Hafele and Keating in a fictious gravitational field

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The so called twin paradox is described in every special relativity (SR) textbook. Obviously, an exact calculation requires the application of the general theory of relativity (GR) but, thanks to some simplifying hypotheses concerning the acceleration and deceleration phases, even in SR it is possible to find the correct solution and that is "the twin who goes on space travel is the one who, returning to Earth, finds the aged brother." In a curved spacetime, instead, we have the algebraic sum between a kinematic and a gravitational effect. What happens in the presence of a gravitational field is, for example, well described by Hafele and Keating's experiment and, in this case, it may happen that the traveling brother can grow older than his brother at rest. Continuing in this pedagogical tradition, we consider the Hafele and Keating experiment in a fictious gravitational field of a rotating frame. The equality of the results between curved spacetime of a real gravitational field and flat spacetime of a fictitious one, in our opinion, is interesting to emphasize even today the great genius of Einstein and that is "*inertia is gravity in disguise*".

Keywords: Special relativity; general theory of relativity; inertia is gravity in disguise.

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1. Introduction

If we have a twin A that stays on Earth and a twin B that takes a journey at relativistic speed, when he returns to Earth, he finds that his brother is older. Naively applying the principle of relativity, all this seems paradoxical as every twin can be considered at rest and, therefore, each should find the other who has aged less. Obviously, as the twin B accelerates and decelerates, it is necessary to apply GR but, with some simplifying assumptions about the acceleration and deceleration phases, it is possible to obtain the correct solution also in SR. Without going into details, as it is beyond the aim of this paper, let us remember that the link between the twin times is

$$\frac{\Delta t_A}{\Delta t_B} = \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx 1 + \frac{v^2}{2c^2} > 1. \tag{1}$$

If we do not neglect the presence of the gravitational field, the gravitational potential affects proper time intervals and also the spatial intervals getting the algebraic sum between a kinematic and a gravitational effect [1]. It is well known that Hafele and Keating carried two atomic clocks on airplanes to detect relativistic effects on the time during terrestrial circumnavigations [2-4]. Let us consider three identical and synchronized clocks. Clock A is stationary on the Earth's surface while clocks H and K are on board two different airplanes that travel in a circular orbit around the Earth at a height h and with velocity v with respect to the ground. Hafele's airplane moves east while Keating's airplane moves west. For simplicity, clock A is at the equator and the orbit of the airplanes are also equatorial. Spacetime around the Earth is described by the following Schwarzschild metric

$$ds^2 = \left(1-\frac{2GM}{c^2 r}\right)c^2dt^2 - \left(1-\frac{2GM}{c^2 r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{2}$$

where θ and ϕ are the usual polar and azimuthal angles, t is the time coordinate measured by a stationary clock located infinitely far from the massive body, $r = C/2\pi$ where C is the circumference of a circle centered on the Earth. Finally, obviously, G is the Newtonian gravitational constant, M is the mass of the Earth and c is the speed of light. The incremental proper time of the clock A is

$$d\tau_A^2 = \frac{ds^2}{c^2}. \tag{3}$$

Choosing the axes so that the equatorial plane corresponds to $\theta = \pi/2$, after a rotation around the Earth's axis we get

$$\Delta\tau_A = \int\sqrt{1-\frac{2GM}{c^2R_E}-\frac{\Omega_E^2R_E^2}{c^2}}dt = \sqrt{1-\frac{2GM}{c^2R_E}-\frac{\Omega_E^2R_E^2}{c^2}}\Delta t\,, \tag{4}$$

where $\Omega_E = d\phi/dt$ is the angular velocity of the Earth of radius R_E. Since we have a non-relativistic speed and a weak gravitational field, we can write with an excellent approximation

$$\Delta\tau_A \approx \left(1-\frac{GM}{c^2R_E}-\frac{\Omega_E^2R_E^2}{2c^2}\right)\Delta t = \left(1+\frac{V}{c^2}\right)\Delta t\,, \tag{5}$$

where $V = -(GM/R_E) - (\Omega_E^2R_E^2/2)$.

2. Hafele and Keating clocks

By considering the Hafele's airplane, instead, we have

$$\Delta\tau_H=\sqrt{1-\frac{2GM}{c^2\,(R_E+h)}-\frac{\left[(R_E+h)\,\Omega_E+v\right]^2}{c^2}}\Delta t\,. \qquad (6)$$

For a precise calculation of the experiment it is necessary to remember the following data

$$\begin{cases} R_E \approx 6.37\cdot 10^6 \quad \text{m} \\ M \approx 5.98\cdot 10^{24} \quad \text{Kg} \\ \Omega_E \approx 7.3\cdot 10^{-5} \quad \text{rad}\cdot\text{s}^{-1} \\ h \approx 10^4 \quad \text{m} \\ v \approx 300 \quad \text{m}\cdot\text{s}^{-1} \end{cases}. \qquad (7)$$

Even for Hafele, we can write

$$\Delta\tau_H \approx \left(1-\frac{GM}{c^2\,(R_E+h)}-\frac{\left[(R_E+h)\,\Omega_E+v\right]^2}{2c^2}\right)\Delta t. \qquad (8)$$

Moveover, the proper distance dl is linked to the Schwarzschild radial coordinate by $dl = (dr/\sqrt{1-[2GM/c^2 r]})$ but, in this situation, we can consider $dl \approx dr$. Of course $h \ll R_E$ and, therefore, with an excellent approximation we can write

$$\Delta\tau_H \approx \left(1-\frac{GM}{c^2 R_E}-\frac{\left[\Omega_E R_E+v\right]^2}{2c^2}\right)\Delta t, \qquad (9)$$

with

$$\Delta\tau_H-\Delta\tau_A \approx -\frac{v^2+2\Omega_E R_E v}{2c^2}\Delta t. \qquad (10)$$

Instead, for Keating's airplane we have

$$\Delta\tau_K=\sqrt{1-\frac{2GM}{c^2\,(R_E+h)}-\frac{\left[(R_E+h)\,\Omega_E-v\right]^2}{c^2}}\Delta t\,, \qquad (11)$$

and with the same procedure we get

$$\Delta\tau_K-\Delta\tau_A \approx \frac{-v^2+2\Omega_E R_E v}{2c^2}\Delta t. \qquad (12)$$

It is always $\Delta\tau_H-\Delta\tau_A<0$ and the clock on the airplane will be delayed as its time runs more slowly. Instead $\Delta\tau_K-\Delta\tau_A$ can be positive or negative depending on the speed of the airplane. Finally we have

$$\Delta\tau_K-\Delta\tau_H \approx \frac{2\Omega_E R_E v}{c^2}\Delta t. \qquad (13)$$

We can also write

$$\frac{\Delta t_A}{\Delta t_H} \approx \left(1-\frac{GM}{c^2 R_E}-\frac{\Omega_E^2 R_E^2}{2c^2}\right) \times\left(1+\frac{GM}{c^2 R_E}+\frac{\left[\Omega_E R_E+v\right]^2}{2c^2}\right). \qquad (14)$$

By neglecting the infinitesimals of higher order, finally we get

$$\frac{\Delta t_A}{\Delta t_H} \approx 1+\frac{v^2}{2c^2}+\frac{\Omega_E R_E v}{c^2}. \qquad (15)$$

Similarly, it is easy to obtain

$$\frac{\Delta t_A}{\Delta t_K} \approx 1+\frac{v^2}{2c^2}-\frac{\Omega_E R_E v}{c^2}. \qquad (16)$$

3. Hafele and Keating on a rotating frame

In this section, let us remember that the Langevin metric, sometimes also called Langevin–Landau–Lifschitz metric (LLL), describes a rotating frame of reference or,in the light of the equivalence principle [5-13], a fictitious gravitational field [14-18]

$$ds^2=\left(1-\frac{\omega^2 r^2}{c^2}\right)c^2dt^2 - dr^2-r^2d\theta^2-dz^2-2r^2\omega d\theta dt. \qquad (17)$$

Obviously, the coordinate θ in LLL metric has a different meaning with respect to the Schwarzschild θ coordinate and the coordinated time is that measured in the inertial frame and therefore outside the fictious gravitational field. We have a stationary metric and, if $x^0=ct$, $x^1=r$, $x^2=\theta$, $x^3=z$, the non-vanishing components of the metric tensor are

$$\begin{cases} g_{00}=1-\frac{\omega^2 r^2}{c^2} \\ g_{11}=g_{33}=-1 \\ g_{02}=g_{20}=-\frac{\omega r^2}{c} \\ g_{22}=-r^2 \end{cases}. \qquad (18)$$

We have a Minkowski spacetime in curvilinear coordinates and, therefore, the curvature tensor vanishes

$$R_{ijkl}(t,r,\theta,z)=0. \qquad (19)$$

Instead, we have the following non-Euclidean spatial metric [14],

$$dl^2=\left(-g_{\alpha\beta}+\frac{g_{0\alpha}g_{0\beta}}{g_{00}}\right)dx^\alpha dx^\beta = dr^2+dz^2+\frac{r^2d\theta^2}{1-\frac{\omega^2 r^2}{c^2}}. \qquad (20)$$

After these premises, let A be an observer at rest on a rotating disk at radius r. The platform is rotating at angular velocity ω. Hafele and Keating are at the same radius r and they move in opposite directions. We could also consider airplanes at a radius $r_1<r$ and, therefore, subject to a lower "centrifugal gravitational force" but we do not do it in order not to burden the calculations. In this situation, the speed v is calculated with respect to the rotating disk, that is, with respect to LLL coordinates. For this reason

$$v=r\frac{d\theta}{dt}. \qquad (21)$$

We can see that the general incremental proper time equation is

$$d\tau^2 = \frac{ds^2}{c^2} = \left(1 - \frac{\omega^2 r^2}{c^2}\right) dt^2 - \frac{r^2 d\theta^2}{c^2} - \frac{2r^2 \omega d\theta dt}{c^2}. \quad (22)$$

Therefore we have

$$dt_A = \sqrt{1 - \frac{\omega^2 r^2}{c^2}} dt \approx \left(1 - \frac{1}{2}\frac{\omega^2 r^2}{c^2}\right) dt. \quad (23)$$

For Hafele clock we have

$$dt_H^2 = \left(1 - \frac{\omega^2 r^2}{c^2} - \frac{r^2 d\theta^2}{c^2 dt^2} - \frac{2r^2 \omega d\theta}{c^2 dt}\right) dt^2\,, \quad (24)$$

getting

$$dt_H \approx \left(1 - \frac{\omega^2 r^2}{2c^2} - \frac{v^2}{2c^2} - \frac{r\omega v}{c^2}\right) dt. \quad (25)$$

By negleting the infinitesimal of higher order, we have

$$\frac{dt_A}{dt_H} \approx \left(1 - \frac{1}{2}\frac{\omega^2 r^2}{c^2}\right)\left(1 + \frac{\omega^2 r^2}{2c^2} + \frac{v^2}{2c^2} + \frac{r\omega v}{c^2}\right) \approx 1 + \frac{v^2}{2c^2} + \frac{r\omega v}{c^2}. \quad (26)$$

Similarly

$$\frac{dt_A}{dt_K} \approx 1 + \frac{v^2}{2c^2} - \frac{r\omega v}{c^2} \quad (27)$$

As it can be seen, we obtain a formal analogy with relations (15) and (16). We conclude by pointing out that the term $\omega r v$ can be interpreted as a gravitational Coriolis potential [19-20].

4. Conclusions

In this manuscript we have observed the analogy between Hafele and Keating's experiment in a curved spacetime or in the flat spacetime of a non-inertial frame. All this could be useful, from a didactic point of view, to further underline, after more than a hundred years, Einstein's intuition that inertial and gravitational mass are physically identical.

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