The problem of the body rotating on a frictionless table, attached to a hanging body, solved partially by conservation theorems

J. Palacios Gómez[a] and A. S. De Ita De la Torre[b]

[a]*Escuela Superior de Física y Matemáticas, IPN,*
Av. Instituto Politécnico Nacional S/N 07738, CDMX, México.
[b]*Universidad Autónoma de México, Unidad Azcapotzalco,*
Av. San Pablo 180, 02200 CDMX, México.

Received 17 October 2022; accepted 10 January 2023

Conservation theorems of Mechanics, have been applied to the problem consisting of a body rotating on a frictionless table, attached to a hanging body, as an illustrative example for students of Physics with no knowledge of sophisticated mathematical methods, how to obtain a description of the physical behavior of a system, when obtaining the equation of motion requires those complicated methods. Applying the conservation of angular momentum it is shown that the angular frequency increases inversely to the square of the radius of motion; then the radius r_c is found at which the centripetal force and the tension of the string compensate each other; then, applying the conservation of energy, turning points are found. At the end, following scenery is obtained: the radial component of motion of the rotating body takes place between two turning points, namely a maximum at $r = l$ given by the initial conditions, and a minimum at $r = r_c\sqrt{r_c/2l}$. With the help of these equations, obtained without the need of solving differential equations, it is possible to obtain a semi quantitative physical behavior of this particular system.

Keywords: Rotation of rigid bodies; conservation theorems.

DOI: https://doi.org/10.31349/RevMexFisE.20.020204

1. Introduction

There are some problems in Mechanics, as well as in many other fields of Physics, whose solution demands the application of complicated differential equations, and of numerical methods. These methods sometimes obscure the physical description of the dynamic behavior of the system, and a comprehensive physical picture is obtained only after results are presented as sequences of graphs. However, in some cases when the equation of motion is not strictly necessary, and higher levels of mathematical knowledge is required to obtain the equation of motion, the application of conservation theorems and some other simple considerations, can deliver a reasonable picture of the physical behavior of the system. This approach has already been used by Lock [1] to study the stability of torque-free rotations of tops, and it is considered here to illustrate the case of a mass m_1 rotating on a frictionless table, attached to a hanging mass m_2 through a massless string of length l passing through a small hole at the center of the table, as shown on (Fig. 1).

For students of science and engineering of the first semesters of the university, this problem has been presented in textbooks [2] for the special case of the stationary state of m_1 moving in a circle, as an application of the equilibrium condition between the outward centripetal force originated on m_1, and the weight of m_2. Furthermore, textbooks which tackle this problem in a more advanced scope applying the Lagrangian method [3,4], arrive only at the differential equation of motion without solving it, leaving actually the solution of the problem without a physical interpretation. To a significant extent, this deficiency intends to be solved here.

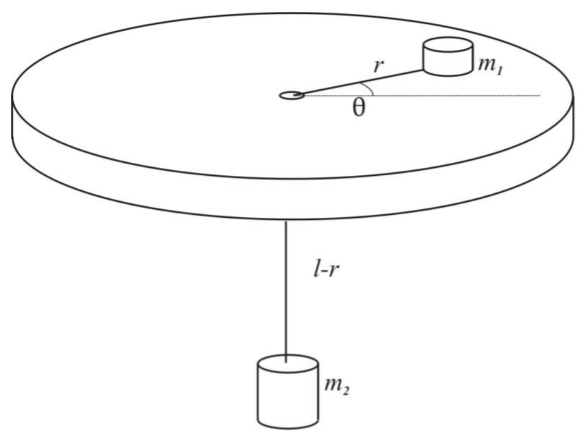

FIGURE 1. Mass m_1 rotates over a frictionless table. Mass m_2 exerts its weight on m_1 through a string of length l. r and θ are polar coordinates of m_1.

2. Methodological process

Let $\boldsymbol{v}_t$ be the tangential and $\boldsymbol{v}_r$ the radial components respectively of the velocity of m_1 at any time. The magnitude of the angular momentum of this system is $L = r\ m_1\ v_t$, and at the initial condition, when mass m_1 starts with angular velocity ω_0 at the distance l from the origin located at the hole, it is $L_0 = m_1 l^2 \omega_0$. Conservation of L and the relation between angular and tangential velocities applied to the initial state, give

$$rm_1 v_t = m_1 r^2 \omega = m_1 l^2 \omega_0,$$

then

$$\omega = \frac{l^2}{r^2}\omega_0, \tag{1}$$

which indicates that the angular speed of mass m_1 accelerates indefinitely as the radius diminishes; however, at some time, equilibrium of the centripetal force and the weight of mass m_2 is reached, and from that time on, motion begins to decelerate. This equilibrium of forces occurs at some radius r_c when

$$m_1 a_c - m_2 g = 0 \tag{2}$$

with a_c the outward centripetal acceleration

$$a_c = \frac{v_t^2}{r_c} = \omega^2 r_c. \tag{3}$$

r_c is an inflection point, and it is expected that after reaching it, the radial acceleration becomes outwards. This behavior is to some extent similar to that of a harmonic oscillator: when the mass attached to a spring is displaced a certain length, the elastic force of the spring tends to restore the mass to its equilibrium position, where the force is zero; however, due to the inertia of the mass, it continues compressing the spring with diminishing speed, until the mass stops, and reverts its motion. The equilibrium position in the harmonic oscillator is an inflection point, and in our case it is equivalent to the point r_c. Using (1), (2) and (3)

$$m_1 \frac{l^4}{r_c^3}\omega_0^2 - m_2 g = 0,$$

and r_c is

$$r_c = \left\{ \frac{m_1}{m_2} \frac{(l^2\omega_0)^2}{g} \right\}^{\frac{1}{3}}. \tag{4}$$

This radius should be less than or equal to l, which gives

$$\frac{m_1}{m_2}\frac{(l^2\omega_0)^2}{g} \le l^3,$$

or

$$\omega_0 \le \sqrt{\frac{m_2}{m_1}\frac{g}{l}}. \tag{5}$$

Equality in this equation means that at the start, there is equilibrium of radial forces:

$$m_2 g = m_1 l \omega_0^2.$$

The second member of this equation is $m_1 a_c$, since $\omega = \omega_0$. The motion is in a circle of radius $r = l$, and the radial speed $v_r = 0$. If (5) is not fulfilled, then at the start of the motion the centripetal force $m_1 l \omega_0^2$ will exceed the tension force $m_2 g$, and there will no stable motion.

The kinetic energy, K_1, of mass m_1 is

$$K_1 = \frac{1}{2}m_1(\boldsymbol{v}_r + \boldsymbol{v}_t)^2 = \frac{1}{2}m_1(\boldsymbol{v}_r + \boldsymbol{v}_t)\cdot(\boldsymbol{v}_r + \boldsymbol{v}_t) = \frac{1}{2}m_1(v_r^2 + v_t^2).$$

Substituting the tangential component of velocity v_t by the angular velocity ω, i.e. $v_t = r\omega$, last equation becomes

$$K_1 = \frac{1}{2}m_1(v_r^2 + r^2\omega^2).$$

The potential energy of mass m_2, V_2, supposing zero level at the height of the table, is $V_2 = -m_2 g(l - r)$, and its kinetic energy is $K_2 = \frac{1}{2}m_2 v_r^2$, since its (vertical) speed equals the radial speed of m_1. At the start, $r_0 = l$ and $v_r = 0$, and the total initial energy is

$$E = \frac{1}{2}m_1 l^2 \omega_0^2.$$

Then at any other moment, since no dissipative forces intervene this initial energy is conserved: $E = K_1 + K_2 + V_2$, *i.e.*:

$$\frac{1}{2}m_1 l^2\omega_0^2 = \frac{1}{2}m_1(v_r^2 + r^2\omega^2) + \frac{1}{2}m_2 v_r^2 - m_2 g(l - r).$$

Applying (1) and rearranging terms

$$\frac{1}{2}m_1 l^2\omega_0^2 = \frac{1}{2}\frac{l^4}{r^2}m_1\omega_0^2 + \frac{1}{2}(m_1 + m_2)v_r^2 - m_2 g(l - r).$$

At the turning points $v_r = 0$, and this equation becomes

$$\frac{1}{2}m_1\omega_0^2 l^2\left(1 - \frac{l^2}{r^2}\right) + m_2 g(l - r) = 0,$$

which can be written as

$$\frac{1}{2}m_1\omega_0^2 l^2(r^2 - l^2) + m_2 g r^2(l - r) = 0.$$

It is evident that $r = l$ is one root of this equation, and its elimination leads to

$$\frac{1}{2}m_1\omega_0^2 l^2(r + l) - m_2 g r^2 = 0,$$

or

$$m_2 g r^2 - \frac{1}{2}m_1\omega_0^2 l^2 r - \frac{1}{2}m_1\omega_0^2 l^3 = 0,$$

whose solutions, using (4), are

$$r_\pm = \frac{1}{4}\frac{r_c^3}{l^2}\left[1 \pm \sqrt{1 + \frac{(2l)^3}{r_c^3}}\right]. \tag{6}$$

The negative root gives $r_- < 0$, which is not allowed since r is a positive quantity between 0 and l, which leaves only two solutions, namely $r = l$ and $r = r_+$. Realizing also that $((2l)^3/r_c^3) \gg 1$, the student can demonstrate that

$$r_+ \cong r_c\sqrt{\frac{r_c}{2l}}, \tag{7}$$

which implies $r_+ < r_c$.

As a result, following picture arises from these solutions: At the start of the motion, *i.e.* at $t = 0$, $r = l$, $\omega = \omega_0$ and $v_r = 0$; from this time on, ω and v_r increase until the inflection point at $r = r_c$ is reached, then the radial speed of the particle slows down, and arrives at a turning point r_+, and from this point on, r increases until $r = l$, and the motion becomes cyclic. No equation of motion is obtained, but the main physical behavior is understood in this way.

3. Conclusions

Using conservation theorems and only the mathematical tools a student of elementary physics curses has, the main physical behavior of the system is obtained, and Eqs. (1), (4) and (7) give the student the opportunity to explore the behavior of mass m_1 for many different particular cases, for example, from Eq. (4), it is readily seen that the relation

$$\frac{r_c}{l} = \left\{ \frac{m_1 l \omega_0^2}{m_2 g} \right\}^{\frac{1}{3}} = \left\{ \frac{L}{m_2 g} \right\}^{\frac{1}{3}},$$

giving the position of the inflection point relative to the length of the string, indicates that the higher the angular momentum, or the lower the weight of the hanging body, the closer is r_c to l, and together with the relation obtained from Eq. (7):

$$\frac{r_+}{r_c} \cong \sqrt{\frac{r_c}{2l}},$$

they indicate that the equilibrium radius for the static case (at $r = r_c$) does not lie at the middle point of the turning points.

Also, these equations will let the student know the appropriate conditions at the laboratory to design an experiment. Additionally, advanced students who have arrived at the differential equations, without its numerical solution, will find this treatment illustrative.

---

1. J. Lock, An alternative approach to the teaching of rotational dynamics, *Am. J. Phys.* **57** (1989) 428, `https://doi.org/10.1119/1.15996`.
2. R. Resnick, D. Halliday, and K.S. Krane, Physics, vol. 1, 5th ed. (John Wiley & Sons, USA, 2002), pp. 113, EX. 39, Ch. 5.
3. D. Morin, Introduction to Classical Mechanics (Cambridge University Press, New York, 2008), p. 240.
4. M. R. Spiegel, Theoretical Mechanics, d ed. (Schaum's Outline Series, McGraw Hill Book Company, USA, 1967), p. 309.