The motion of a particle on the surface of a general cone

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We use the formalism of Lagrange to find the equations of motion of a particle on the inner surface of a "general cone". The equations of motion are challenging to solve, but we can evaluate them numerically with different software, to obtain the particle's trajectory on the surface as a function of parameters such as angular momentum L_θ, cone shape and initial conditions, and then we find the total free-fall time of the particle. The results show a special cone in which the free fall time has a minimum for a fixed angular momentum and fall height. Differences in the free-fall times and the particle's are also analyzed for a two-coordinate (r, z) and three-coordinate (r, θ, z) system. This work shows the importance of learning to use software (*Wolfram Mathematica*, *Python*, *POV-Ray*) to help with some complex theoretical problems. Finally, the results can easily be generalized for other more complex surfaces.

Keywords: Angular momentum; cone; Wolfram mathematica.

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1. Introduction

The problem of finding the equation of motion of a particle sliding on the surface of a cone is studied in elementary and advanced theoretical mechanics courses [1–3]. The differential equations cannot be solved analytically, and only general results can be found for the particle's path [4–6]. In this article, we propose to study the movement of a particle on a cone with different contours [7]:

$$z = ar^n, \tag{1}$$

where n is a positive integer which parameterizes the shape, a is a constant (in this work we set $a = 1$, with units $[L]^{-(n-1)}$). The case $n = 1$ is the one studied in elementary mechanics courses [2]. We use a cylindrical coordinate system (ρ, θ, z) to locate the particle's position on the cone's surface [8]. The main goal of the present work is to find the time it takes for a particle to fall on the cone's surface for different values of the exponent n in the Eq. (1). We consider the particle under a constant gravitational field g, without friction with the surface. The trajectory of a particle is found with the formalism of the Euler-Lagrange equation:

$$\left(\frac{\partial \mathscr{L}}{\partial q_i}\right) - \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right) = 0, \tag{2}$$

where $\mathscr{L}(q_i, \dot{q}_i)$ is the *Lagrangian* of the system in generalized coordinates q_i. The *Lagrangian* is the difference between kinetic energy $K(\dot{q}_1)$ and potential energy $U(q_i)$:

$$\mathscr{L}(q, \dot{q}) = K(\dot{q}_i) - U(q_i), \tag{3}$$

where the index i runs over $i = 1, 2, 3$ for a three-dimensional system, and with $i = 1, 2$ for a two-dimensional one. In particular, we use $q_1 = r$, $q_2 = \theta$, $q_3 = z$ and for kinetic energy we need the square of the velocity:

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2, \tag{4}$$

but if we restrict the particle to the surface of the cone, Eq. (1), we get:

$$v^2 = \dot{r}^2(1 + n^2 r^{2(n-1)}) + r^2\dot{\theta}^2. \tag{5}$$

For potential energy U, we have only the effect of gravity that depends on the height z, we get:

$$U = mgz = mg\, r^n. \tag{6}$$

Then the *Lagrangian* of the system is:

$$\mathscr{L} = \frac{1}{2}m(\,\dot{r}^2(1 + n^2 r^{2(n-1)}) + r^2\dot{\theta}^2) - mg\, r^n. \tag{7}$$

The *Lagrangian* depends on two coordinates, the radial distance r and angular position θ. Then we have two-equation of motion (2).

For r:

$$\ddot{r}(1 + n^2 r^{2(n-1)}) + gnr^{n-1} + \dot{r}^2 n^2 (n-1) r^{2n-3} - r\dot{\theta}^2 = 0. \tag{8}$$

For θ:

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0. \tag{9}$$

From Eq. (9), we get the conservation of angular momentum:

$$mr^2\dot{\theta} = L_\theta. \tag{10}$$

System **A**, the particle's trajectory is calculated by numerically solving Eqs. (8) and (9), which depends on two coordinates $(r, \theta,)$. In this work, we use a program written in *Python*; specific details of the code are available in the link of the Ref. [9] (as an alternative, we also provide code for the software *Wolfram Mathematica*, which is much more flexible and powerful and provides a real-time animations of the particle motion). Using Eqs. (1) and (8), we find an expression for the total distance of fall z_T, when radial velocity is considered equal to zero $\ddot{r} = \dot{r} = 0$, we have:

$$z_T = \left(\frac{L_\theta^2}{g\,n\,m^2}\right)^{\frac{n}{n+2}}. \tag{11}$$

Total energy E is conserved (the system does not dissipate energy), using the Eqs. (5) and (6), we have:

$$E = \frac{1}{2}m(\,\dot{r}^2(1+n^2r^{2(n-1)}) + r^2\dot{\theta}^2) + mg\,r^n, \tag{12}$$

where we can define the effective potential:

$$V_{eff} = \frac{1}{2}mr^2\dot{\theta}^2 + mg\,r^n. \tag{13}$$

Equations (12) and (13) depend on the optional parameters n, E, and L_θ. The particle must fall along the surface of the cone, reaching a minimum value, and then by conservation of energy, the particle tends to rise until it reaches a point of maximum height, *i.e.* the system has two values of return r_1 and r_2 for the values of angular momentum L_θ and energy E. Then, setting the radial velocity equal to zero $\dot{r} = 0$ in Eq. (12), and using Eq. (10), we find:

$$E = \frac{L_\theta^2}{2mr^2} + mg\,r^n. \tag{14}$$

Using this equation for the return points with a fixed angular momentum L_θ, we find.

$$L_\theta^2 = \frac{(r_2^n - r_1^n)(r_1^2 r_2^2)}{r_2^2 - r_1^2}(2m^2g). \tag{15}$$

Then, using relation (1) we have an expression for the value of L_θ that can be adjusted for specific values of r_1 and r_2.

On the other hand, for the case when θ is fixed, that we call system **B**, the angular dependence $\dot{\theta}$ is eliminated in Eq. (8), and then we have only one equation for radial distance, *i.e.* the particle falls along the plane (r, z)

$$\ddot{r}(1+n^2r^{2(n-1)}) + gnr^{n-1} + \dot{r}^2n^2(n-1)r^{2n-3} = 0. \tag{16}$$

2. Exercises

1.- We can use a different equation for the shape of the cone [7]:

$$z = a(r-h)^n, \tag{17}$$

where a and h are arbitrary parameters. Find the equations of motion equivalent to Eqs. (8) and (9) using this new Eq. (17).

2.- The choice of the cylindrical coordinate system is arbitrary. Find the condition of motion (15) for a spherical coordinate system.

3. Results

An analytical solution to the particle trajectory is complicated, so it is necessary to use numerical methods. The system of Eqs. (8) and (9) was solved using the *Python* programming language (which is free) and the *Wolfram Mathematica software* (see Ref. [9]), where the following values of the physical constants were used for all calculations, $g = 9.81$ kg/m s^2, and $m = 1$ kg.

In Fig. 1, we show the free fall height z as a function of time for the system **A** and different values of angular momentum L_θ. The initial conditions for the solution of Eqs. (8) and (9) are $r(0) = 1.0$ m, $\dot{r}(0) = 0$ m/s, $\theta(0) = 0.0$ rad, with $n = 2$. We can see that the free fall height varies depending on the angular momentum used. For $L_\theta = 1/2$ kg m^2/s, we have the most significant displacement in the z coordinate, and for $L_\theta = 3$ kg m^2/s, the smallest. These extreme values can be roughly understood using Eq. (15) with $n = 2$, we find:

$$z = \frac{L_\theta^2}{2m^2r_1^2g}, \tag{18}$$

evaluating for $L_\theta = 1/2$ kg m^2/s and $r_1 = r(0) = 1.0$ m, we have $z = 0.013$ m (and for $L_\theta = 3$ kg m^2/s we have $z = 0.46$ m) values that agree with the results found by the numerical solution. For negative values of angular momentum L_θ, the same results are obtained when we plot the trajectory in the z coordinates (or x coordinates, not shown).

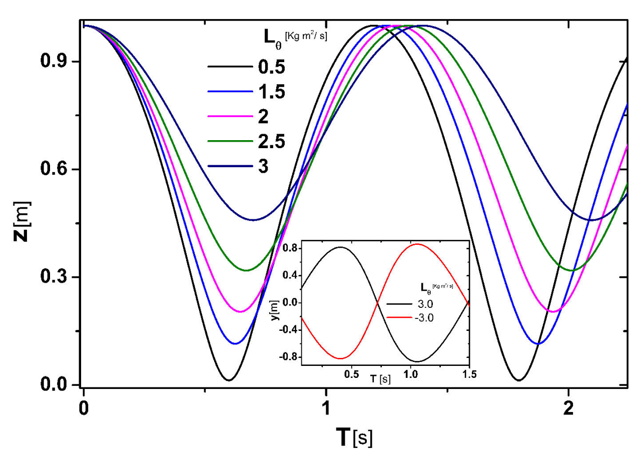

FIGURE 1. The system **A**, the free fall height z as a function of time for different values of angular momentum L_θ and with parameters $n = 2$, $r(0) = 1.0$ m, $\dot{r}(0) = 0$ m/s, $\theta(0) = 0.0$ rad. The inset shows the case of a negative value of angular momentum on the y-axis.

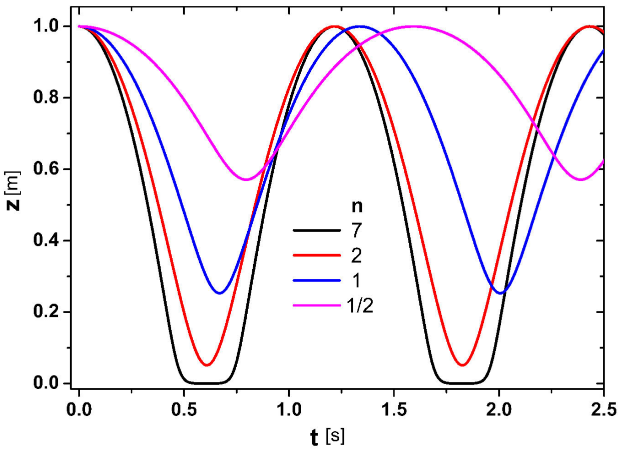

FIGURE 2. The system **A**, the free fall height z as a function of time for different values of the exponent n in the equation of cone and with parameters $L_\theta = 1.0$ kg m^2/s, $r(0) = 1.0$ m, $\dot{r}(0) = 0$ m/s, $\theta(0) = 0.0$ rad.

However, for the direction in y the trajectories are out of phase by $\pi/2$, as seen in the inset in Fig. 1, with $L_\theta = -3$ kg m^2/s. Also, it can be seen that the particle exhibits an oscillatory motion, and the period of oscillation is a function of L_θ.

In Fig. 2, we analyze the system when the exponent n in Eq. (1) is varied, with angular momentum fixed to $L_\theta = 1$ kg m^2/s and initial conditions $r(0) = 1.0$ m, $\dot{r}(0) = 0$ m/s, $\theta(0) = 0.0$ rad/s. We can see how with large values of n ($n = 7$) the free fall height is significant $z \approx 1$ m, while when n is small ($n = 0.5$) the free- fall height is $z \approx 0.4$ m. As expected, the system also shows oscillations and the period of oscillation is inversely proportional to the exponent n in Eq. (1).

Figure 3 shows the trajectories in the $x - y$ and $x - z$ planes for three values of exponent n. In the $x - z$ plane, we can see the system with different profiles, a "parabolic one" with $n = 2$, a "linear one" with $n = 1$, and for $n = 0.5$ one with the profile $z = \sqrt{r}$. Note how the scale on the z-axis is different in the three cases. However, for the $x - y$ plane, the scale of the axes is the same, but the trajectories of the particles show appreciable differences. In Ref. [9], you can download *Wolfram Mathematica*, *Python* and *POV-Ray* codes to visualize the particle trajectory in space. Using the *Wolfram Mathematica* code, one can detect the effect on the particle trajectory of changing parameters such as the value of the exponent in Eq. (1), the value of the angular momentum L_θ, initial conditions in Eq. (8) and the total time of the trajectory, all these changes are updated in real-time to see the motion of the particle in the cone. For example, varying the full-time gives an animation of the motion of the particle. The code of *POV-Ray* is shown to generate images (with high resolution) of the particle trajectory, which can be joined together to create a movie. These software are simple and easy to learn, so you have an essential numerical tool that can be used in various calculations, such as those performed in this article.

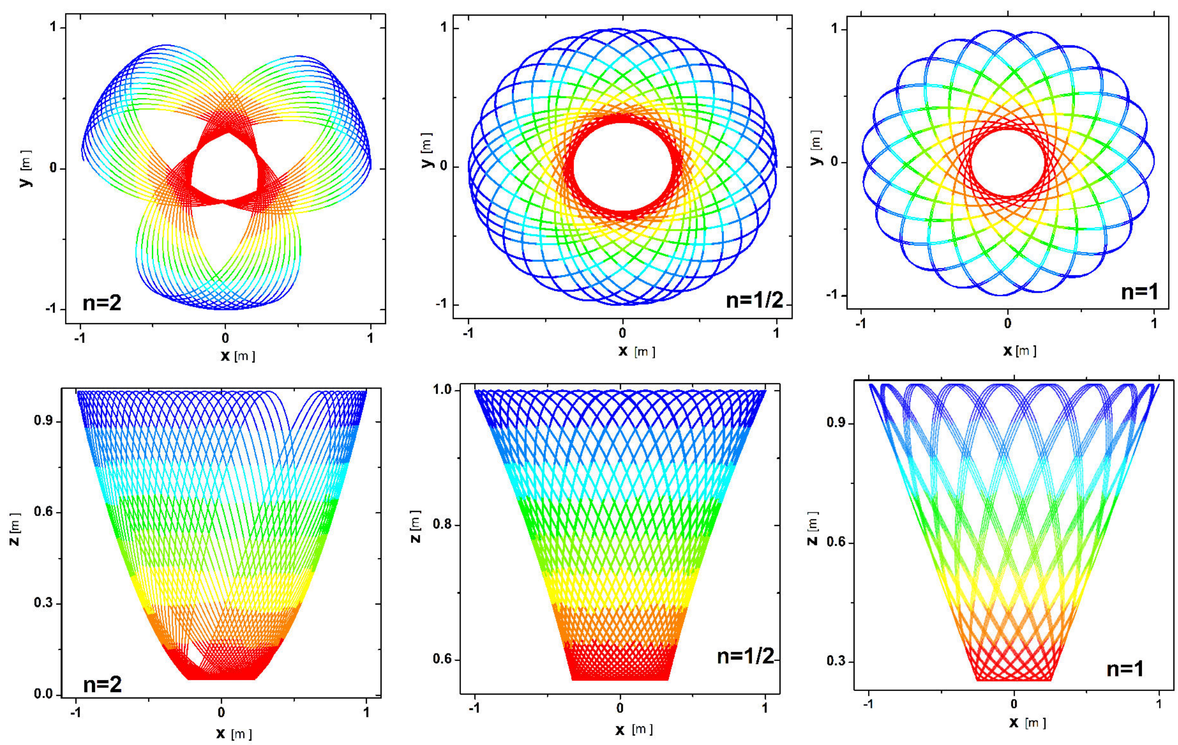

FIGURE 3. The system **A**, the trajectory of the particle for the plane $x - z$ (down) and $x - y$ (top), for different values of the exponent n, and the same parameters as Fig. 2. *Wolfram Mathematica*, *Python* and *POV-Ray* [11] codes for other visualizations of the trajectory can be found in Ref. [9].

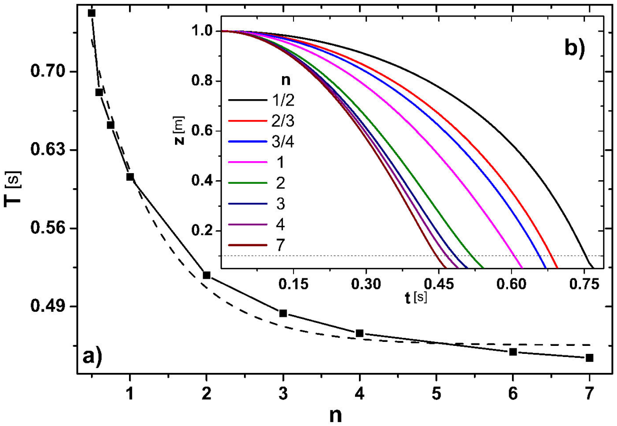

FIGURE 4. The system **A**, a) shows the free fall time as a function of the value of n in the equation of the cone, with parameters $L_\theta = 0.01\ \mathrm{kg\,m^2/s}$, $r(0) = 1.0$ m, $\dot{r}(0) = 0$ m/s, $\theta(0) = 0.0$ rad. b) Shows the fall height z as a function of time for different values of the exponent n and the same parameters. The dotted line sets the cut-off distance $z = 0.1$ m.

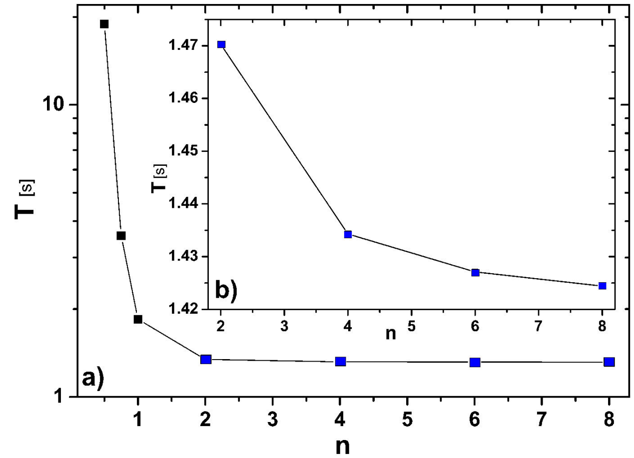

FIGURE 6. The system **B**. a) Shows the free fall time as a function of the value of n in the equation of the cone (1), with parameters $r(0) = 10.0$ m, $\dot{r}(0) = 0$ m/s. b) Shows a zoom for the last four points.

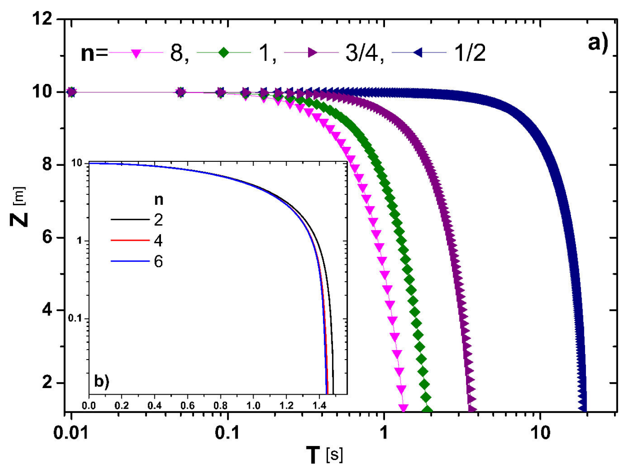

FIGURE 5. The system **B**. a) Shows the the fall height z as a function of time for different values of the exponent n in the Eq. 1) and the parameters $r(0) = 10.0$ m, $\dot{r}(0) = 0$ m/s. b) Is the same but for other values of n.

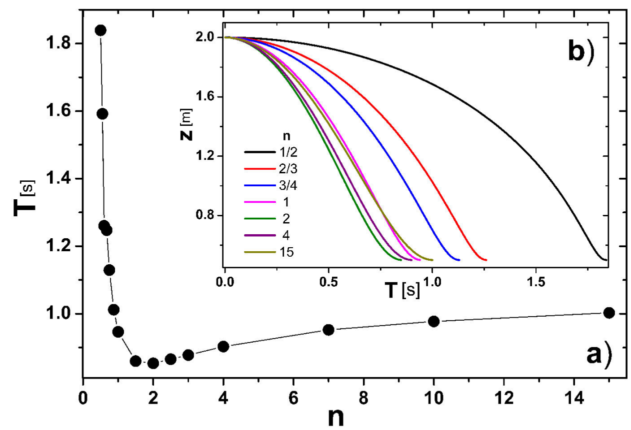

FIGURE 7. The system **A**. a) Shows the free fall time as a function of the value of n in the equation of the cone (1), with parameters $r(0) = 2^{1/n}$ m, $\dot{r}(0) = 0$ m/s, and for L_θ use Eq. (15). b) Shows the fall height z as a function of time for different values of the exponent n in the equation of the cone.

For a better comparison of the free fall time, the particle has to fall the same distance when the value of the exponent n is changed, and this is achieved by choosing a small value of the angular momentum L_θ [Eq. (11)]. Figure 4b) shows the results of the free fall height for a particle with the parameter of $L_\theta = 0.01\ \mathrm{kg\,m^2/s}$. It can be seen how the decay time increases as the value of the exponent n in Eq. (1) decreases. For large values of n, the changes in the free fall height are small. Figure 4a) shows the decay times, selecting a cutoff distance of $z = 0.1$ m [dotted line in Fig. 4b)] as a function of the exponent n in Eq. (1). It is observed how the time decays can be fitted to an exponential function $T = 0.45 + 0.47e^{-1.11n}$, the dotted line in Fig. 4a).

Figure 5 shows the free-fall height z as a function of time for the system **B**, when the equation of motion (16) is solved numerically. The initial conditions are $r(0) = 10.0$ m, $\dot{r}(0) = 0$ m/s. The free-fall height shows the same trend as Fig. 4 (the decay time increases as the value of the exponent n in Eq. (1) decreases). For large values of n, the changes in the decay path are small, as can be seen in Fig. 5b). Figure 6 shows the decay time for the system **B** as a function of the exponent n. For large values of n, the time is approximately constant $t \approx 1.4$ [see Fig. 6b)], while for small values of n, the decay time tends to large values $t \approx 14$.

Finally, in Fig. 7, we set the $z_1 = r_1^n$ and $z_2 = r_2^n$ distances, and then we calculate the value of the angular momentum L_θ with Eq. (15). For the system **A**, the inset in Fig. 7b) shows the free fall height results as a function time for various values of the exponent n. It is observed how the times decreases as the value of n increases until it reaches the value of $n = 2$, where the behavior is reversed. Figure 7a) shows a value in which the free-fall time is minimum for the value $n \approx 1.75$ and then tends to be constant for large values of n, *i.e.*, the decay time is not an exponential function. This result contrasts with the cases mentioned above in Figs. 4a) and 6a).

4. Conclusions

In this work, we find the equations of motion of a particle falling on the surface of a general cone. The equations of motion are numerically solved and then the particle's trajectory can be calculated. We analyze the free fall times as a function of parameters such as the angular momentum L_θ, Eq. (10), the value of n in the equation of the cone, Eq. (1), and for a systems with one (system **B**) or two degrees of freedom (system **A**). We found that by varying the value of n, the free fall time decays continuously, showing that particles with large values of the exponent n in the profile fall first. On the other hand, if we fix the fall distance and adjust the value of the angular momentum L_θ, the system shows a minimum for the free fall time when the cone profile has an exponent $n \approx 1.75$ in Eq. (1). These results show how the numerical solution of the Lagrangian equations generates exciting results, which can give us new information about systems studied in elementary courses.

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