Abundant traveling wave solutions of 3+1 dimensional Boussinesq equation with dual dispersion

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In this study, we consider (3+1)-dimensional Boussinesq equation with dual dispersion. It appears in many models of nonlinear problem such as ocean ecology, weather forecast, wave motion and ocean engineering. We applied the Jacobi elliptic function expansion method in order to construct dark soliton, bright soliton and trigonometric solutions. Depending on the structure of the auxiliary equation $F' = \sqrt{PF^4 + QF^2 + R}$,a wide variety of solutions are obtained when special values are given to P, Q, R. Besides, the figures for some solutions are given. The resulting outcomes verify that the referred method is valid and reliable for analytical technique of an extensive application of nonlinear phenomena.

Keywords: Jacobi elliptic function method; solitary wave solutions; exact solutions.

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1. Introduction

Nonlinear partial differential equations (PDEs) are commonly used to modelling different kinds of fields of nonlinear phenomena, particularly plasma physics, solid state physics, fluid mechanics, and optics, etc. The search of the travelling wave solutions takes a substantial role in the study of physical phenomena which has applications in many science branches.

A unique valid technique for solving all types of NLEEs has not been investigated yet. So many mathematicians and physicists are making efforts to establish more exact solutions of them. Therefore, a great number of methods such as sine-cosine method, tanh method, extended tanh method, sech-tanh method, Backlund transform method, homogeneous balance scheme, Painlev expansion, Exp-function method, rational expansion method, elliptic function method, inverse scattering, and the modified simplest equation method, have been used to investigate exact solutions [1-34].

The classical Boussinesq equation,

$$u_{tt} + au_{xxxx} - u_{xx} - bu^2_{xx} = 0,$$

depicts the propagation of gravity waves on the surface of shallow water. Here $u(x,t)$ and the modified simplest equation method, have been used to investigate exact solutions [1-34]. Of the free surface of the water, and arbitrary constants a and b depend on both the depth of the water and characteristic speed of the long waves.

Two-dimensional Boussinesq equation is expressed by

$$u_{tt} - su_{xxxx} - au_{xx} - bu_{yy} - ru^2_{xx} = 0,$$

a, b, r, and s are arbitrary constants with $r \neq 0$.

In the literature three-dimensional Boussinesq equation appears in the modeling of many problems in physics and engineering area such as in ocean ecology, weather forecast, wave motion, and ocean engineering. It is generally handled [39-42] by the expression

$$u_{tt} - u_{xx} - u_{xxxx} - u_{yy} - u_{zz} - 3(u^2)_{xx} = 0. \quad (1)$$

As stated in [35, 36], it it does not posses three soliton solutions nor passes the painleve test. Utilizing the Hirota bilinear method and the Riemann theta function method on (3+1)-dimensional Boussinesq equation, soliton solutions were found from the periodic wave solutions [43]. Khalique *et al.* constructed exact solutions with the aid of Lie symmetry approach and simplest equation method. Moreover, they also obtained conservation laws using Ibragimov's method [44].

Jacobi elliptic function expansion method is used for finding exact travelling wave solutions of NLEEs in a unified way. This method provides a more comprehensive solution set compared to other exact solution methods. This method was first introduced by Liu *et al*. [45] and Fu *et al*. [46] in which doubly periodic solutions of nonlinear wave equations were constructed. Then the method was developed and applied in many forms. Jacobi elliptic function method is more general than the tanh method, sine-cosine method, extended tanh method and (G'/G) expansion method which are widely used in the literature. The superiority of this method compared with other methods is that it comprises various kinds of exact solutions, including periodic and exponential functions. For this, the trigonometric solutions and soliton-like solutions have also been obtained as limiting cases, when the modulus $m \to 1$ and $m \to 0$, respectively.

The aim of this work is to perform the Jacobi elliptic function expansion method to find travelling wave solutions of (3+1) dimensional Boussinesq equation with dual dispersion

(B-DD). Seadawy *et al*. applied $(G\prime/G)$ expansion method to find solitary wave solutions for this equation and the relevant equation is given in [47] as follows:

$$u_{tt} - v_1(u^2)_{xx} - v_2 u_{xx} - v_3(u_{xxxx} + u_{xxtt}) - v_4 u_{yy} - v_5 u_{zz} = 0, \tag{2}$$

where the coefficients v_i for $i = 1, ..., 5$, are real constants. The structural form of this paper is constructed as follows: In the first section, we present the introduction. Secondly, we give the Jacobi elliptic function method. In the third section we apply the method on the Boussinesq equation and construct new exact solutions. In the last section, we present the conclusions about the obtained solutions.

2. Jacobi's elliptic function procedure

Consider the a general form of the non-linear PDE

$$N(u, u_t, u_x, u_y, u_z, u_{xx}, ...) = 0, \tag{3}$$

which consist of three spatial and one time variables x, y, z and t, respectively. Meanwhile, u is an unknown function. Firstly, we apply the following travelling wave transformation

$$u(x, y, z, t) = u(\xi), \qquad \xi = x + y + z - wt,$$

where w corresponds to wave speed and is to be attained at the end of the process. Consequently, the nonlinear PDE (3) is reduced to an ordinary differential equation (ODE) as follows:

$$N(u, u', u'', u''', ...) = 0. \tag{4}$$

We seek for the exact solutions of the form

$$u(\xi) = \sum_{i=0}^{n} a_i F^i(\xi), \tag{5}$$

where F satisfies the Eq. (3) and n is an integer, which can be determined by using homogeneous balance principle in Eq. (4). $F(\xi)$ ensures the following ansatz equation:

$$F'(\xi) = \sqrt{PF^4(\xi) + QF^2(\xi) + R}, \tag{6}$$

where P,Q, and R are constants. The last equation hence holds for $F(\xi)$:

$$\begin{aligned} &F'' = 2PF^3 + QF, \\ &F''' = (6PF^2 + Q)F', \\ &F''' = 24P^2F^5 + 20PQF^3 + (12PR + Q^2)F, \\ &\vdots \end{aligned} \tag{7}$$

By substituting (6) into (4) along with Eq. (7) and grouping the terms of the same power $F^i(F')^j$ $(j = 0, 1,\ i = 0, 1, 2, ...)$ and setting each of the obtained coefficients to zero we get a set of algebraic equations. From it, it is possible to obtain the values of P, Q, R and ω, and by substituting them into Eq. (6), it yields the hyperbolic and trigonometric solutions. It is well-known [48, 49] the table of ansatz equation solutions:

Case 1. P=m^2, $Q = -(1 + m^2)$, $R = 1$ so $F(\xi) = sn\xi$,

Case 2. $P = -m^2$, $Q = 2m^2 - 1$, $R = 1 - m^2$ so $F(\xi) = cn\xi$,

Case 3. $P = 1$, $Q = -(1 + m^2)$, $R = m^2$ so $F(\xi) = ns\xi$,

Case 4. $P = 1$, $Q = 2 - m^2$, $R = 1 - m^2$ so $F(\xi) = cs\xi$,

Case 5. $P = \frac{1}{4}$, $Q = \frac{1 - 2m^2}{2}$, $R = \frac{1}{4}$ so $F(\xi) = ns\xi + cs\xi$,

Case 6. $P = \frac{1}{4}$, $Q = \frac{1 - 2m^2}{2}$, $R = \frac{1}{4}$ so $F(\xi) = ns\xi$-$cs\xi$,

Case 7. $P = P > 0$, $Q = Q < 0$, $R = \frac{m^2Q^2}{(1 + m^2)^2P}$ so $F(\xi) = \sqrt{\frac{-m^2Q}{(1 + m^2)P}}sn\left(\sqrt{\frac{-Q}{1 + m^2}}\xi\right)$,

Case 8. $P = P < 0$, $Q = Q > 0$, $R = \frac{(1 - m^2)Q^2}{(m^2 - 2)^2P}$ so $F(\xi) = \sqrt{\frac{-Q}{(2 - m^2)P}}dn\left(\sqrt{\frac{Q}{2 - m^2}}\xi\right)$,

Case 9. $P = \frac{1}{4}$, $Q = \frac{1 - 2m^2}{2}$, $R = \frac{1}{4}$ so $F(\xi) = \frac{sn\xi}{1 + cn\xi}$,

Case 10. $P=\dfrac{1}{4},\ Q=\dfrac{1-2m^2}{2},\ \ R=\dfrac{1}{4}$ so $F(\xi)=\dfrac{sn\xi}{1-cn\xi}$,

Case 11. $P=-(m^2+2m+1)B^2,\ Q=2m^2+2,\ \ R=\dfrac{2m^2-m^2-1}{B^2}$ so $F(\xi)=\dfrac{msn^2\xi-1}{B(msn^2\xi+1)}$,

Case 12. $P=1,\ Q=-1-m^2,\ R=m^2$ so $F(\xi)=ns\xi$,

Case 13. $P=1,\ Q=-1-m^2,\ \ R=m^2$ so $F(\xi)=dc\xi$,

Case 14. $P=1-m^2, Q=2-m^2,\ \ R=1$ so $F(\xi)=sc\xi$,

Case 15. $P=1,\ Q=2-m^2,\ \ R=1-m^2$ so $F(\xi)=cs\xi$,

Case 16. $P=\dfrac{1}{4},\ Q=\dfrac{1-2m^2}{2},\ \ R=\dfrac{1}{4}$ so $F(\xi)=ns\xi$+$cs\xi$,

Case 17. $P=\dfrac{1}{4},\ Q=\dfrac{1-2m^2}{2},\ R=\dfrac{1}{4}$ so $F(\xi)=ns\xi$-$cs\xi$,

Case 18. $P=\dfrac{1-m^2}{4},\ Q=\dfrac{1+m^2}{2},\ \ R=\dfrac{1-m^2}{4}$ so $F(\xi)=nc\xi+sc\xi$,

Case 19. $P=\dfrac{1-m^2}{4},\ Q=\dfrac{1+m^2}{2},\ \ R=\dfrac{1-m^2}{4}$ so $F(\xi)=nc\xi-sc\xi$,

Case 20. $P=\dfrac{1}{4},\ Q=\dfrac{1-2m^2}{2},\ \ R=\dfrac{1}{4}$ so $F(\xi)=\dfrac{sn\xi}{1+cn\xi}$,

Case 21. $P=\dfrac{1}{4},\ Q=\dfrac{1-2m^2}{2},\ \ R=\dfrac{1}{4}$ so $F(\xi)=\dfrac{sn\xi}{1-cn\xi}$,

Case 22. $P=\dfrac{1-m^2}{4},\ Q=\dfrac{1+m^2}{2},\ \ R=\dfrac{1-m^2}{4}$ so $F(\xi)=\dfrac{cn\xi}{1+sn\xi}$,

Case 23. $P=\dfrac{1-m^2}{4},\ Q=\dfrac{1+m^2}{2},\ R=\dfrac{1-m^2}{4}$ so $F(\xi)=\dfrac{cn\xi}{1-sn\xi}$,

Case 24. $P=\dfrac{C^2m^4-(B^2+C^2)m^2+B^2}{4},\ Q=\dfrac{m^2+1}{2},\ \ R=\dfrac{m^2-1}{4(C^2m^2-B^2)}$ so $F(\xi)=\dfrac{\sqrt{\frac{(B^2-C^2)}{(B^2-C^2m^2)}}+sn\xi}{Bcn\xi+Cdn\xi}$,

Case 25. $P=\dfrac{B^2+C^2m^2}{4},\ Q=\dfrac{1}{2}-m^2,\ R=\dfrac{1}{4(C^2m^2+B^2)}$ so $F(\xi)=\dfrac{\sqrt{\frac{(C^2m^2+B^2-C^2)}{(B^2+C^2m^2)}}+cn\xi}{Bsn\xi+Cdn\xi}$,

Case 26. $P=\dfrac{B^2+C^2}{4},\ Q=\dfrac{m^2}{2}-1,\ \ R=\dfrac{m^4}{4(C^2+B^2)}$ so $F(\xi)=\dfrac{\sqrt{\frac{(B^2+C^2-C^2m^2)}{(B^2+C^2)}}+dn\xi}{Bsn\xi+Ccn\xi}$.

In literature, commonly used Jacobian elliptic functions are $sn\xi,\ cn\xi$ and $dn\xi$ which correspond to sine function, cosine function and elliptic function of Jacobian type. The other Jacobian functions can be generated by these three kinds of functions, namely

$$ns\xi=\frac{1}{sn\xi},\ \ nc\xi=\frac{1}{cn\xi},\ \ nd\xi=\frac{1}{dn\xi},\ \ sc\xi=\frac{cn\xi}{sn\xi},$$

$$cs\xi=\frac{sn\xi}{cn\xi},\ \ ds\xi=\frac{dn\xi}{sn\xi},\ \ sd\xi=\frac{sn\xi}{dn\xi}.$$

Also, these functions satisfying the following formulas:

$$sn^2\xi + cn^2\xi = 1,\ dn^2\xi + m^2 sn^2\xi = 1,\ ns^2\xi = 1 + cs^2\xi,$$
$$ns^2\xi = m^2 + m^2 ds^2\xi,\ sc^2\xi + 1 = nc^2\xi,$$
$$m^2 sd^2\xi + 1 = nd^2\xi,$$

and the addition derivative properties,

$$sn'\xi = cn\xi dn\xi,\ cn'\xi = -sn\xi dn\xi,\ dn'\xi = -m^2 sn\xi cn\xi.$$

If $m \to 1$, then the Jacobi elliptic functions are reduced to the hyperbolic functions as follows:

If $m \to 0$, then the Jacobi elliptic functions are reduced to the trigonometric functions as follows.

$$\{sn\xi, sd\xi\} \to \sin\xi,\ \{cn\xi, cd\xi\} \to \cos\xi,\ sc\xi \to \tan\xi,$$
$$\{ns\xi, ds\xi\} \to \csc\xi,\ \{nc\xi, dc\xi\} \to \sec\xi,\ cs\xi \to \cot\xi,$$
$$dn\xi, nd\xi \to 1.$$

Traveling Wave Solutions of (2)

To seek traveling wave solutions of Eq. (2), by substituting $u(x, y, z, t) = u(\xi),\ \xi = x + y + z - wt$ into Eq. (2) and integrating it once, taking the integral constant be zero, we get

$$(w^2 - v_2 - v_4 - v_5)u' - 2v_1 u'u - v_3(1 + w^2)u''' = 0, \tag{8}$$

where prime denotes differentiation with respect to ξ. $N = 2$ is attained by balancing nonlinear terms u''' and $u'u$. Hence, from (5), we might select

$$u(\xi) = a_0 + a_1 F(\xi) + a_2 F(\xi)^2 \tag{9}$$

in which a_0, a_1, and a_2 are undetermined constants. Substituting (9) and (6) into (8) and setting the coefficients of $F^i(x)F'(\xi)^j = 0,\ i = 0, 1, 2, 3,\ j = 0, 1$ to zero yields the following set of algebraic equations for a_0, a_1, a_2, and β:

$$\begin{aligned} 24a_2v_3w^2 + 24a_2v_3 + 4a_2^2v_1 &= 0,\\ 6Pa_1v_3w^2 + 6Pa_1v_3 + 6a_1a_2v_1 &= 0,\\ -8Qa_2v_3w^2 - 8Qa_2v_3 - 4a_0a_2v_1 - 2a_1^2v_1 + 2a_2w^2 - 2a_2v_2 - 2a_2v_4 - 2a_2v_5 &= 0,\\ -v_3a_1w^2Q - v_3a_1Q - 2v_1a_0a_1 + a_1w^2 - a_1v_2 - a_1v_4 - a_1v_5 &= 0. \end{aligned} \tag{10}$$

By solving the above algebraic equations, one finds the following results

$$\begin{aligned} a_0 &= -\frac{1}{2}\frac{4Qv_3w^2 + 4Qv_3 - w^2 + v_2 + v_4 + v_5}{v_1},\\ a_1 &= 0, \qquad a_2 = -\frac{6Pv_3(w^2+1)}{v_1}, \qquad w = w. \end{aligned} \tag{11}$$

Substituting these results into (9), we have the following solution of Eq. (8):

$$u(\xi) = -\frac{1}{2}\frac{4Qv_3w^2 + 4Qv_3 - w^2 + v_2 + v_4 + v_5}{v_1} - \frac{6Pv_3(w^2+1)}{v_1}F(\xi)^2. \tag{12}$$

With the help of all the table cases and from the above solution (12), one can induce more general united Jacobian-elliptic function solutions of Eq. (8). Thus, we attain the following exact solutions. Some soliton-like solutions of Eq. (2) can be attained in the limited case when the modulus $m \to 1$. For instance,

Case 1. If we take $P : m^2,\ Q : -(1 + m^2), \quad F(\xi) = sn\xi,$ then we have

$$u_1 = -\frac{1}{2}\frac{-4(1+m^2)v_3w^2 - 4(1+m^2)v_3 - w^2 + v_2 + v_4 + v_5}{v_1} - \frac{6m^2v_3(w^2+1)}{v_1}sn^2\xi.$$

In the limit case when the modulus $m \to 1$, we get soliton-like solutions of Eq. (2) as

$$u_1(x,y,z,t) = -\frac{1}{2}\frac{-8v_3w^2 - 8v_3 - w^2 + v_2 + v_4 + v_5}{v_1} - \frac{6v_3(w^2+1)}{v_1}\tanh(x+y+z-wt)^2. \tag{13}$$

Case 2. Suppose that $P : -m^2,\ \ Q : 2m^2 - 1,\ \ F(\xi) = cn\xi,$ then

$$u_2 = \frac{4(2m^2-1)v_3w^2 + 4(2m^2-1)v_3 - w^2 + v_2 + v_4 + v_5}{-2v_1} + \frac{6m^2v_3(w^2+1)}{v_1}cn^2\xi.$$

Considering $m \to 1$, one of the solitary wave solutions of Eq. (2) has been obtained as

$$u_2(x,y,z,t) = -\frac{1}{2}\frac{4v_3w^2 + 4v_3 - w^2 + v_2 + v_4 + v_5}{v_1} + \frac{6v_3(w^2+1)}{v_1}\operatorname{sech}(x+y+z-wt)^2.$$

Case 3. When $\ P : 1,\ \ Q : -(1+m^2),\ \ F(\xi) = ns\xi\ $ are chosen, then

$$u_3 = \frac{-4(1+m^2)v_3w^2 - 4(1+m^2)v_3 - w^2}{-2v_1} - \frac{v_2 + v_4 + v_5}{2v_1} - \frac{6v_3(w^2+1)}{v_1}ns^2\xi.$$

As $m \to 1$, the solitary wave solution of Eq. (2) has been attained as

$$u_3(x,y,z,t) = -\frac{1}{2}\frac{-8v_3w^2 - 8v_3 - w^2 + v_2 + v_4 + v_5}{v_1} - \frac{6v_3(w^2+1)}{v_1}\coth(x+y+z-wt)^2.$$

Case 4. Choosing $\ P : 1,\ \ Q : (2-m^2),\ \ F(\xi) = cs\xi,\ $ for all the cases in the table, then,

$$u_4 = \frac{4(2-m^2)v_3w^2 + 4(2-m^2)v_3 - w^2 + v_2 + v_4 + v_5}{-2v_1} - \frac{6v_3(w^2+1)}{v_1}cs^2\xi.$$

If $m \to 1$, we get the solitary wave solution of Eq. (2) as

$$u_4(x,y,z,t) = -\frac{1}{2}\frac{4v_3w^2 + 4v_3 - w^2 + v_2 + v_4 + v_5}{v_1} - \frac{6v_3(w^2+1)}{v_1}\operatorname{csch}(x+y+z-wt)^2.$$

Case 5. Supposing $\ P : 1/4,\ \ Q : (1-2m^2)/2\ $ for all the table cases this choices correspond to $F(\xi) = ns\xi + cs\xi,$ therefore

$$u_5 = \frac{2(1-2m^2)v_3w^2 + 2(1-2m^2)v_3 - w^2 + v_2 + v_4 + v_5}{-2v_1} - \frac{3v_3(w^2+1)}{2v_1}\left(ns\xi + cs\xi\right)^2.$$

For $m \to 1$, we acquire one of the solitary wave solutions of

$$u_5(x,y,z,t) = \frac{-2v_3w^2 - 2v_3 - w^2 + v_2 + v_4 + v_5}{-2v_1}\frac{3v_3(w^2+1)}{2v_1}(\coth[x+y+z-wt] + \operatorname{csch}[x+y+z-wt])^2.$$

Case 6. Considering $\ P : 1/4,\ \ Q : (1-2m^2)/2\ $ for all the table cases, $F(\xi) = ns\xi - cs\xi,$ therefore we obtain

$$u_6 = \frac{2(1-2m^2)v_3w^2 + 2(1-2m^2)v_3 - w^2 + v_2 + v_4 + v_5}{-2v_1} - \frac{3v_3(w^2+1)}{2v_1}(ns\xi - cs\xi)^2.$$

As $m \to 1$, the solitary wave solution of Eq. (2) has been obtained as

$$u_6(x,y,z,t) = \frac{-2v_3w^2 - 2v_3 - w^2 + v_2 + v_4 + v_5}{-2v_1} - \frac{3v_3(w^2+1)}{2v_1}(\coth[x+y+z-wt] - \operatorname{csch}[x+y+z-wt])^2.$$

Case 7. If we get $\ P > 0,\ \ Q < 0,\ \ F(\xi) = \sqrt{(-m^2Q)/([1+m^2]P)}sn\left(\sqrt{(-Q/[1+m^2])}\xi\right),$ then

$$u_7 = -\frac{1}{2}\frac{4Qv_3w^2 + 4Qv_3 - w^2 + v_2 + v_4 + v_5}{v_1} - \frac{6Pv_3(w^2+1)}{v_1}\frac{-m^2Q}{(1+m^2)P}sn^2\left(\sqrt{\frac{-Q}{1+m^2}}\xi\right).$$

As long as $m \to 1$, we acquire the solitary wave solution of Eq. (2) as

$$u_7(x,y,z,t) = -\frac{1}{2}\frac{4v_3w^2Q + 4v_3Q - w^2 + v_2 + v_4 + v_5}{v_1} + \frac{3v_3(w^2+1)Q\tanh\left(\frac{1}{2}\sqrt{-2Q}[x+y+z-wt]\right)^2}{v_1}.$$

Case 8. For choices $P<0,\ \ Q>0$ for all the table cases, F is given $F(\xi)=\sqrt{(-Q/[2-m^2]P)}dn\left(\sqrt{Q/(2-m^2)}\xi\right)$, in this way the solution may be expressed as

$$u_8=-\frac{1}{2}\frac{4Qv_3w^2+4Qv_3-w^2+v_2+v_4+v_5}{v_1}-\frac{6Pv_3(w^2+1)}{v_1}\frac{-Q}{(2-m^2)P}dn^2\left(\sqrt{\frac{Q}{2-m^2}}\xi\right).$$

For $m\to 1$, one of the solitary wave solutions of Eq. (2) can be found as

$$u_8(x,y,z,t)=\frac{4v_3w^2Q+4v_3Q-w^2+v_2+v_4+v_5}{-2v_1}+\frac{6v_3(w^2+1)Q\,\mathrm{sech}(\sqrt{Q}[x+y+z-wt])^2}{v_1}.$$

Case 9. Setting $P:1/4,\ \ Q:(1-2m^2)/2,$ then $F(\xi)=(sn\xi)/(1+cn\xi),$ due to this settings,

$$u_9=\frac{2(1-2m^2)v_3w^2+2(1-2m^2)v_3-w^2+v_2}{-2v_1}+\frac{v_4+v_5}{-2v_1}-\frac{3v_3(w^2+1)}{2v_1}\left(\frac{sn\xi}{1+cn\xi}\right)^2.$$

Moreover, when $m\to 1$, then we obtain one of the solitary wave solutions of Eq. (2) as

$$u_9(x,y,z,t)=\frac{-2v_3w^2-2v_3-w^2+v_2+v_4+v_5}{-2v_1}=-\frac{3v_3(w^2+1)\tanh(x+y+z-wt)^2}{2v_1(1+\mathrm{sech}[x+y+z-wt])^2}.$$

Case 10. If we take $P:1/4,\ \ Q:1-2m^2/2$ it may be deducted for all the table cases, $F(\xi)=sn\xi/1-cn\xi$, therefore

$$u_{10}=\frac{2(1-2m^2)v_3w^2+2(1-2m^2)v_3-w^2+v_2}{-2v_1}+\frac{v_4+v_5}{-2v_1}-\frac{3v_3(w^2+1)}{2v_1}\left(\frac{sn\xi}{1-cn\xi}\right)^2.$$

In this case for $m\to 1$, one of the solitary wave solutions of Eq. (2) can be shown as

$$u_{10}(x,y,z,t)=\frac{-2v_3w^2-2v_3-w^2+v_2+v_4+v_5}{-2v_1}-\frac{3v_3(w^2+1)\tanh(x+y+z-wt)^2}{2v_1(1-\mathrm{sech}[x+y+z-wt])^2}.$$

Case 11. Regarding $P:-(m^2+2m+1)B^2,\ \ Q:2m^2+2$, then $F(\xi)=msn^2\xi-1/B(msn^2\xi+1)$, so

$$u_{11}=\frac{4(2m^2+2)v_3w^2+4(2m^2+2)v_3-w^2+v_2+v_4+v_5}{-2v_1}+\frac{6(m^2+2m+1)B^2v_3(w^2+1)}{v_1}\left(\frac{msn^2\xi-1}{B[msn^2\xi+1]}\right)^2.$$

For $m\to 1$, the solitary wave solution of Eq. (2) can be stated as

$$u_{11}(x,y,z,t)=\frac{16v_3w^2+16v_3-w^2+v_2+v_4+v_5}{-2v_1}+\frac{24v_3(w^2+1)(\tanh(x+y+z-wt)^2-1)^2}{v_1(\tanh[x+y+z-wt]^2+1)^2}.$$

In addition to soliton-like solutions of Eq. (2) , we can also obtain some trigonometric-function solutions of Eq. (2) in the limited case when the modulus $m\to 0$. For instance,

Case 12. If we choose $P:1,\ \ Q:-(1+m^2),\ \ F(\xi)=ns\xi,$ then we have

$$u_{12}=\frac{-4(1+m^2)v_3w^2-4(1+m^2)v_3-w^2+v_2}{-2v_1}+\frac{v_4+v_5}{-2v_1}-\frac{6v_3(w^2+1)}{v_1}ns^2\xi.$$

In the limit case when $m\to 0$, the periodic solution of Eq. (2) can be written as

$$u_{12}(x,y,z,t)=\frac{-4v_3w^2-4v_3-w^2+v_2+v_4+v_5}{-2v_1}-\frac{6v_3(w^2+1)}{v_1}\csc(x+y+z-wt)^2.$$

Case 13. Assigning $P:1,\ \ Q:-(1+m^2),$ then $F(\xi)=dc\xi,$ hence

$$u_{13}=\frac{-4(1+m^2)v_3w^2-4(1+m^2)v_3-w^2+v_2}{-2v_1}+\frac{v_4+v_5}{-2v_1}-\frac{6v_3(w^2+1)}{v_1}dc^2\xi.$$

For $m\to 0$, one of the traveling wave solutions of Eq. (2) can be evaluated as

$$u_{13}(x,y,z,t)=\frac{-4v_3w^2-4v_3-w^2+v_2+v_4+v_5}{-2v_1}-\frac{6v_3(w^2+1)}{v_1}\sec(x+y+z-wt)^2.$$

Case 14. For choices $P: 1-m^2,\ Q: 2-m^2,\ F(\xi)=sc\xi,$ then

$$u_{14}=\frac{4(2-m^2)v_3w^2+4(2-m^2)v_3-w^2+v_2}{-2v_1}+\frac{v_4+v_5}{-2v_1}-\frac{6(1-m^2)v_3(w^2+1)}{v_1}sc^2\xi.$$

As long as $m\to 0$, we can evaluate one of the traveling wave solutions of Eq. (2) as

$$u_{14}(x,y,z,t)=\frac{8v_3w^2+8v_3-w^2+v_2+v_4+v_5}{-2v_1}-\frac{6v_3(w^2+1)}{v_1}\tan(x+y+z-wt)^2.$$

Case 15. When $P: 1,\ Q: (2-m^2)$ for all the table cases, F is obtained as $F(\xi)=cs\xi,$ in this way the solution may be expressed as

$$u_{15}=\frac{4(2-m^2)v_3w^2+4(2-m^2)v_3-w^2+v_2}{-2v_1}+\frac{v_4+v_5}{-2v_1}-\frac{6v_3(w^2+1)}{v_1}cs^2\xi.$$

Considering $m\to 0$, one of the periodic solutions of Eq. (2) can be attained as

$$u_{15}(x,y,z,t)=\frac{8v_3w^2+8v_3-w^2+v_2+v_4+v_5}{-2v_1}-\frac{6v_3(w^2+1)}{v_1}\cot(x+y+z-wt)^2.$$

Case 16. Regarding $P: 1/4,\ Q: (1-2m^2)/2$ this choices correspond to $F(\xi)=ns\xi+cs\xi,$ hence

$$u_{16}=\frac{2(1-2m^2)v_3w^2+2(1-2m^2)v_3-w^2+v_2}{-2v_1}-+\frac{v_4+v_5}{-2v_1}-\frac{3v_3(w^2+1)}{2v_1}(ns\xi+cs\xi)^2.$$

For $m\to 0$, we acquire the periodic solution of Eq. (2) as

$$u_{16}(x,y,z,t)=-\frac{1}{2}\frac{2v_3w^2+2v_3-w^2+v_2+v_4+v_5}{v_1}-\frac{3v_3(w^2+1)}{2v_1}(\cot[x+y+z-wt]+\csc[x+y+z-wt])^2.$$

Case 17. If we get $P: 1/4,\ Q: (1-2m^2)/2,$ then $F(\xi)=ns\xi-cs\xi,$ so

$$u_{17}=\frac{2(1-2m^2)v_3w^2+2(1-2m^2)v_3-w^2+v_2}{-2v_1}+\frac{v_4+v_5}{-2v_1}-\frac{3v_3(w^2+1)}{2v_1}(ns\xi-cs\xi)^2.$$

As $m\to 0$, we obtain the traveling wave solution of Eq. (2) as

$$u_{17}(x,y,z,t)=\frac{2v_3w^2+2v_3-w^2+v_2+v_4+v_5}{-2v_1}-\frac{3v_3(w^2+1)}{2v_1}(\csc[x+y+z-wt]-\cot[x+y+z-wt])^2.$$

Case 18. Supposing $P: (1-m^2)/4,\ Q: (1+m^2)/2$ so, $F(\xi)=nc\xi+sc\xi,$ then

$$u_{18}=\frac{2(1+m^2)v_3w^2+2(1+m^2)v_3-w^2+v_2}{-2v_1}+\frac{v_4+v_5}{-2v_1}-\frac{3(1-m^2)v_3(w^2+1)}{2v_1}(nc\xi+sc\xi)^2.$$

For $m\to 0$, we attain one of the traveling wave solutions of Eq. (2) as

$$u_{18}(x,y,z,t)=\frac{2v_3w^2+2v_3-w^2+v_2+v_4+v_5}{-2v_1}-\frac{3v_3(w^2+1)}{2v_1}(\sec[x+y+z-wt]+\tan[x+y+z-wt])^2.$$

Case 19. If we take $P: (1-m^2)/4,\ Q: (1+m^2)/2$ for all the table cases, $F(\xi)=nc\xi-sc\xi,$ therefore

$$u_{19}=\frac{2(1+m^2)v_3w^2+2(1+m^2)v_3-w^2+v_2}{-2v_1}+\frac{v_4+v_5}{-2v_1}-\frac{3(1-m^2)v_3(w^2+1)}{2v_1}(nc\xi-sc\xi)^2.$$

In the limit case when $m\to 0$, the periodic solution of Eq. (2) can be acquired as

$$u_{19}(x,y,z,t)=\frac{2v_3w^2+2v_3-w^2+v_2+v_4+v_5}{-2v_1}-\frac{3v_3(w^2+1)}{2v_1}(\sec(x+y+z-wt)-\tan(x+y+z-wt))^2.$$

Case 20. When $P:1/4,\ Q:(1-2m^2)/2$ it can be deducted for all the table cases, $F(\xi)=sn\xi/(1+cn\xi),$ the solution can be evaluated as

$$u_{20}=\frac{2(1-2m^2)v_3w^2+2(1-2m^2)v_3-w^2+v_2}{-2v_1}+\frac{v_4+v_5}{-2v_1}-\frac{3v_3(w^2+1)}{2v_1}\left(\frac{sn\xi}{1+cn\xi}\right)^2.$$

Considering $m\to 0$, one of the periodic solutions of Eq. (2) can be stated as

$$u_{20}(x,y,z,t)=\frac{2v_3w^2+2v_3-w^2+v_2+v_4+v_5}{-2v_1}-\frac{3v_3(w^2+1)\sin[x+y+z-wt]^2}{2v_1(1+\cos[x+y+z-wt])^2} \tag{14}$$

Case 21. Suppose that $P:1/4,\ Q:1-2m^2/2,\ F(\xi)=sn\xi/1-cn\xi,$ then

$$u_{21}=\frac{2(1-2m^2)v_3w^2+2(1-2m^2)v_3-w^2+v_2}{-2v_1}+\frac{v_4+v_5}{-2v_1}-\frac{3v_3(w^2+1)}{2v_1}\left(\frac{sn\xi}{1-cn\xi}\right)^2.$$

If $m\to 0$, we get the traveling wave solution of Eq. (2) as

$$u_{21}(x,y,z,t)=\frac{2v_3w^2+2v_3-w^2+v_2+v_4+v_5}{-2v_1}-\frac{3v_3(w^2+1)\sin(x+y+z-wt)^2}{2v_1(1-\cos(x+y+z-wt))^2}.$$

Case 22. Setting $P:(1-m^2)/4,\ Q:(1+m^2)/2,$ then $F(\xi)=cn\xi/1+sn\xi,$ because of this setting

$$u_{22}=\frac{2(1+m^2)v_3w^2+2(1+m^2)v_3-w^2+v_2}{-2v_1}+\frac{v_4+v_5}{-2v_1}-\frac{3(1-m^2)v_3(w^2+1)}{2v_1}\left(\frac{cn\xi}{1+sn\xi}\right)^2.$$

For $m\to 0$, one of the periodic solutions of Eq. (2) can be evaluated as

$$u_{22}(x,y,z,t)=\frac{2v_3w^2+2v_3-w^2+v_2+v_4+v_5}{-2v_1}-\frac{3v_3(w^2+1)\cos(x+y+z-wt)^2}{2v_1(1+\sin(x+y+z-wt))^2}.$$

Case 23. Considering $P:(1-m^2)/4,\ Q:(1+m^2)/2,$ then $F(\xi)=cn\xi/1-sn\xi$, so

$$u_{23}=\frac{2(1+m^2)v_3w^2+2(1+m^2)v_3-w^2+v_2}{-2v_1}+\frac{v_4+v_5}{-2v_1}-\frac{3(1-m^2)v_3(w^2+1)}{2v_1}\left(\frac{cn\xi}{1-sn\xi}\right)^2.$$

As long as $m\to 0$, we can obtain one of the traveling wave solutions of Eq. (2) as

$$u_{23}(x,y,z,t)=\frac{2v_3w^2+2v_3-w^2+v_2+v_4+v_5}{-2v_1}-\frac{3v_3(w^2+1)\cos(x+y+z-wt)^2}{2v_1(1-\sin(x+y+z-wt))^2}.$$

Case 24. For choices $P:(C^2m^4-(B^2+C^2)m^2+B^2)/4,\ Q:(m^2+1)/2$ it may be deducted for all the table cases, $F(\xi)=(\sqrt{[(B^2-C^2)/(B^2-C^2m^2)]}+sn\xi)/(Bcn\xi+Cdn\xi),$ therefore

$$u_{24}=\frac{2(m^2+1)v_3w^2+2(m^2+1)v_3-w^2+v_2+v_4+v_5}{-2v_1}$$
$$-\frac{3(C^2m^4-(B^2+C^2)m^2+B^2)v_3(w^2+1)}{2v_1}\left(\frac{\sqrt{\frac{(B^2-C^2)}{(B^2-C^2m^2)}}+sn\xi}{Bcn\xi+Cdn\xi}\right)^2.$$

For $m\to 0$, one of the periodic solutions of Eq. (2) can be evaluated as

$$u_{24}(x,y,z,t)=\frac{2v_3w^2+2v_3-w^2+v_2+v_4+v_5}{-2v_1}-\frac{3B^2v_3(w^2+1)\left(\sqrt{\frac{B^2-C^2}{B^2}}+\sin(x+y+z-wt)\right)^2}{2v_1(B\cos(x+y+z-wt)+C)^2}.$$

Case 25. When $P{:}B^2{+}C^2m^2/4,\ Q{:}(1/2){-}m^2,$ then $F(\xi){=}\left(\sqrt{[C^2m^2{+}B^2{-}C^2]/[(B^2{+}C^2m^2]}{+}cn\xi\right)/(Bsn\xi{+}Cdn\xi),$ hence

$$u_{25} = \frac{4\left(\frac{1}{2}-m^2\right)v_3w^2 + 4\left(\frac{1}{2}-m^2\right)v_3 - w^2 + v_2 + v_4 + v_5}{-2v_1} - \frac{3(B^2+C^2m^2)v_3(w^2+1)}{2v_1}\left(\frac{\sqrt{\frac{(C^2m^2+B^2-C^2)}{(B^2+C^2m^2)}} + cn\xi}{Bsn\xi + Cdn\xi}\right)^2.$$

Moreover for $m \to 0$, one of the travelling wave solutions of Eq. (2) can be obtained as

$$u_{25}(x,y,z,t) = \frac{2v_3w^2 + 2v_3 - w^2 + v_2 + v_4 + v_5}{-2v_1} - \frac{3B^2v_3(w^2+1)\left(\sqrt{\frac{B^2-C^2}{B^2}} + \cos[x+y+z-wt]\right)^2}{2v_1(B\sin[x+y+z-wt]+C)^2}.$$

Case 26. Supposing $P:(B^2+C^2)/4,\ Q: m^2/2-1$, for all the table cases, $F(\xi) = (\sqrt{[(B^2+C^2-C^2m^2]/[(B^2+C^2]} + dn\xi)/(Bsn\xi + Ccn\xi)$, then

$$u_{26} = \frac{4\left(\frac{m^2}{2}-1\right)v_3w^2 + 4\left(\frac{m^2}{2}-1\right)v_3 - w^2 + v_2 + v_4 + v_5}{-2v_1} - \frac{3(B^2+C^2)v_3(w^2+1)}{2v_1}\left(\frac{\sqrt{\frac{(B^2+C^2-C^2m^2)}{(B^2+C^2)}} + dn\xi}{Bsn\xi + Ccn\xi}\right)^2.$$

If $m \to 0$, we attain one of the travelling wave solutions of Eq. (2) as

$$u_{26}(x,y,z,t) = \frac{-4v_3w^2 - 4v_3 - w^2 + v_2 + v_4 + v_5}{-2v_1} - \frac{6(B^2+C^2)v_3(w^2+1)}{v_1(B\sin[x+y+z-wt] + C\cos[x+y+z-wt])^2}.$$

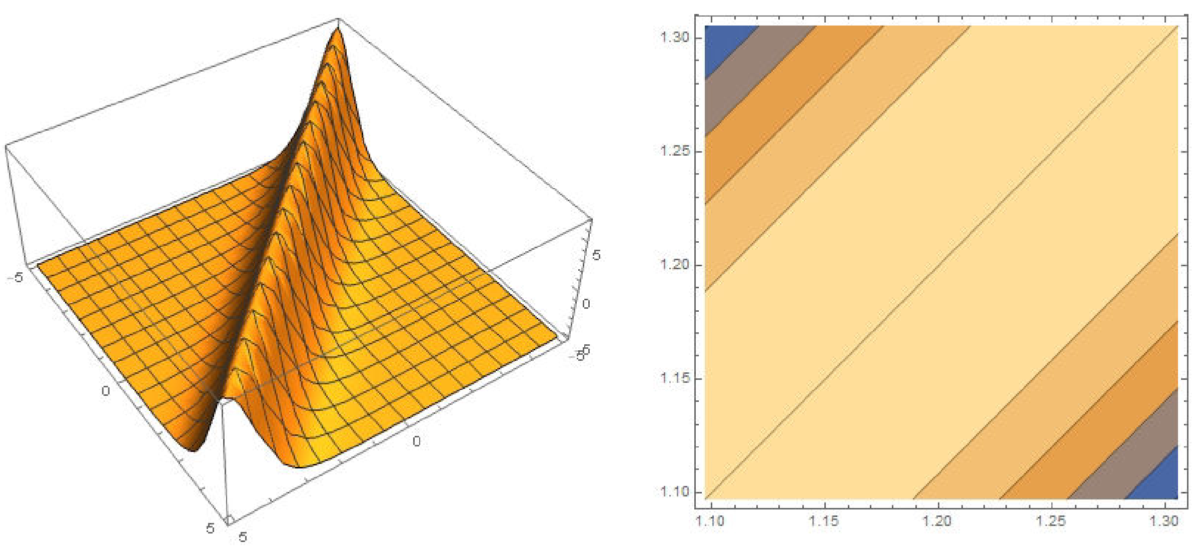

FIGURE 1. 3-D and contour plots of a dark solitary wave solution of Eq. (13) with $v_1 = v_2 = v_3 = v_4 = v_5 = w = 1,$ and in the interval $-5 \le x \le 5$ and $-5 \le t \le 5$.

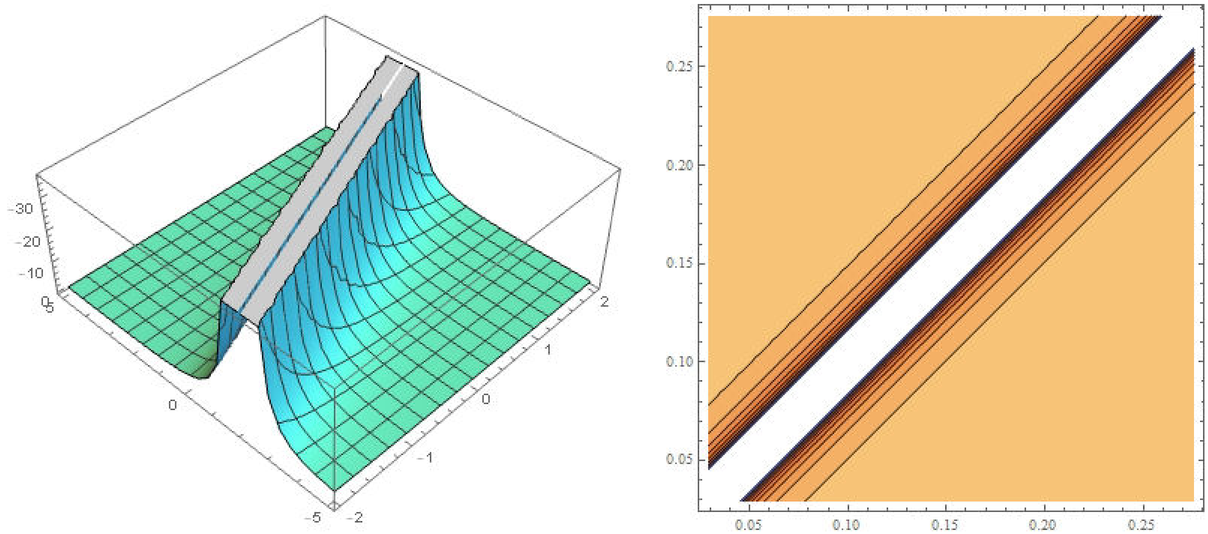

FIGURE 2. 3-D and contour of a hyperbolic function solution of Eq. (14) with $v_1 = v_2 = v_3 = v_4 = v_5 = w = 1,$ and in the interval $-5 \leq x \leq 5$ and $-2 \leq t \leq 2$.

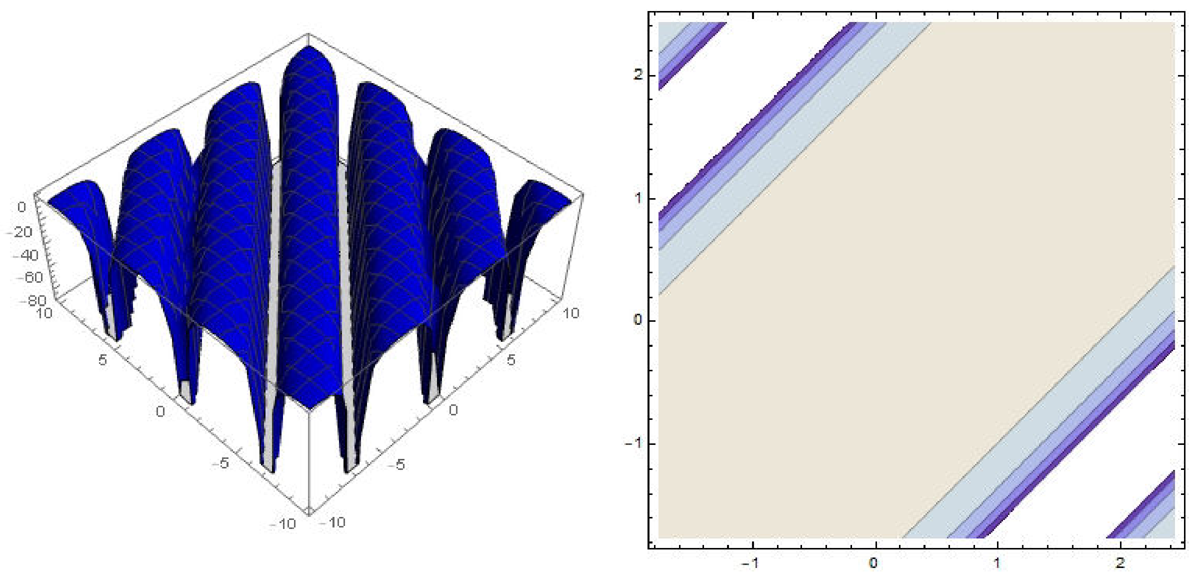

FIGURE 3. 3-D and contour plots of a periodic wave solution of Eq. (15) with $v_1 = v_2 = v_3 = v_4 = v_5 = w = 1,$ and in the interval $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$.

3. Conclusion

This study shows how to apply the Jacobi elliptic function method to seek for the travelling wave solutions of (3+1)- dimensional Boussinesq equation with dual dispersion. By applying wave transform, the equation is reduced to one dimension. When special values are given to P, Q, R in the auxiliary Eq. (6), many solution classes are obtained. In this way, we constructed twenty six sets of solutions such as dark solitons, bright solitons and trigonometric function solutions. To our knowledge, the obtained solutions which appear in two different forms, such as $(m \to 0)$ trigonometric and $(m \to 1)$ hyperbolic, are original. It is especially stated by the authors that obtained solutions satisfy the original equation. Besides, 3D and contour graphs of bright and dark soliton and traveling wave solutions are presented for better clarification. This study shows that this algorithm is powerful and effective to find analytical solutions, so it can be also applied to many other partial differential equations in physical phenomenon and applied sciences. In future studies, this method can be applied to many fractional differential equations of different dimensions.

Data availibility expression: No data were used in this study.

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