different potentials have been combined for wider applications. Examples of such combined potentials include Manning-Rosen plus Hulthen potential [28], Hulthen plus a Ring-Shaped like potential [29], Hulthen plus Yukawa potential [30], Manning-Rosen plus a class of Yukawa potential [31], and others [32, 33].

Before now, various potential models have been employed to study the thermodynamic functions of different quantum systems, with the help of numerous techniques [34–42]. Angelova and Frank [43] employed the Lie algebraic techniques to evaluate the thermodynamic functions of different diatomic molecule systems. They also calculated the number of bosons per electronic state for the diatomic molecules considered. Valencia-Ortega and Arias-Hernandez [44] studied a one-dimensional microscopic system for the generalized Morse potential (GMP), describing local anharmonic effects. The authors obtained some thermodynamic functions for a heteronuclear diatomic system (HCl) and compared these functions with that of the Morse potential, graphically. In addition, a relationship between the GMP and the Eckart potential was established using $\mathrm{SO}(2,1)$ algebras.

Recently, Boumali [45] presented the closed-form expressions of the vibrational partition function and other related thermodynamic functions for the one-dimensional q-deformed Morse potential energy model, using the Euler-Maclaurin method [46]. Also, the thermal properties of three dimensional Morse potential for some selected diatomic molecules were obtained [47] using Euler-Maclaurin method. Similarly, the thermodynamic functions of some potential models have been studied for some diatomic molecule systems recently [35, 48–50]. More recent studies on thermodynamic and thermochemical functions for various diatomic molecule systems have been reported in the literature [51–56].

In this study, we are motivated to extend the work of Ahmadov *et al.* [31] to thermodynamic studies, with respect to temperature. This study has not been carried out before to the best of our knowledge. The applications of this study spans from the vibrations of diatomic molecules in molecular physics to the description of charged particles in plasma physics, etc.

The Manning-Rosen potential considered in this work is defined as [57–60]

$$V_{MR}(r)=\frac{\hbar^2}{2Md^2}\left[\frac{\alpha(\alpha-1)e^{-2r/d}}{(1-e^{-r/d})^2}-\frac{Ae^{-r/d}}{(1-e^{-r/d})}\right], \quad (1)$$

where A and α are two dimensionless parameters and d is the range of the potential. On the other hand, a class of Yukawa potential is defined as [31]

$$V_{CY}(r)=-\frac{V_0e^{-\delta r}}{r}-\frac{V_0'e^{-2\delta r}}{r^2}. \quad (2)$$

Here, $1/\delta$ is the potential range and V_0 is the potential strength. The linear combination of the Manning-Rosen and a class of Yukawa is given as follows:

$$V(r)=\frac{\hbar^2}{2Md^2}\left[\frac{\alpha(\alpha-1)e^{-2r/d}}{(1-e^{-r/d})^2}-\frac{Ae^{-r/d}}{(1-e^{-r/d})}\right] -\frac{V_0e^{-\delta r}}{r}-\frac{V_0'e^{-2\delta r}}{r^2}. \quad (3)$$

This study is organized as follows: In Sec. 2, we solve the radial Schrödinger equation with the Manning-Rosen plus a class of Yukawa potential (MRCYP) and obtained its corresponding eigensolutions via the Nikiforov-Uvarov (NU) method. The procedure of the NU method is presented in Appendix A. The thermodynamic functions of the combined potential model are evaluated in Sec. 3 using the Euler-Maclaurin formula. Section 4 is devoted to results and discussion. Finally, a brief concluding remarks are presented in Sec. 5.

2. Solution of the Schrödinger equation with combined Manning-Rosen and a class of Yukawa potential

In spherical coordinates, the time-independent Schrödinger equation (SE) takes the form [61]:

$$\left[-\frac{\hbar}{2m}\nabla^2+V(r)\right]\psi(r,\theta,\phi)=E_{nl}\psi(r,\theta,\phi), \quad (4)$$

where $\psi(r,\theta,\phi)$ is the wave function, E_{nl} is the energy eigenvalue of the particle and $V(r)$ is the interaction potential. By separation of variables in the wave function $\psi(r,\theta,\phi)=\frac{1}{r}R_{nl}Y_{lm}(\theta,\phi)$, the radial part of the SE reads:

$$\frac{d^2\psi_{nl}(r)}{dr^2}+\frac{2\mu}{\hbar^2}\left[E_{nl}-V(r)-\frac{l(l+1)\hbar^2}{2\mu r^2}\right]\psi_{nl}(r)=0, \quad (5)$$

where ψ_{nl} is the radial wave function, $\hbar$ is the reduced Plank's constant, μ is the reduced mass, $l(l+1)/r^2$ is the centrifugal barrier, Y_{lm} is known as the spherical harmonics, with n and l being the radial and orbital angular momentum quantum numbers, respectively. To get rid of the centrifugal barrier in Eq. (5), we use the Greene and Aldrich approximation scheme [60,63]

$$\frac{1}{r^2}\approx\frac{4\delta^2e^{-2\delta r}}{(1-e^{-2\delta r})^2}; \quad \frac{1}{r}\approx\frac{2\delta e^{-\delta r}}{(1-e^{-2\delta r})}. \quad (6)$$

By employing the Greene and Aldrich approximations scheme of Eq. (6), and an assumption $1/d=2\delta$ on Eq. (3), the combined potential becomes

$$V(r)=\frac{V_{14}e^{-4\delta r}}{(1-e^{-2\delta r})^2}-\frac{V_{23}e^{-2\delta r}}{(1-e^{-2\delta r})}. \quad (7)$$

Here, the following parameters are defined as follows:

$$V_{14}=(V_1+V_4), \quad V_{23}=(V_2+V_3),$$
$$V_1=\frac{2\hbar^2\delta^2\alpha(\alpha-1)}{M},$$
$$V_4=-4\delta^2V_0', \quad V_2=\frac{2\hbar^2\delta^2A}{M}, \quad V_3=2\delta V_0. \quad (8)$$

By substituting Eqs. (6) and (7) into Eq. (5), the radial SE becomes

$$\frac{d^2\psi_{nl}(r)}{dr^2}+\left[\frac{2\mu}{\hbar^2}E_{nl}+\frac{2\mu V_{23}}{\hbar^2}\frac{e^{-2\delta r}}{(1-e^{-2\delta r})} -\frac{2\mu V_{14}}{\hbar^2}\frac{e^{-4\delta r}}{(1-e^{-2\delta r})^2}-4l(l+1)\delta^2 \times\frac{e^{-2\delta r}}{(1-e^{-2\delta r})^2}\right]\psi_{nl}(r)=0. \quad (9)$$

By simplifying Eq. (9), we get

$$\frac{d^2\psi_{nl}(r)}{dr^2}+\frac{1}{(1-e^{-2\delta r})^2}\big[-\varepsilon_{nl}^2(1-e^{-2\delta r})^2 +\xi e^{-2\delta r}(1-e^{-2\delta r}) -\gamma e^{-4\delta r}-l(l+1)e^{-2\delta r}\big]\psi_{nl}(r)=0. \quad (10)$$

The following parameters have been defined in Eq. (10) for bound-state solutions ($E_{nl}<0$):

$$\varepsilon_{nl}^2=-\frac{2\mu}{4\hbar^2\delta^2}E_{nl};$$
$$\xi=\frac{2\mu}{4\hbar^2\delta^2}V_{23}; \quad \gamma=\frac{2\mu}{4\hbar^2\delta^2}V_{14}. \quad (11)$$

By implementing the coordinate transformation $s=e^{-2\delta r}$ with $\psi_{nl}(r)\rightarrow\psi_{nl}(s)$, we obtain the hypergeometric-type

differential equation of the form

$$\frac{d^2\psi_{nl}(s)}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{d\psi_{nl}(s)}{ds} + \frac{1}{s^2(1-s)^2}\left[\left(-\varepsilon_{nl}^2-\gamma-\xi\right)s^2 + \left(2\varepsilon_{nl}^2+\xi-l(l+1)\right)s-\varepsilon_{nl}^2\right]\psi_{nl}(s)=0. \tag{12}$$

By comparing Eq. (A.1) with Eq. (12), we obtain the following parameters:

$$\tilde{\tau}(s)=1-s, \tag{13}$$

$$\sigma(s)=s(1-s), \tag{14}$$

$$\tilde{\sigma}(s)=-\left(\varepsilon_{nl}^2+\gamma+\xi\right)s^2 + \left(2\varepsilon_{nl}^2+\xi-l(l+1)\right)s-\varepsilon_{nl}^2. \tag{15}$$

The function $\pi(s)$ can be obtained by substituting Eqs. (A.8), (14) and (15) in Eq. (A.10). This gives

$$\pi(s)=-\frac{s}{2}\pm\sqrt{(\varsigma_1-k)s^2-(\varsigma_2-k)s+\varsigma_3}, \tag{16}$$

where

$$\varsigma_1=\varepsilon_{nl}^2+\xi+\gamma;$$
$$\varsigma_2=2\varepsilon_{nl}^2+\xi-l(l+1);\quad \varsigma_3=\varepsilon_{nl}^2. \tag{17}$$

To find the expession for k, the discriminant of the expression under the square root of Eq. (16) must be equal to zero. Hence, we have

$$k_{\pm}=-\left(l(l+1)-\xi\right)\pm 2\varepsilon_{nl}\sqrt{l(l+1)+\gamma}. \tag{18}$$

For bound-state solutions, the valid expression for k is in the form

$$k_{-}=-\left(l(l+1)-\xi\right)-2\varepsilon_{nl}\sqrt{l(l+1)+\gamma}. \tag{19}$$

The four possible functions of $\pi(s)$ are obtained by substituting Eq. (18) in Eq. (16). This gives

$$\pi(s)=-\frac{s}{2}\pm\begin{cases}\left(\varepsilon_{nl}-\sqrt{l(l+1)+\gamma}\right)s-\varepsilon_{nl} & : \quad k_+,\\ \left(\varepsilon_{nl}+\sqrt{l(l+1)+\gamma}\right)s-\varepsilon_{nl} & : \quad k_-.\end{cases} \tag{20}$$

Also, we choose the expression $\pi(s)_-$ where the function $\tau(s)$ has a nagative derivative. This is given as

$$\pi(s)_-=-\frac{s}{2}-\left[\left(\varepsilon_{nl}-\sqrt{l(l+1)+\gamma}\right)s-\varepsilon_{nl}\right], \tag{21}$$

with $\tau(s)$ obtained as

$$\tau(s)=1-2s-2\left(\sqrt{l(l+1)+\gamma}+\varepsilon_{nl}\right)s+\varepsilon_{nl}. \tag{22}$$

The first derative of Eq. (22) satisfies the bound-state condition, as given in Eq. (A.9)

$$\tau'(s)=-2\left(1+\sqrt{l(l+1)+\gamma}+\varepsilon_{nl}\right)<0. \tag{23}$$

By employing Eq. (A.11), we define the λ parameter as

$$\lambda=-l(l+1)+\xi-2\varepsilon_{nl}\sqrt{l(l+1)+\gamma} -\frac{1}{2}-\sqrt{l(l+1)+\gamma}-\varepsilon_{nl}. \tag{24}$$

By determining $\sigma''(s)=-2$ and using Eq. (A.13), the constant λ_n can be defined as

$$\lambda_n=n^2+2n\left[\frac{1}{2}+\varepsilon_{nl}+\sqrt{l(l+1)+\gamma}\right]. \tag{25}$$

By comparing Eqs. (24) and Eq. (25), we obtain the energy spectrum with the linear combination of Manning-Rosen and a class of Yukawa potentials (MRCYP) as

$$E_{nl}=-\frac{\hbar^2\delta^2}{2\mu}\left[\frac{(n+\zeta)^2-(\xi+\gamma)}{(n+\zeta)}\right]^2, \tag{26}$$

where

$$\zeta=\frac{1}{2}+\sqrt{l(l+1)+\gamma}. \tag{27}$$

To calculate the radial wave function in Eq. (A.2), we first evaluate $\phi(s)$ by substituting $\sigma(s)$ and $\pi(s)$ from Eq. (14) and (21), respectively into Eq. (A.4) and solving the first-order differential equation gives

$$\phi(s)=s^{\varepsilon_{nl}}\left(1-s\right)^{\frac{1}{2}+\sqrt{l(l+1)+\gamma}}. \tag{28}$$

The weight function $\rho(s)$ from Eq. (A.7) can be obtained using Eq. (A.5) as

$$\rho(s)=s^{2\left(\varepsilon_{nl}-\sqrt{l(l+1)+\gamma}\right)}\left(1-s\right)^{2\sqrt{l(l+1)+\gamma}}. \tag{29}$$

From Rodrigues relation of Eq. (A.5), we obtain

$$\chi_n(s)=B_n s^{-2\left(\varepsilon_{nl}-\sqrt{l(l+1)+\gamma}\right)}\left(1-s\right)^{-2\sqrt{l(l+1)+\gamma}} \times\frac{d^n}{ds^n}\left[s^{n+2\left(\sqrt{l(l+1)+\gamma}-\varepsilon_{nl}\right)}\left(1-s\right)^{n+2\sqrt{l(l+1)+\gamma}}\right]. \tag{30}$$

From Jacobi polynomials, we have

$$\chi_n(s)\equiv B_n P_n^{2\left(\sqrt{l(l+1)+\gamma}-\varepsilon_{nl}\right),2\left(\sqrt{l(l+1)+\gamma}\right)}\left(1-2s\right). \tag{31}$$

By substituting $\phi(s)$ and $\chi_n(s)$ from Eq. (28) and Eq. (31) respectively into Eq. (A.2), we obtain the unnormalized radial wave function as

$$\psi(s)=N_{nl}(s)s^{\varepsilon_{nl}}\left(1-s\right)^{\left(\frac{1}{2}+\sqrt{l(l+1)+\gamma}\right)} \times P_n^{2\left(\sqrt{l(l+1)+\gamma}-\varepsilon_{nl}\right),2\left(\sqrt{l(l+1)+\gamma}\right)}\left(1-2s\right), \tag{32}$$

where $N_{nl}(s)$ is the normalization constant.

3. Thermodynamic properties calculations using Euler-MacLaurin formula

The thermodynamic functions of MRCYP can be obtained from the partition function, being the starting point to derive all thermodynamic functions of any system [64]. The bound-state contributions to the partition function of any system at a given temperature T is given as [51]

$$Z(\beta,\sigma)=\sum_{n=0}^{\sigma} e^{-\beta(E_{nl}-E_{0l})}, \quad \beta=(k_BT)^{-1}. \tag{33}$$

Here, σ is the maximum vibration quantum number, E_{nl} are the energies of the combined potential model, k_B is the Boltzmann's constant and T is the absolute temperature. It is well known that the partition function cannot be calculated exactly in a closed form. As such, reasonable approximation can only be obtained for high temperature where $T \longrightarrow \infty$ and for low temperature where $T \longrightarrow 0$ [65]. Hence, the partition function can be obtained using the Euler-Maclaurin formula. This is defined as [45–47]

$$\sum_{n=0}^{\infty} f(x)=\frac{1}{2}f(0)+\int_0^{\infty} f(x)dx-\sum_{p=1}^{\infty}\frac{B_{2p}}{(2p)!}f^{(2p-1)}(0), \tag{34}$$

where B_{2p} are the Bernoulli numbers, $f^{(2p-1)}$ is the derivative of order $(2p-1)$. Taking p up to 3, we obtain

$$Z(\beta,\sigma)=\frac{1}{2}+\int_0^{\sigma} f(n)dn-\sum_{p=1}^{\infty}\frac{B_{2p}}{(2p)!}f^{(2p-1)}(0), \tag{35}$$

where $f(n)=e^{-\beta(E_{nl}-E_{0l})}$, $B_2=1/6$ and $B_4=-1/30$.

By employing the Euler-Maclaurin formula, the expression for the partition function of Manning-Rosen plus a class of Yukawa potential (MRCYP) is obtained as

$$\begin{aligned} Z(\beta,\sigma)=&\frac{1}{2}-\frac{\sqrt{\pi}}{4\sqrt{-H\beta}}\exp\left[E_{0l}-2GH\beta-2A_0\right]\\ &\times\left[\operatorname{erf}(A_1)+\operatorname{erf}(A_2)\right.\\ &\left.+\,\mathrm{e}^{4A_0}\left(\operatorname{erf}(A_3)-\operatorname{erf}(A_4)\right)\right]-J, \end{aligned} \tag{36}$$

where erf(x) is the error function and

$$A_0=\sqrt{-H\beta}\sqrt{-G^2H\beta}, \tag{37}$$

$$A_1=\zeta\sqrt{-H\beta}-\frac{\sqrt{-G^2H\beta}}{\zeta},$$

$$A_2=\frac{\sqrt{-G^2H\beta}}{\zeta+\sigma}-\sqrt{-H\beta}\,(\zeta+\sigma), \tag{38}$$

$$A_3=\zeta\sqrt{-H\beta}+\frac{\sqrt{-G^2H\beta}}{\zeta};$$

$$A_4=\frac{\sqrt{-G^2H\beta}}{\zeta+\sigma}+\sqrt{-H\beta}\,(\zeta+\sigma), \tag{39}$$

$$H=\frac{\hbar^2\delta^2}{2\mu}; \quad G=\xi+\gamma;$$

$$\sigma=\sqrt{G}-\zeta; \quad J=\frac{\hbar^2\delta^2\beta}{12\mu}\left[\frac{\zeta^4-G^2}{\zeta^3}\right]. \tag{40}$$

Other thermodynamic properties including Helmholtz free energy $F(\beta,\sigma)$, entropy $S(\beta,\sigma)$, mean energy $U(\beta,\sigma)$, and specific heat capacity $C_v(\beta,\sigma)$ can be obtained from the partition function as given below [66]:

$$F(\beta,\sigma)=-\frac{1}{\beta}\ln Z(\beta,\sigma);$$

$$S(\beta,\sigma)=k_B\ln Z(\beta,\sigma)-k_B\beta\frac{\partial}{\partial\beta}\ln Z(\beta,\sigma);$$

$$U(\beta,\sigma)=-\frac{\partial\ln Z(\beta,\sigma)}{\partial\beta};$$

$$C_v(\beta,\sigma)=k_B\beta^2\frac{\partial^2}{\partial\beta^2}\ln Z(\beta,\sigma). \tag{41}$$

4. Results and discussion

In this study, we solved the time-independent SE with the MRCYP and obtained the analytical expression for its energy eigensolutions in closed form using Nikiforov-Uvarov method, as given in Eqs. (26) and (32), respectively. Numerical results of the energies have been presented for different screening parameters and quantum states in Table I. Throughout our analysis, the following parameters have been employed: $\hbar=1$; $V_0=1$; $M=1$; $\mu=1$; $\alpha=0.75$ and $V_0'=0.1$. The bounded energies shown in Table I are seen to increase with an increase in quantum states, when considering a particular screening parameter. Also, an increase in the screening parameter results in a decrease in energy for a specific quantum state.

The variation of the bounded energies with certain parameters and quantum numbers is displayed in Fig. 1. In Fig. 1a), the energies decrease with an increase in δ for different quantum states. At a particular δ, E_{nl} increases while increasing n and l. The energies decrease with an increase in V_0 for different quantum states, as shown in Fig. 1b). At a specific

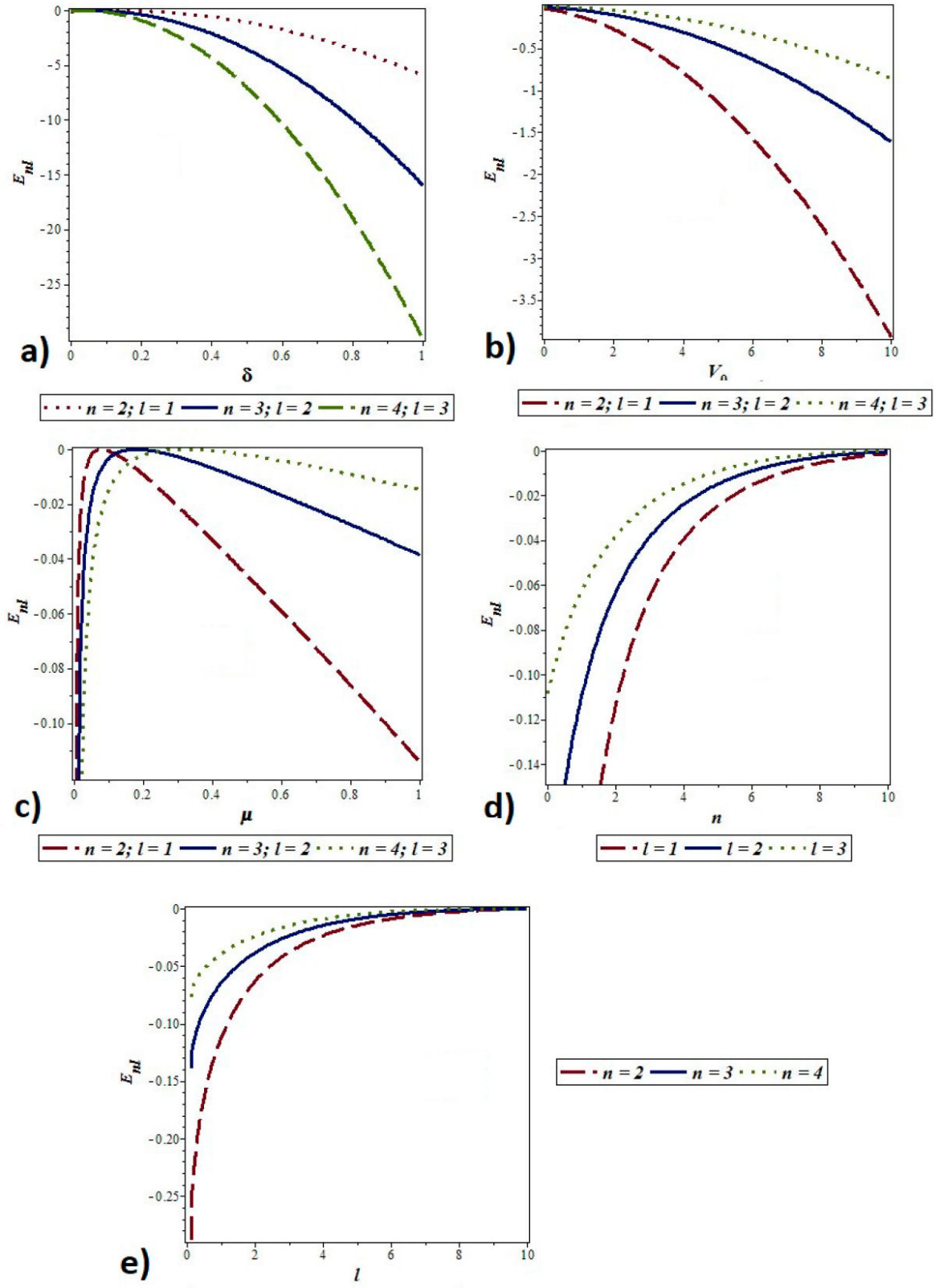

FIGURE 1. Variation of energy eigenvalues of MRCYP with restepct to a) δ; b) V_0; c) μ; d) n; and e) l, for various quantum states.

TABLE I. Energy eigenvalues of MRCYP for different screening parameters and quantum states.

n	l	$-E_{nl}$		
		$\delta = 0.0001$	$\delta = 0.001$	$\delta = 0.01$
0	0	−0.2047860794	−0.2056459966	−0.6940114185
1	0	0.1618605870	0.1607692012	0.5135365490
2	0	0.2659258174	0.2640362978	0.2454376935
	1	0.1405236150	0.1386819594	0.1209335323
3	0	0.1483481473	0.1465026002	0.1286106270
	1	0.08770339890	0.08588439030	0.06874269910
	2	0.05785981880	0.05605700625	0.03959817075
4	0	0.09306815480	0.09124584530	0.07394246325
	1	0.05987399025	0.05806990360	0.04154688244
	2	0.04218571795	0.04039467676	0.02462098684
	3	0.03178274934	0.03000293314	0.01502520083
5	0	0.06342586895	0.06161884650	0.04491373596
	1	0.04343734854	0.04164520098	0.02580015834
	2	0.03209761056	0.03031738282	0.01530837738
	3	0.02500506086	0.02323593189	0.009114176240
	4	0.02008839780	0.01833050216	0.005177749845

V_0, the energies increase with an increase in the quantum states. In Fig. 1c), the energies increase sharply at a zero reduced mass. Further enhancement of μ results in a monotonous decrease in energies for the quantum states considered. The monotonous decrease in energy is seen to be sharper for $n = 2$ and $l = 1$, as compared to other quantum states. Figure 1d) shows a monotonous increase in energies as the principal quantum number, n, increases for different orbital angular momentum quantum number, l. For a specific n, E_{nl} increase with an increase in l. Figure 1e) also shows a monotonous increase in E_{nl} as l increase for different n. For a specific l, E_{nl} increase while increasing n.

The analytical expression for energy given in Eq. (26) is employed to obtain the expressions for partition function and other thermodynamic functions with the help of the Euler-Maclaurin formula. The partition function and other thermodynamic function expressions are given in Eqs. (36) and (41), respectively. The variation of the partition function with temperature for different screening parameter and quantum states are given in Figs. 2a) and b), respectively. It is observed that the partition function first increases sharply at zero temperature and later remains constant at unique values corresponding to the screening parameters and quantum states considered, as the temperature is increased. In Figs. 3a) and b), the free energy respectively decreases linearly while increasing temperature for different screening parameters and quantum states considered. There exists a monotonous increase in mean energy as temperature is increased first for different screening parameter and quantum states, as shown in Figs. 4a) and b), respectively. Further enhancement of tem-

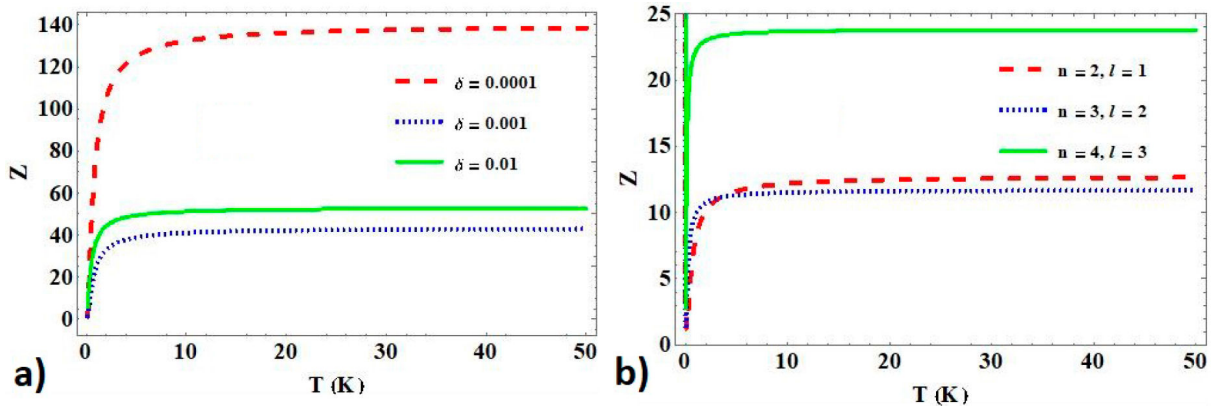

FIGURE 2. Variation of partition function with temperature for a) various δ values; b) various n and l values.

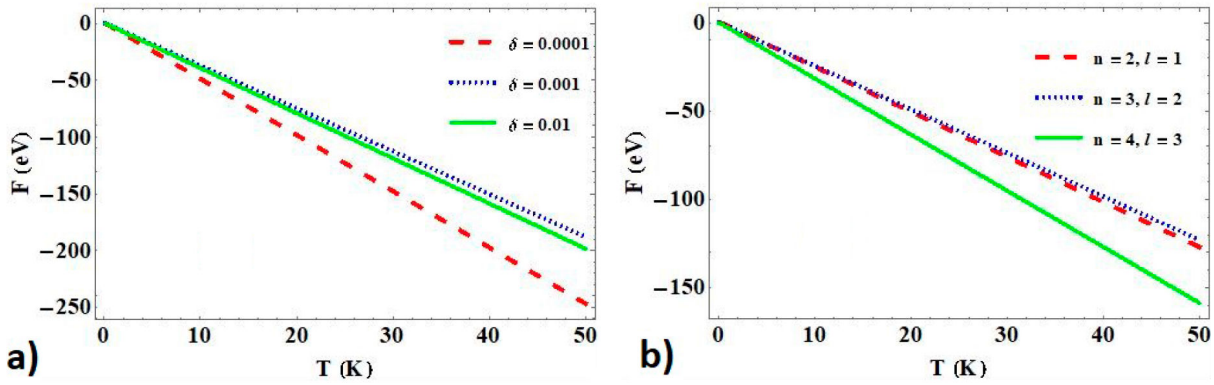

FIGURE 3. Variation of free energy with temperature for a) various δ values; b) various n and l values.

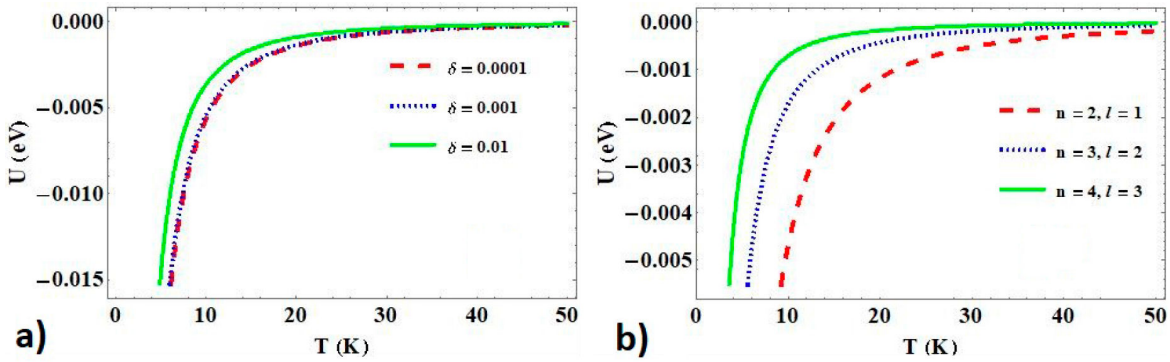

FIGURE 4. Variation of mean energy with temperature for a) various δ values; b) various n and l values.

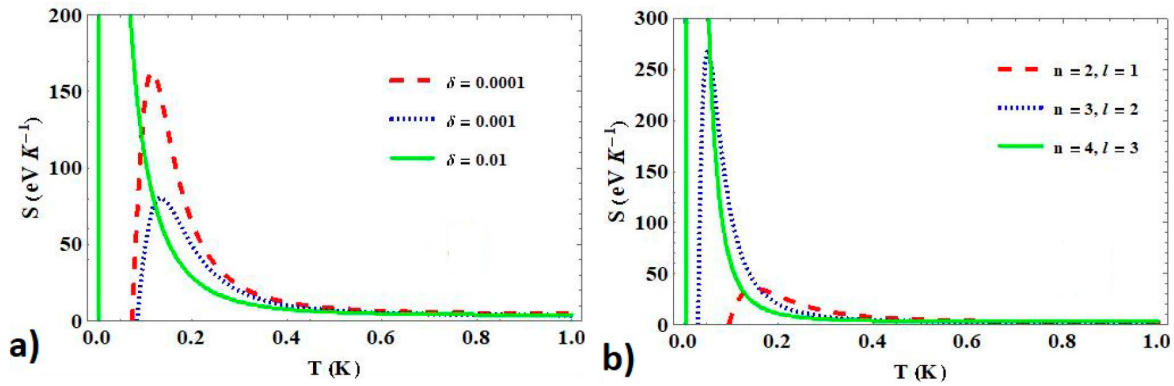

FIGURE 5. Variation of free energy with temperature for a) various δ values; b) various n and l values.

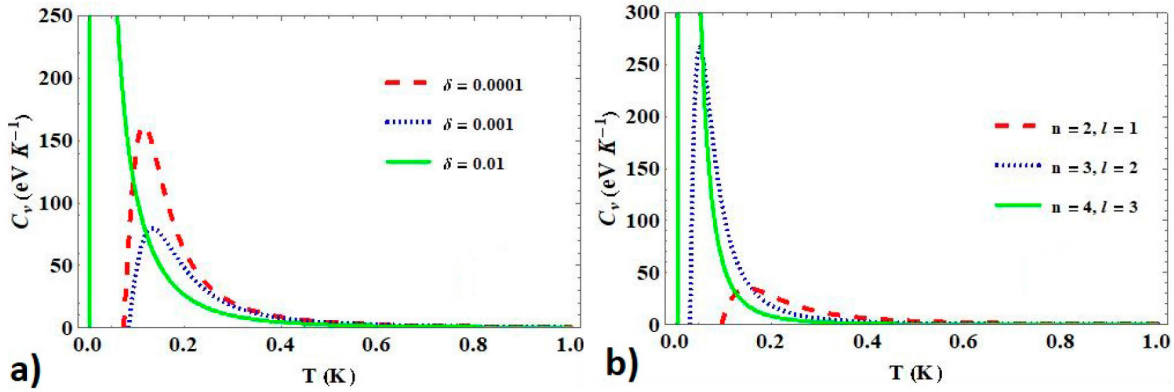

FIGURE 6. Variation of specific heat capacity with temperature for a) various δ values; b) various n and l values.

perature in both plots results in the mean energy becoming saturated at the origin.

In Figs. 5 and 6, we observe a sharp increase in entropy and specific heat capacity to a unique value at a temperature value called Critical Temperature, for different screening parameters and quantum states. Furthermore, the entropy and specific heat capacity begin to reduce monotonously as the temperature is increased. Beyond a temperature of 0.4 K, the entropy and specific heat capacity for the screening parameters and quantum states considered becomes saturated at the origin. At this point, much energy can no longer be absorbed. This is because the excited states of the system are fully occupied.

5. Conclusions

In this research, we first solved the radial part of the time-independent Schrödinger equation (SE) containing manning-Rosen plus a class of Yukawa potential (MRCYP) model using the Nikiforov-Uvarov (NU) method. Analytical energy eigensolutions were obtained and their corresponding numerical results discussed with respect to different screening parameters and quantum states. The energy expression for the combined potential was used to obtain the partition function expression and other thermodynamic function expressions with the help of Euler-Maclaurin formula in closed form. Effect of temperature on the thermodynamic functions for different screening parameters and quantum states were discussed extensively. It is seen from our discussion that the screening parameter and quantum states have a strong influence on the energy and thermodynamic functions of MRCYP considered. This study promises to be very relevant in the fields of atomic, particle and nuclear physics. In addition, our results are very similar to results obtained in literature.

Appendix

A. Nikiforov-Uvarov Method

The Nikifirov-Uvarov (NU) framework is given by the second-order differential equation of the form [14]

$$\psi''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s)=0, \tag{A.1}$$

where $\psi(s)$ is the function of hypergeometric-type, $\sigma(s)$ and $\tilde{\sigma}(s)$ are at most second-order polynomials whereas, on the other hand, $\tilde{\tau}(s)$ is at most a first-degree polynomial. To obtain the exact solution for Eq. (A.1), we set

$$\psi(s)=\phi(s)\chi_n(s). \tag{A.2}$$

In order to transform in Eq. (A.1) into a hypergeometric type equation, we substitute Eq. (A.2) into Eq. (A.1) to get

$$\sigma(s)\chi_n''(s)+\tau(s)\chi_n'(s)+\lambda\chi_n(s)=0, \tag{A.3}$$

where the wave function $\phi(s)$ in Eq. (A.2) is defined as a logarithm derivative [14]

$$\frac{d\phi(s)}{\phi(s)}=\frac{1}{\rho(s)}\frac{d\rho(s)}{d(s)}=\frac{\pi(s)}{\sigma(s)}, \tag{A.4}$$

and $\pi(s)$ is at most a first-degree polynomial. Similarly, the second wave function $\chi_n(s)$ in Eq. (A.2) is a hypergeometric function given by the Rodrigues relation

$$\chi_n(s)=\frac{B_n}{\rho(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\rho(s)\right], \tag{A.5}$$

where B_n is the normalization constant and $\rho(s)$ is the weight function given by

$$\frac{d\rho(s)}{d(s)}=\frac{1}{\rho(s)}\frac{d\rho(s)}{d(s)}=\frac{\tau(s)-\sigma'(s)}{\sigma(s)}, \tag{A.6}$$

satisfying the condition

$$\frac{d}{ds}\left[\sigma(s)\rho(s)\right]=\tau(s)\rho(s), \tag{A.7}$$

where

$$\tau(s)=\tilde{\tau}(s)+2\pi(s). \tag{A.8}$$

For the wave function $\rho(s)$ to satisfy the (NU), the polynomial $\tau(s)$ should be equal to zero at some point within the interval (a,b) and its derivative $\tau'(s)$ should be negative within this interval, that is

$$\tau'(s)<0. \tag{A.9}$$

As a result, we obtain the function $\pi(s)$ and parameters λ required by the (NU) defined as

$$\pi(s)=\frac{\sigma'(s)-\tilde{\tau}(s)}{2} \pm\sqrt{\left(\frac{\sigma'(s)-\tilde{\tau}(s)}{2}\right)^2-\tilde{\sigma}(s)+k\sigma(s)}, \tag{A.10}$$

$$\lambda=k+\pi'(s). \tag{A.11}$$

The value of k in Eq. (A.11) can be obtained by setting the discriminant of the square root in Eq. (A.10) equal to zero. This gives

$$\Delta=b^2-4ac=0. \tag{A.12}$$

The energy eigenvalue equation can be obtained using the relation

$$\lambda+n\tau'(s)+\frac{n(n-1)}{2}\sigma''(s)=0,\ (n=0,1,2,...). \tag{A.13}$$

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