Retarded potentials and radiation of a rotating charged rod

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In this paper we determine the electromagnetic properties of a rotating charged rod, which rotates around the x-axis with an angular frequency that it's not constant. As the charge is changing its position over time, the electromagnetic information reaches us with a certain time lag. Therefore, it is necessary to obtain the time-dependent(retarded) potentials at any point in space, for which we need multipolar expansion to determine the field at any point in space and do a Taylor expansion of the distance at which we want to measure the electromagnetic fields, in the static and induction zone. Then, we determine the Poynting vector in the corresponding radiation zone. Finally, we determine the radiated power, emphasizing the symmetries of the problem and showing how we can approach its solution considering these symmetries.

Keywords: Classical electrodynamics; multipole expansion; retarded potentials; electromagnetic radiation.

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1. Introduction

In electrodynamics we have many techniques to solve a problem, for instance, to find the electric potential we can use direct integration, multipolar expansion or either the Laplace or Poisson equation, or we can even find it using the electric field found through Gauss's law. However, among all this range of possibilities, it is sometimes difficult to decide which is the best way to attack and solve the problem, due to the symmetries it presents.

Emphasizing the symmetries and all the possibilities mentioned above, in this work we attack a "typical problem", which we have sectioned into two parts, static and dynamic. The Static problem consists in determining the electric potential at any point in space for a finite rod that is in the z-axis Fig. 1a), with a homogeneous linear distribution of charge. Due to the cylindrical symmetry of the rod, one could decide to use Gauss's law. However, since the bar is finite and the potential is required in the whole space, it's necessary to do a previous analysis about what symmetries there are.

In the dynamics case, the rod pivots on the center of coordinates and rotates around the x-axis, Figs. 1b)-f), and rotates with a variable angular velocity. Since the charge is moving, we have a time dependent charge distribution, and we want to determine the potentials at any point in space, therefor, the information from where the charge is will take now some time to arrive to where the observer is, hence there will be no longer any symmetry. Therefore, it is necessary to make use of the retarded potentials, and the behavior of these potentials depends on the distance at which we want to measure it. For this, it will be necessary to make use of a Taylor expansion in the distance.

On the other hand, since the rod is rotating with a variable angular speed, then, it is very likely that it will generate an electromagnetic radiation, therefore, we also need to compute the corresponding Poynting vector.

This paper is organized as follows. In Sec. 2, we present the problem and the physical implications, then we find the potential Φ and the electric field $\mathbf{E}$ when the rod is at rest. Later we consider that the rod is in motion, so it is necessary to determine the retarded potentials, both in the near zone and in the intermediate zone as well. In Sec. 2.3 we calculate the Poynting vector and the radiated power. Finally, the conclusions and remarks are presented in Sec. 3.

2. The Problem

In this work we will solve the following question.

A rod with a uniform charge density $\lambda(r')$ lies on the z-axis from $z=0$ to $z=a$. The rod turns around the x-axis, with an angular frequency $\cos(\omega t)$. a) First consider that the rod is at rest Fig. 1a), and determine the scalar potential Φ, and the electric field $\mathbf{E}$ at all points of space. b) Now, consider that the rod turns around the x-axis Figs. 1b), f), with a variable angular frequency, determine the potentials. c) Find the Poynting vector and calculate the radiated power.

2.1. Electrostatic problem

To solve this problem, we could think of using cylindrical coordinates bearing in mind that the rod has a uniform charge density, then it has symmetry in the radius r and angle ϕ. Because of these symmetries, we could think in Gauss's law, $\oint \mathbf{E}\cdot d\mathbf{a} = q_{\text{in}}/\epsilon_0$, with $q_{\text{in}} = \int \lambda(r')\, dl$. However, the electric field in all space is required, $i.e$, at any point (x,y,z). Since the rod is finite the observer can be either at a point above the rod, along the rod, or below the rod. The previous fact indicates a dependence on the polar angle θ, but the

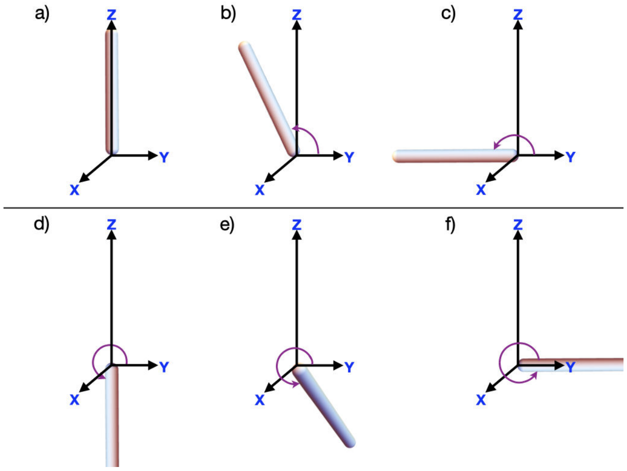

FIGURE 1. a) The rod is at rest, on z-axis, b) the rod starts to rotate around the x-axes, c)-f) the rod continue rotating with different angular velocity. Always pivoting at the origin.

Gauss's law does not have this dependence in cylindrical coordinates. Therefore, the only way to have such dependency is through a multipolar expansion (r, θ), in spherical coordinates, through the scalar potential.

Since the rod is at rest, there is symmetry in the azimuthal angle ϕ, and the dependence is only on (r, θ). It's known from multipole expansion [1], Eq. (4.2) that

$$\Phi(r,\theta) = k\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n\, P_n(\cos\theta)\rho(r')\, dV' ,$$

It is important to remember that r is where we need to measure the field, and r' is where the charge distribution is located. For our case $\rho(r')\, dV' \to \lambda(r')\, dz' = (Q/a)\, dz'$ and $r' = z'$, substituting

$$\Phi(r,\theta) = k\sum_{n=0}^{\infty}\frac{\lambda(r')}{r^{(n+1)}}\int_0^a (z')^n\, P_n(\cos\theta)\, dz'$$
$$= k\lambda(r')\sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{r^{(n+1)}}\cdot\frac{a^{n+1}}{n+1} .$$

Finally, replacing $\lambda(r') = Q/a$.

$$\Phi(r,\theta) = kQ\sum_{n=0}^{\infty}\frac{1}{n+1}\frac{a^n}{r^{n+1}}\, P_n(\cos\theta) ,$$

substituting some Legendre polynomials

$$\Phi(r,\theta) \approx kQ\Bigg[\frac{1}{r} + \frac{a}{2r^2}P_1(\cos\theta) + \frac{a^2}{3r^3}P_2(\cos\theta) + \cdots\Bigg]$$
$$\approx kQ\Bigg[\frac{1}{r} + \frac{a}{2r^2}\cos\theta + \frac{a^2}{6r^3}\Big(3\cos^2\theta - 1\Big) + \cdots\Bigg] . \qquad (1)$$

Where the first term is defined as the monopole $\Phi \sim 1/r$, the second term is the dipole $\Phi \sim 1/r^2$, and the third term is the quadrupole $\Phi \sim 1/r^3$, see [2] for more multipoles, and how generate this.

For the electric field, $\mathbf{E} = -\nabla\Phi$, with ∇, in spherical coordinates, from Eq. (1).

$$\mathbf{E}(r,\theta) \approx kQ\Bigg[\Big(\frac{1}{r^2}+\frac{a}{r^3}\cos\theta+\frac{a^2}{2r^4}\Big(3\cos^2\theta-1\Big)+\cdots\Big)\hat{r} + \Big(\frac{a}{2r^3}\sin\theta + \frac{a^2}{r^4}\cos\theta\sin\theta + \cdots\Big)\hat{\theta}\Bigg] . \qquad (2)$$

2.2. The Retarded Potentials.

Now, if we consider that the rod is rotating with angular frequency $\cos(\omega t)$ Fig. 1b), f), then, to determine the potentials anywhere in space, we need the potentials to depend on time, and this is achieved through the retarded potentials [3], Eq. (10.26),

$$\Phi(r,t) = k\int\frac{\lambda(r',t_r)}{R}dl', \quad \mathbf{A}(r,t) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{I}(r',t_r)}{R}dl',$$

where $R = |\vec{r} - \vec{r}'|$, and $t_r = t - (R/c)$, is the retarded time. These potentials satisfy the Lorenz gauge [4]. The electric potential in all space is required but, in this case, there is no azimuthal symmetry since the rotation is along the x-axis.

Therefore, the use of spherical harmonics is necessary. Once more, we do multipolar expansion in R, but now with dependence on (r, θ, ϕ), Eq. (3.70) of [1].

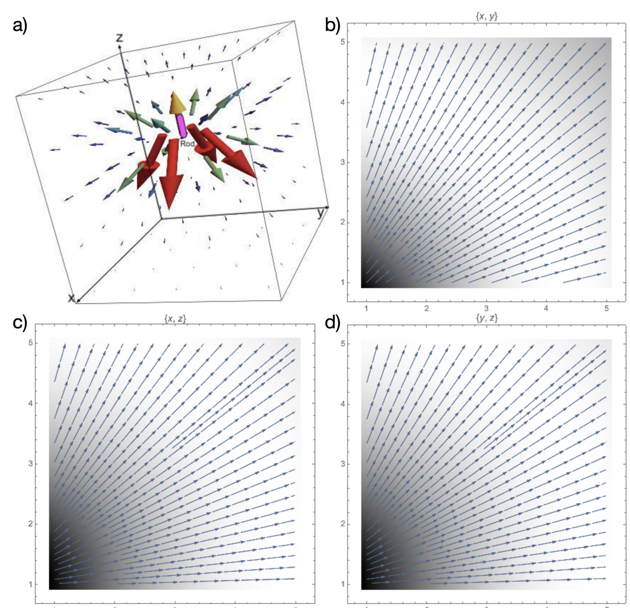

FIGURE 2. a) Rod's electric field Eq(2), with the rod in the center. Subsequently, some cross sectional planes of the field b) plane (x, y), c) plane (x, z), d) plane (y, z). Brightness reflects closeness to the rod.

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{(r')^l}{r^{l+1}} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi)\,,$$

then

$$\Phi(r,t) = \frac{\alpha}{\epsilon_0}\int (z')^l\,\lambda(r',t_r)\,dz'\,, \tag{3}$$

$$\mathbf{A}(r,t) = \alpha\mu_0\int (z')^l\,\mathbf{I}(r',t_r)\,dz'\,, \tag{4}$$

were

$$\alpha = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{1}{r^{l+1}} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi)\,,$$

since α, is a function that does not have depend on z', has been left out of the integral.

The sum in Eq. (4), starts at $l=1$, because $l=0$ is the magnetic monopole. And the charge and current distributions, respectively, will be

$$\lambda(r',t_r) = \lambda(r')\,\cos(\omega\,t_r), \tag{5}$$

$$\begin{aligned}\mathbf{I}(r',t_r) &= \lambda(r')\vec{v} = \lambda(r')\,\vec{\omega}\times\vec{r}\,' \\ &= \lambda(r')\omega_0 a\cos(\omega\,t_r)(\sin\phi\,\hat{r}+\sin\theta\cos\phi\,\hat{\phi}).\end{aligned} \tag{6}$$

From the mathematic form of equation Eq. (3) and Eq. (4), we can write $\mathbf{A}$, in terms of Φ.

$$\mathbf{A}(r,t) = \frac{\omega_0 a}{c^2}(\sin\phi\,\hat{r}+\sin\theta\cos\phi\,\hat{\phi})\,\Phi(r,t)\,. \tag{7}$$

Therefore, we will only concentrate on determining $\Phi(r,t)$, since the retarded magnetic vectorial potential is completely determined through $\Phi(r,t)$.

Now we wish to establish certain simple, but general, properties of the fields in the limit where the sources are very small compared to a wavelength. As the source size is a, and the wavelength is $\lambda' = 2\pi c/\omega$, then if $a \ll \lambda'$, there are three spatial regions of interest; the near, the intermediate, and the far zones. We will only explore the near and the intermediate zone because the far zone is the radiation zone, and that will be explored in Sec. 2.3. As we will see, the fields have very different properties in the different zones.

2.2.1. The near (static) zone, $a \ll r \ll \lambda'$

In this zone the fields have the character of a static field, with a radial component that changes according to the distance, it also depends on the properties of the source.

From the definition of R.

$$R = |\vec{r}-\vec{r}\,'| = \Big[r^2 + {r'}^2 - 2rr'\cos\theta\Big]^{1/2} = r\Big[1 - 2\Big(\frac{r'}{r}\Big)\cos\theta + 2\Big(\frac{r'}{r}\Big)^2\Big]^{1/2},$$

and remember that we have already defined $r'=z'$, to integrate with respect to dz', subsequently.

$$R = r\Big[1 - 2\Big(\frac{z'}{r}\Big)\cos\theta + 2\Big(\frac{z'}{r}\Big)^2\Big]^{1/2}.$$

Expanding in Taylor to first order respect to z'.

$$R \cong r\bigg(1 - \frac{z'}{2r}\cos\theta\bigg), \tag{8}$$

substituting Eq. (8) in $\cos(\omega\,t_r)$, up to the first order in z'

$$\cos(\omega\,t_r) = \cos[\omega(t-R/c)] \approx \cos[\omega(t-r/c)]\cos\left(\frac{\omega z'}{2c}\cos\theta\right) - \sin[\omega(t-r/c)]\sin\left(\frac{\omega z'}{2c}\cos\theta\right). \tag{9}$$

Finally, replacing Eq. (9) in Eq. (5), and subsequently substituting these equations into Eq. (3), the retarded potential takes the form

$$\Phi(r,t) = \alpha\frac{\lambda(r')}{\epsilon_0}\left[\cos[\omega(t-r/c)]\int_0^a (z')^l\cos\left(\frac{\omega z'}{2c}\cos\theta\right)dz' - \sin[\omega(t-r/c)]\int_0^a (z')^l\sin\left(\frac{\omega z'}{2c}\cos\theta\right)dz'\right]. \tag{10}$$

For illustration integrating for $l=0$, $\lambda(r') = Q/a$, and $\alpha = (1/\sqrt{4\pi})(1/r)$ the retarded potential is

$$\Phi(r,t) = \frac{Q}{\epsilon_0 a\sqrt{4\pi}}\frac{1}{r}\left[\cos[\omega(t-r/c)]\frac{2c}{\omega}\sec\theta\sin\left(\frac{a\omega}{2c}\cos\theta\right) - \sin[\omega(t-r/c)]\frac{4c}{\omega}\sec\theta\sin^2\left(\frac{a\omega}{4c}\cos\theta\right)\right], \tag{11}$$

and analogously for the retarded magnetic vector potential Eq. (7).

In this zone the sources are inside their own near zone at optical frequencies. In the case of atoms, if the atoms are in a liquid or solid state, there is a near field interaction that may be important in determining optical dispersion and other observable phenomena. Only for microwave frequencies or less does the near zone become relevant on a macroscopic scale.

2.2.2. *The intermediate (induction) zone, $a \ll r \sim \lambda'$*

Since waves of frequency ω have a wavelength $\lambda' = 2\pi c/\omega$, this result in the requirement $z' \ll \lambda'$. Under this condition, $\sin\theta \approx \theta$. Applying this to Eq. (9)

$$\cos(\omega\, t_r) = \cos[\omega(t - R/c)] \cong \cos[\omega(t - r/c)] - \frac{\omega z'}{2c}\cos\theta\sin[\omega(t - r/c)]$$

Again, substituting the previous expansion, in Eq. (5), and subsequently substituting these equations into Eq. (3), the retarded potential takes the form

$$\Phi(r,t) = \alpha\frac{\lambda(r')}{\epsilon_0}\left[\cos[\omega(t - r/c)]\int_0^a (z')^l\, dz' - \frac{\omega}{2c}\cos\theta\sin[\omega(t - r/c)]\int_0^a (z')^{l+1}\, dz'\right],$$

Finally, integrating the retarded potential

$$\Phi(r,t) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{1}{r^{l+1}}Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi)\frac{Q}{\epsilon_0}a^l\left[\frac{\cos[\omega(t - r/c)]}{l+1} - \frac{\omega a}{2c(l+2)}\cos\theta\sin[\omega(t - r/c)]\right], \qquad (12)$$

and basically, the same for the retarded magnetic vector potential Eq. (7). Recall that for $\mathbf{A}(r,t)$ the sum starts at $l = 1$. Again, for illustration let's take $l = 0$, the retarded potential Eq. (12) take the form

$$\Phi(r,t) = \frac{Q}{\epsilon_0\sqrt{4\pi}}\frac{a}{r}\left[\cos[\omega(t - r/c)] - \frac{\omega a}{4c}\cos\theta\sin[\omega(t - r/c)]\right], \qquad (13)$$

If we compare the equations Eq. (11) and Eq. (13), we can observe that both decay as $1/r$, which is typical for electromagnetic radiation [5]. However, the dependence on frequency is inverse.

In the induction zone most of the simple approximations fail. The waves change character completely inside this zone. This zone is important in molecular dynamics and condensed matter theory because these objects interact in this zone.

2.3. c) The Poynting vector and power radiated

This is the third zone (the far (radiation) zone) $a \ll \lambda' \ll r$, generally, the sources are smaller, much smaller than a wavelength. In the far zone, the emitted EM fields are characteristically transverse and fall off in amplitude as $1/r$ or faster, and often far enough away they look locally like plane waves This is typical of radiation fields from compact sources.

To get the electromagnetic radiation, we can observe from Eq. (1) that, the most dominant term (after monopole), in the series, is the dipole. Therefore, we need to construct the dipole moment of the rod. To do so, we match the potential of a dipole to the term $1/r^2$, in Eq. (1).

$$k\frac{Qa}{2}\frac{\cos\theta}{r^2} = k\,p\frac{\cos\theta}{r^2}\,, \quad p = \frac{Qa}{2} = p_0\,.$$

To get the time-dependent dipole moment we multiply by the spin frequency around the x-axis.

$$\mathbf{p} = p_0\cos(\omega t)\,\hat{x} = p_0\cos(\omega t) \times\left(\sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}\right),$$

Since we are interested in determining the radiation of the rotating rod, we use the generalization of electric dipole radiation. From Eq. (11.59) of [3], for the Poynting vector

$$\mathbf{S} = \frac{\mu_0}{c}\left[\frac{\ddot{p}(t_0)}{4\pi r}\right]^2\sin^2\theta\ \hat{r}\,,$$

bear in mind that this equation can only be applied to the oscillating electric dipole, and only is valid for $r' \ll r$. The dot in $\mathbf{S}$, denotes derivative with respect to time.

$$\mathbf{S} = \frac{\mu_0}{c}\left(\frac{p_0\omega^2}{4\pi r}\right)^2\cos^2(\omega t)\sin^2\theta\ \hat{r},$$

bear in mind that the intensity is obtained by averaging (in time) over a complete cycle of the Poynting vector

$$\langle\mathbf{S}\rangle = \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\frac{\sin^2\theta}{r^2}\,\hat{r}. \qquad (14)$$

In Fig. 3, we can see that the Poynting vector is exactly a donut which was expected due to the rotation and the angular frequency of the rod. However, its direction of propagation is well defined.

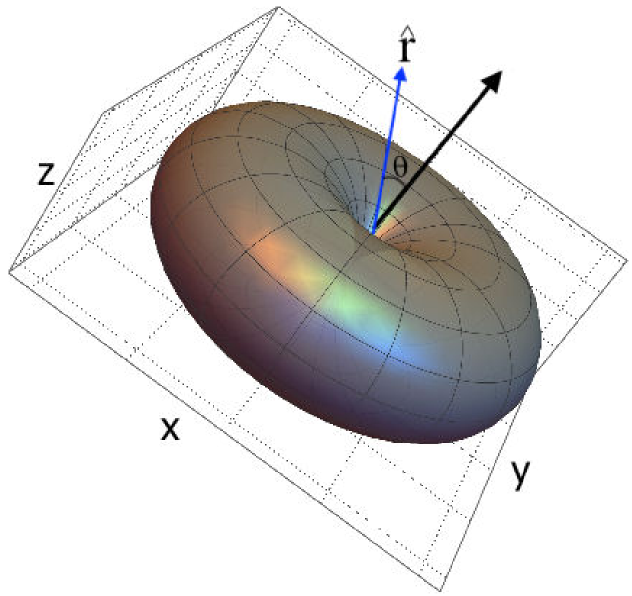

FIGURE 3. Poynting vector distribution. No power is radiated neither in the forward nor in the backward direction, rather it is emitted in a donut about the direction of instantaneous acceleration.

The total power radiated is found by integrating $\langle \mathbf{S} \rangle$, over a sphere of radius r.

$$P(r,t) = \oint \mathbf{S} \cdot d\mathbf{a} = \frac{\mu_0 p_0^2}{12\pi c}\omega^4.$$

As it is known, the power radiated by an electric dipole varies as the fourth power of the frequency for fixed dipole moments [6, 7]. Therefore, this system is emitting electromagnetic radiation.

3. Conclusion and remarks

It was possible to find the potential Φ, and the electric field $\mathbf{E}$, at any point in space using the multipolar expansion for a rod that rotates around the x-axis. It is important to emphasize that, when we calculate Φ, in the static case, we only need to make use of Legendre's polynomials, since the rod is fixed, and we have azimuthal symmetry. However, when the rod is rotating around the x-axis, there is no longer azimuthal symmetry then the use of spherical harmonics was necessary. Furthermore, since the charge was in motion, it was necessary to introduce the temporal information into the potential and this is achieved through the retarded potentials. If the rod was moving along an axis with a certain speed, then instead of the retarded potentials we would need the Liénard-Wiechert potentials.

The retarded potentials were obtained in the static and in the intermediate zone. The decays in distance in these zones are the same $1/r^{l+1}$. However, the oscillatory behavior of the fields is what changes. For the static zone $\Phi \propto (1/\omega)$, while in the intermediate zone $\Phi \propto \omega$.

The corresponding Poynting vector was obtained at the radiation zone. It is also important to emphasize the direction of this vector since it always points out in the radial direction, regardless of the shape of the object that produces it, which is to be expected due to the intensity profile which takes the form of a donut. In addition, the power radiated was obtained and we can guarantee that the system is indeed radiating since it is known that dipole radiation is proportional to the fourth power of the frequency.

Recall that many electromagnetic waves exhibit the characteristics of electric dipole radiation. An important characteristic of this type of radiation is that the intensity of the electromagnetic wave radiated by a dipole antenna is zero along the antenna axis and it presents a maximum in directions perpendicular to the antenna axis.

Finally, in this work we wanted to emphasize some symmetries that could be present in a problem, and not contemplating them correctly would lead us to an erroneous solution.

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