Local time in Lagrangian mechanics

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Received 16 February 2022; accepted 16 July 2022

We show that under the replacement of the time by a local time in the Lagrange equations, the form of the equations is maintained if the Lagrangian does not depend explicitly on the time. We also study the corresponding modifications in the Hamilton equations and in the Hamilton–Jacobi equation.

Keywords: Lagrangian formulation; local time; Hamiltonian formulation; Schrödinger equation.

DOI: https://doi.org/10.31349/RevMexFisE.19.020209

1. Introduction

In the standard approach followed in classical dynamics, the equations of motion of a mechanical system form a set of ordinary differential equations that determine the configuration of the system as a function of time. However, in some cases it is useful to make use of another parameter, sometimes as an intermediate step. A well-known and very important example is the Kepler problem, where, using polar coordinates (r, θ) in the plane of the orbit, it is considerably simpler to find the expression for r in terms of θ than in terms of the time.

The advantages of replacing some variables by others are profusely exploited in mathematics and theoretical physics. Probably, the most common method employed in the calculation of integrals is that of substitution; an adequate change of the integration variable transforms a given integral into a simpler one. Very often, when dealing with ordinary differential equations, by suitably transforming the variables, an apparently complicated equation is converted into another equation whose solutions are already known. In the context of the complex variable theory, the conformal transformations are employed to relate different boundary value problems related to the Laplace equation.

In this article we investigate the modifications required in the Lagrange equations if the time, t, is replaced by another parameter, τ. Usually, such a replacement is called *global* if τ is some function of t [$\tau = F(t)$] or *local* if the relation between $\mathrm{d}\tau$ and $\mathrm{d}t$ may depend on the configuration, that is, $\mathrm{d}t = f(q_i)\mathrm{d}\tau$, where f is some function of the coordinates (hence, the relation between t and τ depends on the path followed in the configuration space). The case of global transformations is relatively simple: If one changes t by $\tau = F(t)$, the Lagrange equations maintain their form with the Lagrangian, L, replaced by $L\,\mathrm{d}t/\mathrm{d}\tau$ (see, *e.g.*, Ref. [1], Sec. 2.4). In this paper, we show that if the time is replaced by a local time, the Lagrange equations maintain their form if the Lagrangian does not depend explicitly on the time.

The idea of a local time arises in a natural manner in the study of the so-called Liouville systems (see, *e.g.*, Refs. [2–5]) and, in the case of a Stäckel system, it is convenient to introduce a different local time for each coordinate (see, *e.g.*, Refs. [4, 6]). These two classes of systems are characterized by time-independent Lagrangians of certain specific forms, which appear in connection with the separability of the Hamilton–Jacobi equation. The results established here are applicable to any system with a Lagrangian that does not depend explicitly on the time, without any further restriction. In the path integral approach to quantum dynamics, it is useful to introduce a local time accompanying a change of coordinates (see, *e.g.*, Ref. [7] and the references cited therein).

In Sec. 2 we show that in the case of a Lagrangian that does not depend explicitly on the time, the replacement of the real time by a local time maintains the form of the Lagrange equations if the Lagrangian is suitably modified, and we give two examples. In Sec. 3 we analyze the implications of these replacements on the Hamilton equations and the Hamilton–Jacobi equation. Finally, the possibility of using a local time in the Schrödinger equation is considered.

2. The Lagrange equations with a local time

In this section we shall study the effect on the Lagrange equations of the substitution of the time, t, by a local time, τ, assuming that

$$\mathrm{d}t = f(q_i)\mathrm{d}\tau, \tag{1}$$

where f is some function that may depend on the coordinates only. We begin by writing down the Lagrange equations for an arbitrary Lagrangian $L(q_i, \dot{q}_i, t)$ in the form

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{f}\frac{\partial (fL)}{\partial \dot{q}_i}\right] - \frac{1}{f}\frac{\partial (fL)}{\partial q_i} + \frac{L}{f}\frac{\partial f}{\partial q_i}. \end{aligned} \tag{2}$$

With the introduction of the local time τ we have new velocities, $q_i' \equiv \mathrm{d}q_i/\mathrm{d}\tau$, which, according to Eq. (1) and the

chain rule, are related to the usual generalized velocities, $\dot{q}_i$, by means of

$$q_i' = f\dot{q}_i, \tag{3}$$

and it is necessary to distinguish the partial derivatives calculated using the set of variables $(q_i, \dot{q}_i)$ from those calculated using the set of variables (q_i, q_i'). Making use of the formula for the total differential of a function of several variables and Eq. (3) we find that, for an arbitrary function $F(q_i, q_i')$,

$$\begin{aligned}
\mathrm{d}F &= \left(\frac{\partial F}{\partial q_i}\right)_{q,q'} \mathrm{d}q_i + \left(\frac{\partial F}{\partial q_i'}\right)_{q,q'} \mathrm{d}q_i' \\
&= \left(\frac{\partial F}{\partial q_i}\right)_{q,q'} \mathrm{d}q_i + \left(\frac{\partial F}{\partial q_i'}\right)_{q,q'} \left(f\mathrm{d}\dot{q}_i + \dot{q}_i \frac{\partial f}{\partial q_j}\mathrm{d}q_j\right) \\
&= \left[\left(\frac{\partial F}{\partial q_i}\right)_{q,q'} + \dot{q}_j \left(\frac{\partial F}{\partial q_j'}\right)_{q,q'} \frac{\partial f}{\partial q_i}\right] \mathrm{d}q_i \\
&\quad + f\left(\frac{\partial F}{\partial q_i'}\right)_{q,q'} \mathrm{d}\dot{q}_i,
\end{aligned}$$

with sum over repeated indices and where we have indicated with subscripts the coordinates kept fixed during the differentiation (as is commonly indicated in thermodynamics). This last expression must coincide with

$$\mathrm{d}F = \left(\frac{\partial F}{\partial q_i}\right)_{q,\dot{q}} \mathrm{d}q_i + \left(\frac{\partial F}{\partial \dot{q}_i}\right)_{q,\dot{q}} \mathrm{d}\dot{q}_i.$$

Hence,

$$\left(\frac{\partial F}{\partial q_i}\right)_{q,\dot{q}} = \left(\frac{\partial F}{\partial q_i}\right)_{q,q'} + \dot{q}_j \left(\frac{\partial F}{\partial q_j'}\right)_{q,q'} \frac{\partial f}{\partial q_i}, \tag{4}$$

and

$$\left(\frac{\partial F}{\partial \dot{q}_i}\right)_{q,\dot{q}} = f\left(\frac{\partial F}{\partial q_i'}\right)_{q,q'}. \tag{5}$$

(Equations (4) and (5) can also be obtained making use of the chain rule.)

Multiplying Eq. (2) by f, with the aid of Eqs. (5), (4) and (1) we find

$$\begin{aligned}
0 &= f\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial(fL)}{\partial q_i'}\right)_{q,q'} - \left[\left(\frac{\partial(fL)}{\partial q_i}\right)_{q,q'} + \dot{q}_j\left(\frac{\partial(fL)}{\partial q_j'}\right)_{q,q'}\frac{\partial f}{\partial q_i}\right] + L\frac{\partial f}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial(fL)}{\partial q_i'}\right)_{q,q'} - \left(\frac{\partial(fL)}{\partial q_i}\right)_{q,q'} \\
&\quad - \frac{\dot{q}_j}{f}\left(\frac{\partial(fL)}{\partial \dot{q}_j}\right)_{q,\dot{q}}\frac{\partial f}{\partial q_i} + L\frac{\partial f}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial(fL)}{\partial q_i'}\right)_{q,q'} - \left(\frac{\partial(fL)}{\partial q_i}\right)_{q,q'} - \left[\dot{q}_j\left(\frac{\partial L}{\partial \dot{q}_j}\right)_{q,\dot{q}} - L\right]\frac{\partial f}{\partial q_i} \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial(fL)}{\partial q_i'}\right)_{q,q'} - \left(\frac{\partial(fL)}{\partial q_i}\right)_{q,q'} - J\frac{\partial f}{\partial q_i},
\end{aligned} \tag{6}$$

where

$$J \equiv \dot{q}_j\left(\frac{\partial L}{\partial \dot{q}_j}\right)_{q,\dot{q}} - L. \tag{7}$$

As is well known, if L does not depend explicitly on the time, J is a conserved quantity (frequently it coincides with the total energy). Thus, if we restrict ourselves to Lagrangians that do not depend explicitly on t, denoting by E the *constant* value of J (determined by the initial conditions), from Eqs. (6) it follows that the Lagrange equations (2) are equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial[(L+E)f]}{\partial q_i'}\right)_{q,q'} - \left(\frac{\partial[(L+E)f]}{\partial q_i}\right)_{q,q'} = 0, \tag{8}$$

which also have the form of the Lagrange equations. Thus, if $\partial L/\partial t = 0$, the Lagrange equations are form-invariant under changes of parameter (1), with the Lagrangian L replaced by

$$\tilde{L}_E \equiv (L+E)f. \tag{9}$$

Since L does not depend explicitly on the time, the Lagrangian $\tilde{L}_E$ does not depend explicitly on τ and, therefore, $\tilde{J} \equiv q_i'(\partial\tilde{L}_E/\partial q_i') - \tilde{L}_E$ is also conserved [*cf.* (7)]. Making use of Eqs. (3), (5), (9) and (7) we see that

$$\begin{aligned}
\tilde{J} &= f\dot{q}_i\frac{1}{f}\frac{\partial[(L+E)f]}{\partial \dot{q}_i} - (L+E)f \\
&= f\left(\dot{q}_i\frac{\partial L}{\partial \dot{q}_i} - L - E\right) = (J-E)f = 0,
\end{aligned}$$

since we imposed the condition $J = E$ in order to get Eqs. (8).

2.1. Example. The Morse potential

A simple example of the use of a local time is provided by the Morse potential

$$V(q) = V_0(1 - \mathrm{e}^{-\alpha q})^2,$$

where V_0 and α are positive constants. In order to eliminate the exponential function appearing in this potential one introduces the new coordinate

$$x = \mathrm{e}^{-\alpha q/2},$$

in terms of which the Lagrangian $L = (1/2)m\dot{q}^2 - V(q)$ takes the form

$$L = \frac{m}{2}\left(\frac{2}{\alpha x}\right)^2 \dot{x}^2 - V_0(1-x^2)^2. \qquad (10)$$

As we have shown, under the substitution of the time by a local time τ defined by Eq. (1), L has to be replaced by the Lagrangian $\tilde{L}_E = (L+E)f$ which in this case is given by [see (3)]

$$\tilde{L}_E = \left[\frac{m}{2}\left(\frac{2}{\alpha x}\right)^2 \dot{x}^2 - V_0(1-x^2)^2 + E\right] f$$
$$= \left[\frac{m}{2}\left(\frac{2}{\alpha x}\right)^2 \frac{1}{f^2}(x')^2 - V_0(1-x^2)^2 + E\right] f,$$

where E is a possible value of the Jacobi integral J. The first term of this Lagrangian acquires the usual form of a kinetic energy, $(1/2)m(x')^2$, if we take $f = (2/\alpha x)^2$. Thus,

$$\tilde{L}_E = \frac{m}{2}(x')^2 + \frac{4(E-V_0)}{\alpha^2 x^2} + \frac{8V_0}{\alpha^2} - \frac{4V_0 x^2}{\alpha^2}.$$

It may be noticed that, if $E < V_0$, this Lagrangian has the form of the radial Lagrangian of a two-dimensional isotropic harmonic oscillator. The only equation of motion obtainable from $\tilde{L}_E$ is equivalent to $\tilde{J} = 0$, which in this case reads

$$\frac{m}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 - \frac{4(E-V_0)}{\alpha^2 x^2} - \frac{8V_0}{\alpha^2} + \frac{4V_0 x^2}{\alpha^2} = 0.$$

With the aid of the change of variable $u = x^2$, from this last equation one readily finds that, if $E < V_0$ (which corresponds to a bounded motion),

$$x^2 = 1 + \sqrt{\frac{E}{V_0}}\cos\left(2\sqrt{\frac{8V_0}{m\alpha^2}}\,\tau\right).$$

Then, making use of this expression one can find the relation between t and τ.

2.2. Example. Particle in a central force field

Another illustrative example is given by the problem of a particle in a central field of force. In spherical coordinates, (r,θ,ϕ), the standard Lagrangian is given by

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) - V(r), \qquad (11)$$

where $V(r)$ is the potential. Making use of Eq. (3) we have

$$\tilde{L}_E = (L+E)f = \frac{r^2}{f}\frac{m}{2}\left[\frac{(r')^2}{r^2} + (\theta')^2 + \sin^2\theta\,(\phi')^2\right]$$
$$+ \big(E - V(r)\big)f.$$

By inspection, one finds that a convenient choice is $f = r^2$. In this manner, the equation of motion (in terms of τ) for the radial coordinate does not involve the angles, and vice versa. In fact, $\tilde{L}_E$ is the sum of a function of (r,r') and a function of (θ,θ',ϕ')

$$\tilde{L}_E = \frac{m}{2}\frac{(r')^2}{r^2} + \big(E - V(r)\big)r^2 + \frac{m}{2}\left[(\theta')^2 + \sin^2\theta(\phi')^2\right].$$

Furthermore, the Jacobi integral $\tilde{J}$, which has to be equal to zero, is the sum of two separately conserved quantities:

$$\frac{m}{2}\frac{(r')^2}{r^2} - \big(E - V(r)\big)r^2, \qquad (12)$$

and [see (3)]

$$\frac{m}{2}\left[(\theta')^2 + \sin^2\theta\,(\phi')^2\right] = \frac{m}{2}\big(r^4\dot{\theta}^2 + r^4\sin^2\theta\,\dot{\phi}^2\big).$$

This last expression can be recognized as $|\mathbf{L}|^2/2m$, where $\mathbf{L}$ is the angular momentum of the particle about the origin. (In fact, we know that in a central force field, each Cartesian component of $\mathbf{L}$ is conserved, not only its magnitude.) Then, equating (12) to $-|\mathbf{L}|^2/2m$ we obtain the well-known equation for the orbit

$$\pm\mathrm{d}\tau = \frac{\mathrm{d}r}{r^2\sqrt{\dfrac{2E}{m} - \dfrac{|\mathbf{L}|^2}{m^2r^2} - \dfrac{2V(r)}{m}}}.$$

Hence, in this example, the local time τ is the angle swept by the position vector of the particle, *measured in the plane of the orbit*, multiplied by the constant factor $m/|\mathbf{L}|$.

3. The Hamiltonian version

The Lagrangian $\tilde{L}_E$ is regular if and only if L is regular and, following the usual definitions, the Hamiltonian, $\tilde{H}_E$, corresponding to $\tilde{L}_E$, is

$$\tilde{H}_E = \tilde{p}_i q_i' - \tilde{L}_E,$$

with [see Eq. (5)]

$$\tilde{p}_i \equiv \frac{\partial\tilde{L}_E}{\partial q_i'} = \frac{1}{f}\frac{\partial[(L+E)f]}{\partial\dot{q}_i} = \frac{\partial L}{\partial\dot{q}_i} = p_i,$$

i.e., the canonical momenta determined by $\tilde{L}_E(q_i,q_i')$ coincide with those determined by $L(q_i,\dot{q}_i)$. Hence [see Eq. (3)]

$$\tilde{H}_E = p_i(f\dot{q}_i) - (L+E)f = (p_i\dot{q}_i - L - E)f = (H-E)f,$$

and the coordinates and momenta, in terms of τ, are determined by the Hamilton equations

$$\frac{\mathrm{d}q_i}{\mathrm{d}\tau} = \frac{\partial\tilde{H}_E}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}\tau} = -\frac{\partial\tilde{H}_E}{\partial q_i}. \qquad (13)$$

It may be pointed out that even though we are only interested in the hypersurface where $\tilde{H}_E = 0$ (which corresponds to $\tilde{J} = 0$), the vanishing of this function does not mean that the partial derivatives of $\tilde{H}_E$ are equal to zero.

Equations (13) are a special case of the results presented in Sec. 4.3 of Ref. [1], where, by considering directly the Hamilton equations, it is shown that, in the case where the Hamiltonian does not depend explicitly on the time, the time can be replaced by a local time in such a way that $\mathrm{d}t$ and $\mathrm{d}\tau$ are related by an arbitrary non-vanishing function of coordinates and momenta. In the approach followed there, the Hamilton equations are treated directly without going through the Lagrangian formalism first.

3.1. The Hamilton–Jacobi equation

Following the standard steps, once we have a Hamiltonian we can write down the corresponding Hamilton–Jacobi equation. In its usual form (that is, for the generating function of a canonical transformation depending on the old coordinates and the new momenta), the Hamilton–Jacobi equation corresponding to $\tilde{H}_E(q_i, p_i')$ is

$$\tilde{H}_E\left(q_i, \frac{\partial W}{\partial q_i}\right) + \frac{\partial W}{\partial \tau} = 0. \tag{14}$$

By virtue of Eqs. (4), we do not have to specify the set of coordinates being used, since W is a function of q_i and τ only.

Using the fact that $\tilde{H}_E = (H - E)f$ and that the only value of interest for $\tilde{H}_E$ is zero, Eq. (14) reduces to

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) = E, \tag{15}$$

which is just the "time-independent" Hamilton–Jacobi equation for the Hamilton characteristic function, W. As is well known, Eq. (15) follows from the Hamilton–Jacobi equation,

$$H\left(q_i, \frac{\partial S}{\partial q_i}\right) + \frac{\partial S}{\partial t} = 0$$

if

$$S = W(q_i) - Et.$$

Thus, we recover the standard formulas, but *all reference to the local time has disappeared* in Eq. (15), in spite of the fact that we started with $\tilde{L}_E$ and $\tilde{H}_E$, whose definitions arose from the introduction of the local time τ. However, we can have τ back into play. For instance, for the Lagrangian (11), Eq. (15) takes the form

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial W}{\partial \phi}\right)^2\right] + V(r) = E$$

and this equation admits separable solutions of the form $W = A(r) + B(\theta) + l\phi$, where l is a separation constant. In fact, we obtain

$$W = \int \sqrt{2m\big(E - V(r)\big) - \frac{M}{r^2}}\,\mathrm{d}r + \int \sqrt{M - \frac{l^2}{\sin^2\theta}}\,\mathrm{d}\theta + l\phi,$$

where M is another separation constant. The partial derivatives of W with respect to M and l are now equated to two constants (interpreted as the new coordinates after performing the canonical transformation generated by W),

$$\frac{\partial W}{\partial l} = -\int \frac{l\,\mathrm{d}\theta}{\sin^2\theta\sqrt{M - \frac{l^2}{\sin^2\theta}}} + \phi = \text{const.}$$

and

$$\frac{\partial W}{\partial M} = -\int \frac{\mathrm{d}r}{2r^2\sqrt{2m\big(E - V(r)\big) - \frac{M}{r^2}}} + \int \frac{\mathrm{d}\theta}{2\sqrt{M - \frac{l^2}{\sin^2\theta}}} = \text{const.}$$

These last two equations are equivalent to

$$\frac{l\,\mathrm{d}\theta}{\sin^2\theta\sqrt{M - \frac{l^2}{\sin^2\theta}}} = \mathrm{d}\phi$$

and

$$\frac{\mathrm{d}r}{2r^2\sqrt{2m\big(E - V(r)\big) - \frac{M}{r^2}}} = \frac{\mathrm{d}\theta}{2\sqrt{M - \frac{l^2}{\sin^2\theta}}},$$

respectively. Hence, combining these equations we obtain

$$\frac{m\mathrm{d}r}{r^2\sqrt{2m\big(E - V(r)\big) - \frac{M}{r^2}}} = \frac{m\mathrm{d}\theta}{\sqrt{M - \frac{l^2}{\sin^2\theta}}} = \frac{m\sin^2\theta\mathrm{d}\phi}{l}, \tag{16}$$

and now we *introduce* the parameter τ by equating these expressions to $\mathrm{d}\tau$. In this way, we obtain three differential equations that determine r, θ and ϕ as functions of τ and one can verify that these equations are *precisely* those given by the first half of Eqs. (13). We can see directly that this parameter indeed coincides with the one introduced in Sec. 2.2. The partial derivative of the principal function, S, with respect to E also has to be a constant

$$\frac{\partial S}{\partial E} = \frac{\partial W}{\partial E} - t = \int \frac{m\mathrm{d}r}{\sqrt{2m\big(E - V(r)\big) - \frac{M}{r^2}}} - t = \text{const.},$$

which is equivalent to

$$\mathrm{d}t = \frac{m\mathrm{d}r}{\sqrt{2m\big(E - V(r)\big) - \frac{M}{r^2}}}$$

and, comparing with Eq. (16) we verify that $\mathrm{d}t = r^2\mathrm{d}\tau$.

It may be remarked that we have the liberty of multiplying all fractions appearing in (16) by *any* function of (r, θ, ϕ) and, then, define the resulting expressions as $\mathrm{d}\tau$. This freedom is related to the fact that any function f contained in $\tilde{L}_E$ disappears when we arrive at the Hamilton–Jacobi equation and, conversely, any function f can be recovered in this way.

3.2. Local time in the Schrödinger equation

A natural, and seemingly nontrivial, question is related to the possibility of using a local time in the Schrödinger equation, taking into account that, in classical mechanics, the relation between the real time and a local time depends on the path followed by the particle or system of particles, and that in quantum mechanics we cannot speak of the path of a particle. (However, as we know, in the path integral formulation of the quantum dynamics one makes use of classical paths. See below.)

If the Hamiltonian operator, H, does not depend explicitly on the time, the Schrödinger equation admits solutions of the form

$$\psi = \phi(q_i)\exp(-\mathrm{i}Et/\hbar),$$

where ϕ is a solution of the time-independent Schrödinger equation, $H\phi = E\phi$, which is the analog of Eq. (15) and, as the latter, does not contain any reference to the real or any local time. That this trivial approach is the appropriate one is justified by applying the path integral approach, as can be seen, *e.g.*, in Ref. [7]. In fact, in order to simplify the computations required in the path integral method, it is highly convenient, and natural, to use a local time transformation in conjunction with a coordinate transformation.

4. Concluding remarks

The results presented above are applicable when the Lagrangian does not depend explicitly on the time, even though, of course, the introduction of a local time is always possible; the conservation of J makes it possible to express the equations of motion parameterized by a local time in the form of the Lagrange equations.

As we have shown in Sec. 3.1, among the properties of the underrated Hamilton–Jacobi equation is its flexibility to use it in conjunction with local times.

A natural question is if, as in the case of global transformations, we can associate conserved quantities with local transformations depending on a continuous parameter that leave invariant the form of the Lagrangian.

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