# The one-dimensional Coulomb oscillator

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Received 23 September 2022; accepted 6 December 2022

We consider the one-dimensional oscillations of a charged point particle under the restoring Coulomb force. We find that for very small amplitudes, the speed of the particle remains almost constant and approximately equal to the speed of light. We also obtain the exact analytical expression for finding the oscillation period. It turns out to be a monotonically increasing function of the amplitude. For small amplitudes, this quantity is directly proportional to the amplitude itself.

*Keywords:* One-dimensional oscillations; Coulomb force; genuine nonlinear oscillations; oscillation period.

DOI: https://doi.org/10.31349/RevMexFisE.20.010213

## 1. Introduction

The idea of non-linearity is one of the key ideas of modern physics and has great scientific sense [1,2]. The development of science, based on the study of the phenomena of a new class of complexity, *i.e.*, non-linear systems and processes, leads to the development of deeper methods of scientific analysis and the formation of a new vision of the world. For example, the modern theory of non-linear oscillations [3] is both an applied and fundamental science. The applied character of this theory is determined by its multiple applications in physics, mechanics, automated control, radio-engineering, electronics, instrumentation, and so on. In this field of science, a great deal of research on different systems and phenomena has been done by using the methods of the theory of non-linear oscillations. Furthermore, new technical directions have arisen on the basis of this theory, namely, vibrational engineering and vibrational diagnostics, biomechanics, etc.

Contrary to the general claim that small oscillations in any system can be approximately treated in terms of simple harmonic motion (that is, the motion under the restoring force proportional to the displacement), there are infinitely many oscillating systems for which this approximation is not valid. Such oscillations are often called intrinsically non-linear oscillations. For these oscillations, the isochronous property (the period of oscillations does not depend on the amplitude) fails even for small amplitudes. Some other features and examples of the intrinsically non-linear oscillations are analyzed in the excellent paper by Mohazzabi [4].

In this paper, we consider the properties of one-dimensional oscillations of a charged particle under the restoring Coulomb force. This type of oscillations can be realized for confined electrons in hydrogen atoms in high magnetic fields, semiconductor quantum wires and carbon nanotubes, polymers, and plasma (Langmuir oscillations) [5,6]. The issues covered in this paper will be useful to undergraduates studying the theory of non-linear oscillations.

## 2. Phase portrait of the oscillations

Let us consider a point particle with the rest mass $m$ and charge $q_1$ that can move only along the $Ox$-axis under the restoring Coulomb force arising due to the presence of fixed point charge $q_2$ placed at the origin $O$. We assume that at the initial time $t = 0$ $x = A > 0$ ($A$ is the oscillation amplitude) and $v_x = 0$, where $\vec{v}$ is the particle velocity.

As the Coulomb potential has a singularity (in Ref. [4] only positive degrees of the displacement in the expression for the potential are considered) at the point where the point-like charge rests, we use the laws of relativistic dynamics, so that the velocity of the oscillating particle everywhere takes a finite value less than the speed of light $c$. Another question concerns the form of the Coulomb potential in the framework of relativistic dynamics. We are entitled to consider the motion of a charged particle in an arbitrary reference frame, including the one, where the attracting center is at rest. In this reference frame, the Coulomb potential has the usual classical form. In other words, the special theory of relativity does not correct the classical Coulomb potential of a fixed charge in any way (see Landau and Lifshitz course [7], in which a similar problem of the three-dimensional motion of a relativistic particle in an external Coulomb field is considered).

In this paper, we neglect the radiation damping force [8], that causes the weak damping of the oscillations over time. We also note that there are no the retarded effects [9], since the electric field of fixed charge $q_2$ is stationary (electrostatic). Due to the omitting of the radiation damping force, we can apply the relativistic conservation law of energy. In our case, it has the following form:

$$\frac{mc^2}{\sqrt{1 - v_x^2/c^2}} - \frac{|q_1 q_2|}{4\pi\varepsilon_0 |x|} = mc^2 - \frac{|q_1 q_2|}{4\pi\varepsilon_0 A}. \qquad (1)$$

We can rewrite Eq. (1) in the following dimensionless form:

$$\left(\frac{v_x}{c}\right)^2 = 1 - \frac{1}{(1 - a^{-1} + |\rho|^{-1})^2}, \qquad (2)$$

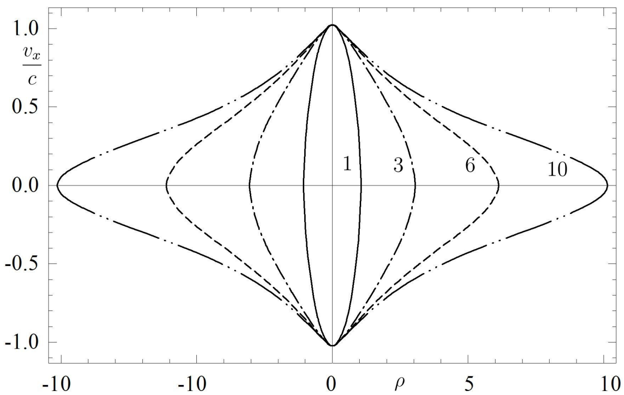


FIGURE 1. The phase trajectories at $a = 1, 3, 6, 10$.

where $a = 4\pi\varepsilon_0 mc^2 A/|q_1 q_2| > 0$; $\rho = 4\pi\varepsilon_0 mc^2 x/|q_1 q_2|$ ($|\rho| < a$). In Fig. 1 we present the set of phase portraits constructed using Eq. (2) for different values of $a$.

At $\rho = 0$ $v_x = c$. For $\rho \to 0$ $v_x \approx c(1 - \rho^2/2) \approx c$. Therefore, near the singularity of the potential the particle move with almost constant speed approximately equal to the speed of light. It is also seen that with an increase in the parameter $a$, the inflection points appear on the phase trajectories (exploring function $v_x(\rho, a)$, we find that it takes place for $a > 3.4$). At a fixed value of $v_x$ the value of $\rho$ increases with the increasing of amplitude $a$.

## 3. Time dependence of the displacement

Using Eq. (2) and relation $v_x = \mathrm{d}x/\mathrm{d}t$, we get:

$$t(\rho, a) = \pm\tau \int\limits_a^\rho \frac{\mathrm{d}\rho'}{\sqrt{1 - \frac{1}{(1-a^{-1}+|\rho'|^{-1})^2}}}, \tag{3}$$

where $\tau = |q_1 q_2|/(4\pi\varepsilon_0 mc^3)$. Since, we will consider only the positive instants of time $t > 0$, then we must choose the sign ”$-$” before the integral in Eq. (3) (because $\mathrm{d}\rho' < 0$). Further, due to the fact that the Coulomb potential is symmetric, it is sufficient for us to restrict ourselves to consideration of time moments less than a quarter of the oscillation period $T$ and omit the absolute value of $\rho^{'-1}$ . At that rate:

$$t(\rho, a) = \tau \int\limits_\rho^a \frac{\mathrm{d}\rho'}{\sqrt{1 - \frac{1}{(1-a^{-1}+\rho'^{-1})^2}}}, \tag{4}$$

for $0 \le t \le T/4$ and $0 \le \rho \le a$.

Let us consider the definite integral of the following form:

$$I(\rho, a) = \int\limits_\rho^a \frac{\mathrm{d}\rho'}{\sqrt{1 - \frac{1}{(1-a^{-1}+\rho'^{-1})^2}}}. \tag{5}$$

If we put $\rho'^{-1} - k = \cosh z'$, where $k = a^{-1} - 1$ ($k \in (-1, \infty)$), then Eq. (5) will be as follows:

$$I(\rho, a) = \int\limits_0^z \frac{\cosh z' \mathrm{d}z'}{(k + \cosh z')^2} = R + k\frac{\partial R}{\partial k}, \tag{6}$$

where

$$R = \int\limits_0^z \frac{\mathrm{d}z'}{k + \cosh z'}, \tag{7}$$

$z = \cosh^{-1}\left(\rho^{-1} - k\right)$.

Now, we make the substitution $\tanh(z'/2) = u'$. Therefore,

$$R = \frac{2}{1-k} \int\limits_0^u \frac{\mathrm{d}u'}{\frac{1+k}{1-k} + u'^2}, \tag{8}$$

where

$$u = \sqrt{\frac{\rho^{-1} - k - 1}{\rho^{-1} - k + 1}}. \tag{9}$$

at that at $\rho = 0$, we have: $u = 1$. Using list of the indefinite integrals of rational functions and Eq. (8), we finally derive:

$$R = \frac{2}{\sqrt{1-k^2}} \tan^{-1}\left(u\sqrt{\frac{1-k}{1+k}}\right), \tag{10}$$

for $|k| < 1$. If $k > 1$

$$R = \frac{2}{\sqrt{k^2-1}} \tanh^{-1}\left(u\sqrt{\frac{k-1}{k+1}}\right). \tag{11}$$

It should be noted that when taking the partial derivative $\partial R/\partial k$, we should consider the variable $u$ (see Eqs. (9)-(11)) as a constant.

Taking into account Eqs. (4)-(6) and (9)-(11), we obtain:

$$t(\rho, a) = \tau I(\rho, a). \tag{12}$$

The time dependence of the displacement $\rho$ can be found as the inverse function of function $t(\rho, a)$. In Figs. 2 and 3 we plot $\rho(t/\tau)$ at two different values of $a$. It is seen that in the case of very small amplitudes, the displacement changes almost linearly (sawtooth) with time. It means that, for this

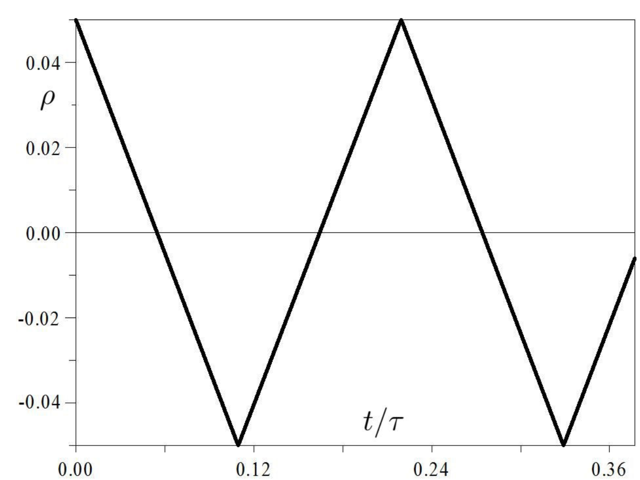


FIGURE 2. Dependence $\rho(t/\tau)$ at $a = 0.05$.

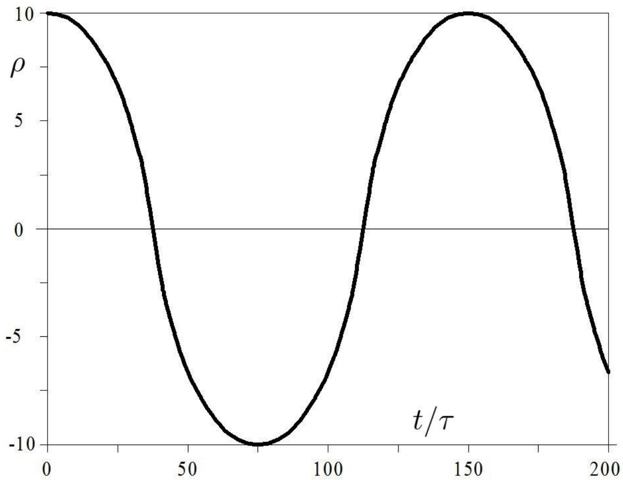


FIGURE 3. Dependence $\rho(t/\tau)$ at $a = 10$.

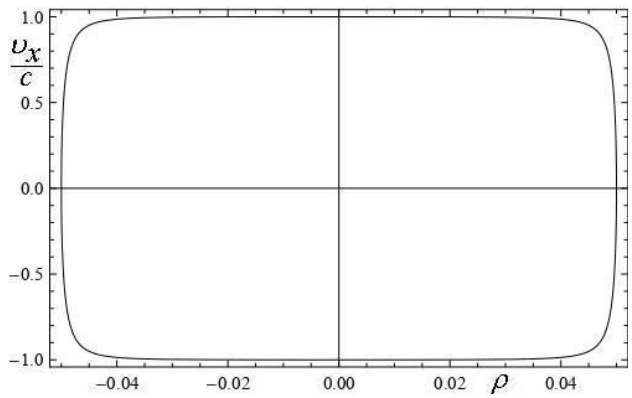


FIGURE 4. The phase trajectory at $a = 0.05$.

condition, the speed of the oscillating particle remains almost constant (see also Fig. 4), from which it is visible that this constant speed is approximately equal to the speed of light).

Indeed, at small amplitudes near $x = \pm A$ the walls of the potential well become so steep that the particle reaches relativistic velocities, having moved from these turning points even by a small (compared to $A$) distance to the center. Wherein, when the speed becomes close to the speed of light, derivative $\mathrm{d}v/\mathrm{d}\mathcal{E}$ (where $\mathcal{E}$ is the sum of rest energy and kinetic energy) becomes small. In other words, the total energy $\mathcal{E}$ can increase significantly with a small change in speed.

## 4. Period of the oscillations

The period of the oscillation can be found using Eq. (12) as:

$$T(a) = 4\tau I(0, a). \tag{13}$$

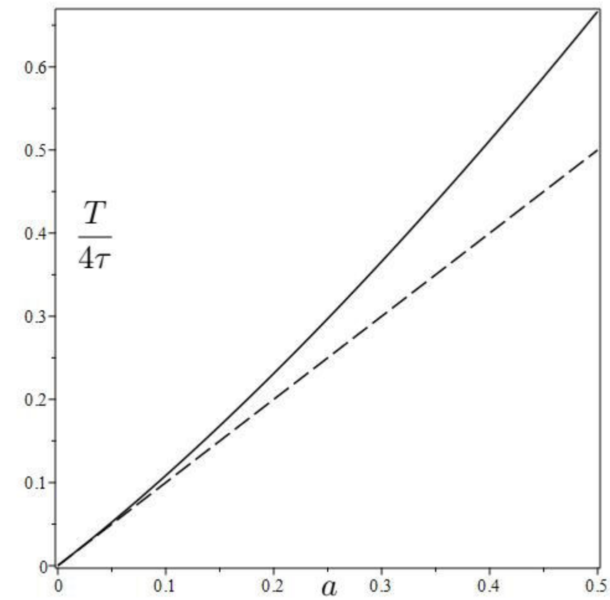


FIGURE 5. The amplitude dependence of the oscillation period. Solid line - the exact dependence; dashed line - the approximate linear dependence $T \approx 4\tau a$.

It follows from Fig. 5 that $T(a)$ is a monotonically increasing function of the amplitude.

For very small values of $a$, we can put the integrand in Eq. (4) approximately equal to 1. Then, using Eq. (13) we have $T \approx 4\tau a$. Figure 5 also helps us to evaluate the upper bound $a_{\max}$ that determines the smallness of parameter $a$ ($a_{\max} \approx 0.1$).

## 5. Conclusions

It is interesting to compare the results obtained in this paper with those derived within the framework of the one-dimensional relativistic linear oscillator. In our case, for most of the region between turning points, the speed of the oscillator differs only negligibly from the speed of light if the oscillation amplitude is very small. The same feature takes place for the linear oscillator, but for very large amplitudes [10]. For the linear oscillator the isochronous property takes place even in the relativistic regime [11]. Our study shows that for small amplitudes, the oscillation period of the Coulomb oscillator depends linearly on its amplitude.

We hope that our consideration should help readers better understand such an important concept as the intrinsically non-linear oscillations, and can be used in undergraduate courses or projects.

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