Application of the double numbers in the representation of the Lorentz transformations

G. F. Torres del Castillo

Instituto de Ciencias, Benemérita Universidad Autónoma de Puebla, 72570 Puebla, Pue., México

Received 31 August 2022; accepted 4 October 2022

We show that the orthochronous proper Lorentz transformations that leave one of the Cartesian coordinates fixed can be represented by 2×2 unitary matrices with determinant equal to 1, whose entries are double numbers. This representation is employed in the calculation of the Wigner angle, which arises in the composition of two boosts in arbitrary directions.

Keywords: Lorentz transformations; double numbers; Wigner angle; Thomas precession.

DOI: https://doi.org/10.31349/RevMexFisE.20.010204

1. Introduction

Despite the initial resistance to employ the complex numbers, the necessity and usefulness of the complex numbers have been recognized and exploited in the last two hundred years and, in the present day, in many areas of physics, the complex numbers are a standard tool applied with fluidity. In many of these applications, such as the solution of differential equations (*e.g.*, in electrodynamics) and the calculation of some definite integrals, the complex numbers are not really indispensable, but highly convenient.

Another example of the use of the complex numbers in physical or geometrical problems is the representation of the rotations in the three-dimensional Euclidean space by means of complex 2×2 matrices. The purely geometrical study of these rotations leads to the consideration of complex unitary 2×2 matrices with determinant equal to 1 (see, *e.g.*, Ref. [1]) and, maybe surprisingly, these matrices are required in quantum mechanics in order to represent the effect of any rotation on a spin-1/2 particle.

On the other hand, there exist two additional sets of numbers, somewhat analogous to the complex ones, called double and dual numbers (though they also receive other names in the literature), that are scarcely employed in physics or mathematics. However, Refs. [2–4] contain some applications of the double and the dual numbers in the standard mathematical physics.

The aim of this paper is to give another example of the suitability of the double numbers. We show that some subgroups of the restricted Lorentz group (*e.g.*, that formed by the transformations leaving z invariant) can be conveniently represented by 2×2 matrices whose entries are double numbers, and we show the advantageousness of this fact by calculating the so-called Wigner angle associated with the composition of two boosts in arbitrary directions.

In Sec. 2 we show in an elementary manner that the restricted Lorentz transformations that leave one of the Cartesian coordinates invariant can be represented by the 2×2 unitary matrices with determinant equal to 1, whose entries are double numbers. In Sec. 3 we make use of this representation finding the effect of two boosts in arbitrary directions.

2. Double numbers in the Lorentz transformations

Roughly speaking, the double numbers are the expressions of the form $a + \mathrm{j}b$, where a and b are real numbers and j is an "imaginary" unit such that $\mathrm{j}^2 = 1$, but $\mathrm{j} \neq \pm 1$. The double numbers are summed and multiplied among themselves by imposing the commutativity and associativity of the sum and the product, as well as the distributivity of the product over the sum. Since $(1 + \mathrm{j})(1 - \mathrm{j}) = 0$, but none of the factors is equal to zero, the double numbers are not a field.

As we know from the elementary special relativity theory, the homogeneous Lorentz transformations can be defined as those coordinate transformations that leave invariant

$$x^2 + y^2 + z^2 - (ct)^2,$$

where x, y, z, ct are the coordinates of an event with respect to some inertial frame. In order to make use of the double numbers we restrict ourselves to the space-time points with $z = 0$; in that way, the homogeneous Lorentz transformations that preserve the condition $z = 0$ leave invariant

$$x^2 + y^2 - (ct)^2, \tag{1}$$

and, therefore, it is convenient to introduce the double 2×2 matrix (that is, the entries of P are double numbers)

$$P \equiv \begin{pmatrix} y & x + \mathrm{j}ct \\ x - \mathrm{j}ct & -y \end{pmatrix}, \tag{2}$$

which has the properties

$$P^\dagger = P, \qquad \operatorname{tr} P = 0, \tag{3}$$

where, as in the case of complex matrices, $P^\dagger$ denotes the transpose of the conjugate of P, with the conjugate of the double number $a + \mathrm{j}b$ (denoted by $\overline{a + \mathrm{j}b}$) being defined as $a - \mathrm{j}b$. The relevance of the definition of P comes from the fact that

$$\det P = -y^2 - x^2 + (ct)^2 \tag{4}$$

[*cf.* Eq. (1)]. (Note that Eq. (4) remains valid if we omit the unit j in the entries of (2) but, as we shall see, the presence of j is essential in what follows.)

Any space-time point with $z = 0$ gives rise to a unique matrix P given by Eq. (2), which satisfies Eqs. (3). Conversely, any double 2×2 matrix satisfying Eqs. (3) must be of the form (2) and defines a unique set of values of x, y and ct. In other words, any space-time point with $z = 0$ can be represented by the array (ct, x, y) or by the tracefree Hermitean matrix (2).

On the other hand, any transformation $(ct, x, y) \mapsto (ct', x', y')$ that leaves invariant (1) must be linear and, therefore, can be represented by a (real) 3×3 matrix or, equivalently, making use of the one-to-one correspondence established above, by

$$P' = KPM, \tag{5}$$

where P' is the matrix corresponding to (ct', x', y'), and K and M are double 2×2 matrices in such a way that $P'^{\dagger} = P'$ and $\mathrm{tr}\, P' = 0$ [see Eqs. (3)]. Since, as in the case of complex matrices, $(KPM)^{\dagger} = M^{\dagger} P^{\dagger} K^{\dagger}$, the condition $P'^{\dagger} = P'$ is satisfied if $M = K^{\dagger}$. Then, $\mathrm{tr}\, P' = \mathrm{tr}\,(KPK^{\dagger}) = \mathrm{tr}\,(PK^{\dagger}K)$, so that the condition $\mathrm{tr}\, P' = 0$ is satisfied if $K^{\dagger} = K^{-1}$ (that is, K is a unitary matrix). Thus, for any double 2×2 matrix K such that $K^{\dagger} = K^{-1}$, the mapping $P \mapsto P'$ given by

$$P' = KPK^{\dagger}, \tag{6}$$

will represent a Lorentz transformation [which follows from the fact that $\det P' = \det K \det P \det K^{\dagger} = \det P \det(KK^{\dagger}) = \det P$ and Eq. (4)].

Finally, since $\det(KK^{\dagger}) = \det K \det K^{\dagger} = \det K\, \overline{\det K}$, we notice that if a double matrix K satisfies $KK^{\dagger} = I$, then $\det K\, \overline{\det K} = 1$, which is equivalent to the existence of a real number α such that $\det K = \pm\, \mathrm{e}^{\mathrm{j}\alpha}$ (we have here an analog of Euler's formula: $\mathrm{e}^{\mathrm{j}\alpha} = \cosh\alpha + \mathrm{j}\sinh\alpha$, as can be readily verified with the aid of the Taylor expansions). Then, the matrix $\tilde{K} \equiv \mathrm{e}^{-\mathrm{j}\alpha/2} K$ satisfies $\det \tilde{K} = \pm 1$ and $\tilde{K} P \tilde{K}^{\dagger} = (\mathrm{e}^{-\mathrm{j}\alpha/2} K)\, P\, (\mathrm{e}^{\mathrm{j}\alpha/2} K^{\dagger}) = KPK^{\dagger}$, which means that K and $\tilde{K}$ produce the same Lorentz transformation (6).

The unitary double matrices with determinant equal to 1 give rise to orthochronous proper Lorentz transformations (that is Lorentz transformations that do not change the direction of the time and do not change the orientation of the spatial axes). In fact, in the following subsection we shall give the explicit form of the special unitary double 2×2 matrices corresponding to a boost in an arbitrary direction and to an arbitrary rotation in the xy-plane. (Special means that the determinant is equal to 1.)

It may be pointed out that K and $-K$ represent the same Lorentz transformation [see Eq. (6)] and that $-K$ is a special unitary matrix if and only if K is. However, this ambiguity is not an inconvenience. (A similar behavior is encountered in the case of the representation of the rotations in the three-dimensional Euclidean space by complex 2×2 matrices mentioned at the Introduction.)

If now (ct'', x'', y'') are related to (ct', x', y') by a second orthochronous proper Lorentz transformation, represented by some special unitary matrix L in a form analogous to (6) (that is $P'' = LP'L^{\dagger}$, where P'' is the Hermitean trace-free matrix corresponding to (ct'', x'', y'')), then

$$P'' = L(KPK^{\dagger})L^{\dagger} = (LK)\, P\, (LK)^{\dagger},$$

meaning that the composition of the two Lorentz transformations is represented by the product LK (and also by $-LK$).

2.1. Explicit forms

The basic example of a Lorentz transformation considered in the elementary textbooks on special relativity is that corresponding to two inertial frames whose Cartesian axes coincide at $t = 0$ and the primed axes move with respect to the unprimed ones with velocity v along the x-axis (the so-called standard configuration, see, *e.g.*, Ref. [5]). The coordinates of any event with respect to these frames are related by

$$\begin{aligned} ct' &= \gamma\left(ct - \frac{v}{c}x\right), \\ x' &= \gamma\left(x - \frac{v}{c}ct\right), \\ y' &= y, \end{aligned} \tag{7}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

It is convenient, especially in what follows, to make use of the *rapidity*, w, instead of the velocity, v; these two parameters are related by

$$\tanh w = \frac{v}{c}, \tag{8}$$

then,

$$\gamma = \cosh w, \qquad \gamma\frac{v}{c} = \sinh w, \tag{9}$$

and the transformation (7), being linear, can be written in terms of matrices

$$\begin{pmatrix} ct' \\ x' \\ y' \end{pmatrix} = \begin{pmatrix} \cosh w & -\sinh w & 0 \\ -\sinh w & \cosh w & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \end{pmatrix}.$$

Since $x' + \mathrm{j}ct' = (x + \mathrm{j}ct)(\cosh w - \mathrm{j}\sinh w) = (x + \mathrm{j}ct)\mathrm{e}^{-\mathrm{j}w}$, this Lorentz transformation can be expressed in the form (6) with K given by

$$\begin{pmatrix} \mathrm{e}^{-\mathrm{j}w/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{j}w/2} \end{pmatrix}, \tag{10}$$

or its negative.

In a similar manner, if the primed axes move along the y-axis with respect to the unprimed ones, instead of (7), we have

$$ct' = \gamma \left(ct - \frac{v}{c} y \right),$$
$$x' = x,$$
$$y' = \gamma \left(y - \frac{v}{c} ct \right),$$

which, using the expressions (9), is represented by the 3×3 matrix

$$\begin{pmatrix} \cosh w & 0 & -\sinh w \\ 0 & 1 & 0 \\ -\sinh w & 0 & \cosh w \end{pmatrix},$$

and, alternatively, by the special unitary matrix K given by

$$\begin{pmatrix} \cosh(w/2) & \mathrm{j}\sinh(w/2) \\ \mathrm{j}\sinh(w/2) & \cosh(w/2) \end{pmatrix}, \qquad (11)$$

or its negative.

The ordinary (spatial) rotations are also included in the Lorentz transformations. Since we are considering here two spatial coordinates only, we only have rotations in the xy-plane:

$$ct' = ct,$$
$$x' = x\cos\theta + y\sin\theta,$$
$$y' = -x\sin\theta + y\cos\theta,$$

and this linear transformation is represented by the real matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix},$$

which corresponds to the special unitary matrix

$$\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \qquad (12)$$

and its negative.

Even though the special unitary matrices (10), (11), and (12) are simple enough to easily execute any required computation (especially in comparison with the 4×4 matrices regularly employed, see, *e.g.*, Ref. [6]), it is convenient to introduce the skew-Hermitean matrices

$$\sigma_1 \equiv \begin{pmatrix} -\mathrm{j} & 0 \\ 0 & \mathrm{j} \end{pmatrix}, \qquad \sigma_2 \equiv \begin{pmatrix} 0 & \mathrm{j} \\ \mathrm{j} & 0 \end{pmatrix},$$
$$\sigma_3 \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad (13)$$

(that is, ${\sigma_i}^\dagger = -\sigma_i$, for $i = 1, 2, 3$). One can readily verify that these matrices, which are analogous to the Pauli matrices appearing in connection with the group of rotations in the three-dimensional Euclidean space, satisfy the relations

$${\sigma_1}^2 = I = {\sigma_2}^2, \qquad {\sigma_3}^2 = -I, \qquad (14)$$

where I is the unit 2×2 matrix, and

$$\sigma_1\sigma_2 = \sigma_3, \quad \sigma_2\sigma_3 = -\sigma_1, \quad \sigma_3\sigma_1 = -\sigma_2, \qquad (15)$$

with $\sigma_i\sigma_j = -\sigma_j\sigma_i$, when $i \neq j$. (A minor difference, however, is that the usual Pauli matrices are Hermitian, in order to facilitate their relation with observables, namely the components of the spin.) For instance, the matrix (12) is equivalent to

$$\cos\frac{\theta}{2}\, I + \sin\frac{\theta}{2}\, \sigma_3,$$

and the matrix corresponding to the case where the primed axes move with respect to the unprimed ones in an arbitrary direction defined by the unit vector $\mathbf{n} = (n_1, n_2)$ is

$$\begin{pmatrix} \cosh(w/2) - n_1\mathrm{j}\sinh(w/2) & n_2\mathrm{j}\sinh(w/2) \\ n_2\mathrm{j}\sinh(w/2) & \cosh(w/2) + n_1\mathrm{j}\sinh(w/2) \end{pmatrix} = \cosh\frac{w}{2}\, I + \sinh\frac{w}{2}(n_1\sigma_1 + n_2\sigma_2), \qquad (16)$$

and its negative, assuming that the unprimed axes coincide with the primed ones at $t = 0$. This expression reduces to (10) and (11) when $\mathbf{n} = (1, 0)$ and $\mathbf{n} = (0, 1)$, respectively. The expression (16) can be compared with the corresponding standard 3×3 matrix

$$\begin{pmatrix} \gamma & -\gamma\frac{v}{c}n_1 & -\gamma\frac{v}{c}n_2 \\ -\gamma\frac{v}{c}n_1 & 1 + (\gamma - 1){n_1}^2 & (\gamma - 1)n_1n_2 \\ -\gamma\frac{v}{c}n_2 & (\gamma - 1)n_1n_2 & 1 + (\gamma - 1){n_2}^2 \end{pmatrix} \qquad (17)$$

[see, *e.g.*, Ref. [6], Eq. (11.98)].

In the following section we apply the representation of the Lorentz transformations obtained above in the analysis of the composition of two Lorentz transformations in arbitrary directions.

3. The Wigner angle and the Thomas precession

A well-known fact is that the composition of two "pure boosts," that is, two homogeneous Lorentz transformations relating inertial frames with parallel axes, may not be a pure boost, but it is equivalent to the composition of a boost and a rotation of (spatial) axes. The subject is discussed in many textbooks and in a long list of papers (see, *e.g.*, Refs. [8–12]). One of the physical implications of this fact is the so-called Thomas precession which is relevant in atomic physics (see, *e.g.*, Refs. [6, 7, 13, 14] and the references cited therein).

Let S be an inertial frame and let S$'$ be a second inertial frame moving with respect to S with rapidity w in the direction defined by the unit vector $\mathbf{n} = (n_1, n_2)$ so that the coordinates of any event with $z = 0$ with respect to S and S$'$ are related by means of $P' = KPK^\dagger$, with K given by Eq. (16). (Note that, in this approach, a boost in an arbitrary direction is almost as simple as a boost along one of the coordinate axes.) Let S$''$ a third inertial frame with its axes also parallel to those of S, moving with respect to S with rapidity w' in the direction defined by the unit vector $\mathbf{n}' = (n'_1, n'_2)$. Then, the coordinates of any event with $z = 0$ with respect to S and S$''$ are related by means of $P'' = LPL^\dagger$ with L given by an expression similar to Eq. (16) with w and (n_1, n_2) replaced by w' and (n'_1, n'_2), respectively. Hence, the coordinates with respect to S$'$ and S'' of any event with $z = z' = z'' = 0$ are related by $P'' = L\,(K^\dagger P' K)\,L^\dagger = (LK^\dagger)\,P'\,(LK^\dagger)^\dagger$. Thus, the Lorentz transformation relating the coordinates measured in S$'$ and S$''$ is represented by the double 2×2 matrix

$$LK^\dagger = \left[\cosh\frac{w'}{2}\, I + \sinh\frac{w'}{2}(n'_1\sigma_1 + n'_2\sigma_2)\right]\left[\cosh\frac{w}{2}\, I - \sinh\frac{w}{2}(n_1\sigma_1 + n_2\sigma_2)\right], \tag{18}$$

which may not be of the form (16) [owing to the possible presence of terms propotional to σ_3 in the result of the multiplication (18)] and this means that the axes of S$'$ and S$''$ need not be parallel. In fact, the transformation represented by (18) can be expressed as the composition of a boost followed by a rotation of the spatial axes; that is, the product (18) must be equivalent to

$$\left[\cos\frac{\Omega}{2}\, I + \sin\frac{\Omega}{2}\,\sigma_3\right]\left[\cosh\frac{\tilde{w}}{2}\, I + \sinh\frac{\tilde{w}}{2}(\tilde{n}_1\sigma_1 + \tilde{n}_2\sigma_2)\right], \tag{19}$$

where Ω is the angle between the axes of S$'$ and S$''$ (the Wigner angle), $\tilde{w}$ is the rapidity of S$''$ with respect to S$'$ and $(\tilde{n}_1, \tilde{n}_2)$ is a unit vector defining the direction of motion of S$''$ with respect to S$'$.

Multiplying the matrices appearing in Eqs. (18) and (19) with the aid of (14) and (15), making use of the linear independence of the set $\{I, \sigma_1, \sigma_2, \sigma_3\}$ one gets the four equations

$$\cos\frac{\Omega}{2}\cosh\frac{\tilde{w}}{2} = \cosh\frac{w}{2}\cosh\frac{w'}{2} - (\mathbf{n}\cdot\mathbf{n}')\sinh\frac{w}{2}\sinh\frac{w'}{2}, \tag{20}$$

$$\sin\frac{\Omega}{2}\cosh\frac{\tilde{w}}{2} = (n_1 n'_2 - n_2 n'_1)\sinh\frac{w}{2}\sinh\frac{w'}{2}, \tag{21}$$

$$\tilde{n}_1\cos\frac{\Omega}{2}\sinh\frac{\tilde{w}}{2} + \tilde{n}_2\sin\frac{\Omega}{2}\sinh\frac{\tilde{w}}{2} = n'_1\cosh\frac{w}{2}\sinh\frac{w'}{2} - n_1\sinh\frac{w}{2}\cosh\frac{w'}{2}, \tag{22}$$

$$\tilde{n}_2\cos\frac{\Omega}{2}\sinh\frac{\tilde{w}}{2} - \tilde{n}_1\sin\frac{\Omega}{2}\sinh\frac{\tilde{w}}{2} = n'_2\cosh\frac{w}{2}\sinh\frac{w'}{2} - n_2\sinh\frac{w}{2}\cosh\frac{w'}{2}, \tag{23}$$

which determine the four unknowns $\Omega, \tilde{w}, \tilde{n}_1$ and $\tilde{n}_2$. By combining Eqs. (20) and (21) one obtains

$$\begin{aligned}\tan\frac{\Omega}{2} &= \frac{(n_1 n'_2 - n_2 n'_1)\sinh\frac{w}{2}\sinh\frac{w'}{2}}{\cosh\frac{w}{2}\cosh\frac{w'}{2} - (\mathbf{n}\cdot\mathbf{n}')\sinh\frac{w}{2}\sinh\frac{w'}{2}} \\ &= \frac{(n_1 n'_2 - n_2 n'_1)\left[\cosh(\frac{w+w'}{2}) - \cosh(\frac{w-w'}{2})\right]}{\left[\cosh(\frac{w+w'}{2}) + \cosh(\frac{w-w'}{2})\right] - (\mathbf{n}\cdot\mathbf{n}')\left[\cosh(\frac{w+w'}{2}) - \cosh(\frac{w-w'}{2})\right]},\end{aligned} \tag{24}$$

and, summing the squares of the left-hand sides of (22) and (23) one finds the surprisingly simple relation

$$\cosh\tilde{w} = \cosh w\cosh w' - (\mathbf{n}\cdot\mathbf{n}')\sinh w\sinh w'. \tag{25}$$

It may be remarked that Eqs. (24) and (25) are valid without restrictions on the relative velocities of the frames. The last expression in (24) for the Wigner angle can be considerably simplified if we assume that the velocities of S$'$ and S$''$ with respect to S differ "slightly." Writing $w' = w + \delta w$, with $\delta w \ll w$, and assuming that the angle between $\mathbf{n}$ and $\mathbf{n}'$, denoted by $\delta\theta$, is small, $\delta\theta \ll 1$, keeping up to first order terms in δw and $\delta\theta$, we have $n_1 n'_2 - n_2 n'_1 = \sin\delta\theta \simeq \delta\theta$, $\mathbf{n}\cdot\mathbf{n}' \simeq 1$, $\cosh([w + w']/2) \simeq \cosh w + \sinh w\,(\delta w/2)$, and $\cosh([w - w']/2) \simeq 1$, so that

$$\delta\Omega \simeq (\cosh w - 1)\delta\theta.$$

4. Final remarks

Apart from the procedure followed in Sec. 2 to find a representation of the subgroup of the orthochronous proper Lorentz transformations preserving the condition $z = 0$ by double 2×2 matrices, there are some alternative methods that can be applied to obtain the basic results presented above. One of them consists in the use of the stereographic projection: each point (ct, x, y) on the surface M defined by the equation $x^2 + y^2 - (ct)^2 = 1$, with $y \geq 1$, is joined with the point $(0, 0, -1)$ by means of a straight line. The intersection of this line with the plane $y = 0$ is the point with coordinates $(ct, x, 0)/(1 + y)$, which can be identified with the double number $\xi \equiv (x + \mathrm{j}ct)/(1 + y)$. Since $x^2 + y^2 - (ct)^2$ is invariant under any homogeneous Lorentz transformation that preserves the condition $z = 0$, under such a transformation any point of M is mapped into another point of this surface, which corresponds to a double number ξ'. Then one finds that ξ' is given in terms of ξ through a linear fractional transformation, that can be associated with a 2×2 matrix, which is, up to a factor, the matrix K defined above. (All this construction is similar to that presented in Sec. 1.4 of Ref. [1], this time making use of the double number plane instead of the complex plane in the stereographic projection.)

Another method consists in applying the more algebraic approach given in Chap. 5 of Ref. [1], with the definition of the appropriate Infeld–van der Waerden symbols, which now are double numbers (see also Ref. [4]).

It may be remarked that even though the double matrices employed above represent a restricted class of the homogeneous Lorentz transformations, for problems where only two spatial directions are relevant (such as the calculation of the Wigner angle and the Thomas precession), this restriction does not signify a loss of generality.

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