The isothermal process peculiarities in the presence of gas leak from a vessel

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We construct a simple mathematical model which uses only the equation of state of an ideal gas and allows one to find the relationship between the gas pressure and volume in the case of a gas leak during its isothermal compression. Among other results, the pressure may have a saturation effect, when it remains constant under compression. We present a simple physical explanation of this effect.

Keywords: Ideal gas; isothermal process; gas leak; mathematical model.

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1. Introduction

The study of ideal gas laws is an integral part of any thermodynamics course [1]. It is well known that all these laws are of a model nature, that is, they are valid only under certain conditions. One of these conditions is the constancy of the mass of the gas in the vessel. Our long teaching experience shows that students often "absolutize" (in other words, they overuse) these laws, forgetting about the need to fulfill this condition. In most cases, such a requirement is met automatically. However, there are examples when the mass of a gas changes during a certain thermodynamic process.

Firstly, the mass of a gas can change due to chemical reactions. These phenomena are the subject of the study of chemical thermodynamics [2]. The mass can also change when gas is pumped into the container. The mass of a gas can change due to phase transformations. A striking consequence of such effects is the Antoine equation [3] describing the non-linear increase in saturated steam pressure with increasing temperature (at the same time, we often notice that students use by default the linear relationship between pressure and temperature of saturated steam).

Finally, the mass of gas can reduce due to its leakage from a vessel (for example, a cylinder of an internal combustion engine) into the environment. For instance, even such a simple problem can cause serious difficulties for students. Let us assume that the pressure and temperature of the air inside a building are equal, respectively, P_1 and T_1. What can we say about the air pressure in the room if the temperature rises to a value $T_2 > T_1$? The students commonly misuse Gay-Lussac's law and find the new pressure as $P_2 = (T_2/T_1)P_1$. In fact, due to the leakage of the room, part of the air will come out, so that the pressure will remain the same and equal to atmospheric pressure. In this case, only the ideal gas equation of state is applicable. Using this equation, we can calculate the decrease in air mass in the house.

For fictitious quasi-static transfers with a change in the number of particles in the system, the first law of thermodynamics can be generalized by introducing a chemical potential [4, 5]. However, the calculation of such a chemical potential is a separate and rather difficult problem.

In this paper, we construct a simple mathematical model which uses only the ideal gas equation of state and allows one to find the relationship between the gas pressure and volume in the case a gas leak during its isothermal compression. The issues covered in this paper will be useful to advanced undergraduate students studying thermodynamics or can be used in undergraduate projects.

2. The model

Let us consider a cylinder with base area A, inside which there is an ideal gas with molar mass M that is compressed by a piston. The temperature inside and outside the cylinder is constant all the time (the isothermal condition) and is equal to T_0. We assume that when compressing gas the speed of the piston motion is also constant and equal to v (v is so low as to satisfy the isothermal condition). There is also a hole in the wall of the cylinder with area $A_{\rm out}$ (Fig. 1).

Taking into account the equation of state of the ideal gas ($PV = (m/M)RT$), we get for the gas inside the cylinder:

$$\mathrm{d}P = \frac{RT_0}{M}\left(\frac{\mathrm{d}m}{V} - \frac{m\mathrm{d}V}{V^2}\right) = \frac{RT_0\mathrm{d}m}{MV} - \frac{P\mathrm{d}V}{V}, \qquad (1)$$

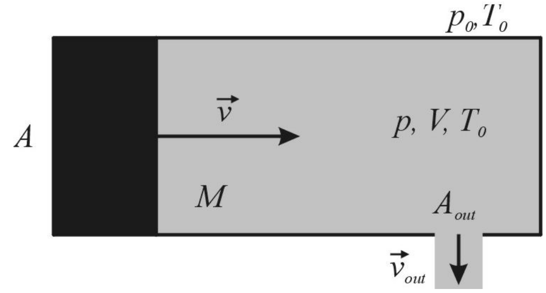

FIGURE 1. Geometry of the problem.

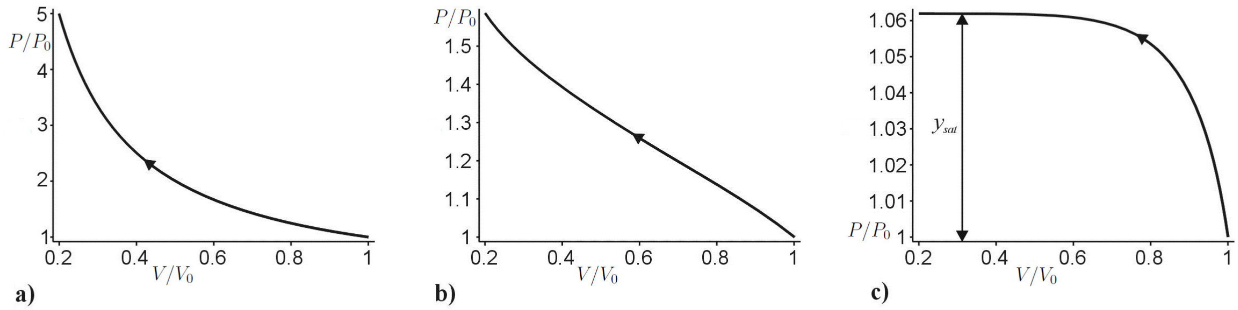

FIGURE 2. Dependence $y(x)$ at $\gamma = 1.4$. a) $k = 0$ (hyperbola); b) $k = 3.4$; c) $k = 8$.

where m, P, and V are the mass, pressure, and volume of the gas at the time point t; $\mathrm{d}m$ is the change of the gas mass due to its leak at time $\mathrm{d}t$. On the other hand

$$\mathrm{d}m = -\rho_{\text{out}} v_{\text{out}} A_{\text{out}} \mathrm{d}t < 0, \tag{2}$$

where $\rho_{\text{out}} = P_0 M/(RT_{\text{out}})$, $T_{\text{out}} < T_0$, and v_{out} are the density, temperature, and velocity of the gas exiting from the hole; P_0 is the external (atmospheric) pressure. Moreover,

$$\mathrm{d}V = -Av\mathrm{d}t < 0. \tag{3}$$

Assuming the gas outflow process is adiabatic, we have [6]:

$$v_{\text{out}} = \sqrt{\frac{RT_0}{M}\frac{2\gamma}{\gamma-1}\left[1-\left(\frac{P_0}{P}\right)^{\frac{\gamma-1}{\gamma}}\right]}, \tag{4}$$

$$T_{\text{out}} = T_0\left(\frac{P_0}{P}\right)^{\frac{\gamma-1}{\gamma}}. \tag{5}$$

Using Eqs. (1-5), we get:

$$\frac{\mathrm{d}V}{V} = -\frac{\mathrm{d}P}{P_0\left[\frac{P}{P_0} - k\sqrt{\left(\frac{P}{P_0}\right)^{2\frac{\gamma-1}{\gamma}} - \left(\frac{P}{P_0}\right)^{\frac{\gamma-1}{\gamma}}}\right]}, \tag{6}$$

where

$$k = \frac{A_{\text{out}}}{vA}\sqrt{\frac{RT_0}{M}\frac{2\gamma}{\gamma-1}}. \tag{7}$$

Let us estimate k value. Putting $A_{\text{out}}/A = 10^{-5}$, $v = 1\,\mathrm{cm/s}$, $T_0 = 300\,\mathrm{K}$, $M = 0.029\,\mathrm{kg/mol}$ (air), $\gamma = 1.4$, we obtain: $k \approx 0.8$. If we introduce the dimensionless variables $x = V/V_0$ (V_0 is the initial gas volume), $y = P/P_0$ and assume that at the initial time point $V = V_0$, $P = P_0$, then, integrating both sides of equation (6), we derive:

$$x = \exp\left(-\int_1^y \frac{\mathrm{d}y'}{y' - k\sqrt{y'^{2\frac{\gamma-1}{\gamma}} - y'^{\frac{\gamma-1}{\gamma}}}}\right). \tag{8}$$

3. The numerical results

In the limiting case $k = 0$ ($A_{\text{out}} = 0$), using Eq. (8), we have: $yx = 1 = \text{const}$, that is, the usual equation of Boyle's law. In the presence of the gas leakage ($k \neq 0$), the integral in Eq. (8) is not expressed in terms of elementary functions, but it can be easily evaluated using various mathematical software [7]. The results of such calculations at different values of k and $\gamma = 1.4$ are presented in Fig. 2.

At the relatively small values of k, the curve $y(x)$ has an inflection point [Fig. 2b)] unlike the limiting case of a hyperbola [Fig. 2a)]. As k increases, this point shifts to the region of the smaller volumes. The most interesting situation occurs at k greater than some critical value $k_{\text{cr}} \approx 4.5$. In this case, function $y(x)$ first increases with a decrease in volume x and then stops changing, reaching a "horizontal plateau" [Fig. 2c)].

Such an effect is physically explained by the fact that the rate of increase in pressure due to a decrease in volume is fully compensated by its rate of decrease due to a decrease in the gas mass [see Eqs. (1) and (6)]. With the increase of k ($k > k_{\text{cr}}$) the value of saturation pressure y_{sat} decreases asymptotically to zero (Fig. 3).

On the other hand, the gas volume x ($x < 1$) corresponding to the beginning of the saturation section increases tending to 1.

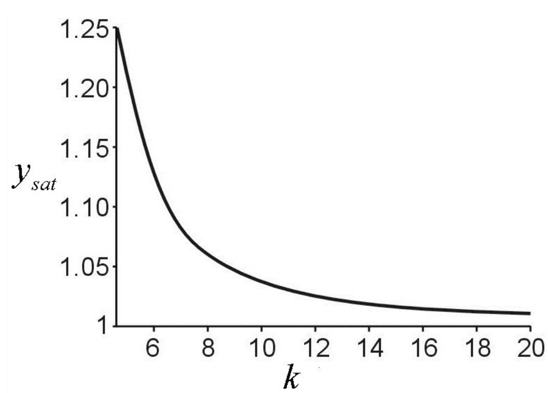

FIGURE 3. Dependence $y_{\text{sat}}(k)$ at $\gamma = 1.4$.

We hope that consideration of the problem described in this article will help students realize that an isothermal process is possible even with a change in the gas mass in the vessel. In addition, the described model of gas behavior can be used as the basis for one more method of test control of the degree of tightness of internal combustion engines. Finally, this practical exercise in thermodynamics has a research pattern (including numerical calculations and analysis) and can be successfully used in undergraduate courses or projects.

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