Wavelet eXtropy of fractal signals

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Recently, the concept of eXtropy was proposed as a complementary dual of Shannon entropy. This article extends the standard time-domain eXtropy concept to the time-scale domain and then obtains a closed-form expression for this wavelet eXtropy for fractal signals of parameter α. A didactic study of the wavelet eXtropy of fractal signals reveals that this infomation-theory quantifier increases for short-memory fractal signals, is maximum for white noise ($\alpha = 0$) and decreases for long-memory fractal processes. Compared to the standard wavelet entropy, wavelet eXtropy performs similar, however has lower variability for stationary fractal signals and higher variability for nonstationary ones. Moreover, the computation of fractality based eXtropy planes allows to highlight further properties and also potential applications for the analysis/estimation of fractals.

Keywords: Fractal; eXtropy; wavelets; wavelet entropy; wavelet eXtropy; fractal analysis.

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1. Introduction

The analysis of fractals by means of information theory quantifiers permits to extend and complement traditional time, frequency and time-scale approaches [1] such as detrended fluctuation analysis (DFA) [2], rescaled range(R/S) statistic [3], Periodogram [4], wavelets [5,6], etc., by more accurate and efficient approaches such as Shannon entropy and Fisher's information measure (FIM) [7,8]. Information theory-based methods not only allow to characterize the information content and complexity within a fractal signal but also to provide a deeper understanding of the signal such as classifying them as stationary/nonstationary [9], short-memory/long-memory and even a more general classification within some scaling range. The analysis of time series by traditional Shannon and Tsallis q-entropies have characterized complexity in electroencephalogram (EEG) signals [10], identified structural damage [11] and analyzed event related potentials in neuroelectrical signals [12,13]. In the fractal analysis context, wavelet entropy has been used for the classification of fractal signals(as stationary/nostationary, long-memory/short-memory, etc) [9], detection of level-shifts [14], estimation of the fractality parameter α among others. Recently, the concept of eXtropy has been proposed in the literature as a complementary dual of Shannon entropy [15] and is currently considered a research hotspot in physics. The purpose of the present article is twofold, first, to extend the standard time domain eXtropy concept to the wavelet domain and second to obtain a closed-form expression for the wavelet eXtropy of fractal signals of parameter α. The latter, will not only allow to compare wavelet eXtropy planes for fractal signals but also to completely characterize wavelet eXtropy values for the different fractals. In addition, wavelet eXtropy planes permits to perform a comparison with wavelet entropy and to highlight potential applications. When compared to wavelet entropy, wavelet eXtropy presents similar behaviour; increasing for stationary fractal signals and decreasing for nonstationary ones, however a key difference is that wavelet eXtropy presents lower variability within fractal stationary processes. The rest of the present contribution is structured as follows. Section 2 reviews some wavelet related results for fractal procesess, specifically, quantities that allow to compute probability mass functions (pmfs) from wavelet representations. Section 3 proposes a wavelet eXtropy definition based on a discrete-wavelet transform(DWT)-based wavelet spectrum, obtains a closed-form expression of this eXtropy for fractal processes and computes wavelet eXtropy planes. Furthermore, this section details eXtropy behaviour for the variety of fractal processes and discusses potential applications for the analysis/estimation of fractality. Section 4 concludes the paper.

2. Wavelet analysis of fractal signals

Wavelet transforms play a special role in the analysis of signals in physics [16] and other scientific areas [17,18]. Specifically adequate for nonstationary signal analysis, wavelet transforms can be computed in continuosly varying time/scale settings (CWT) or discrete ones (DWT). The discrete wavelet transform (DWT) of a signal X_t is given by:

$$d_X(j,k) = \int_{-\infty}^{\infty} X_t\, \psi_{j,k}(t)\, dt, \tag{1}$$

where $\psi_{j,k}(t) = 2^{-j/2}\psi(2^{-j}t - k)$ is a dyadically scaled and integer-shifted mother wavelet which satisfies admissibility condition. For random signals, the first and second moments of the DWT are usually employed, the first moment is defined as follows,

$$\mathbb{E}d_X(j,k) = \mathbb{E}\left\{\int_{-\infty}^{\infty} X_t \psi_{j,k}(t)dt\right\} = 0, \quad (2)$$

while, the second, also known in the literature as the wavelet spectrum [19], takes the form given by the following relation:

$$\mathbb{E}d_X^2(j,k) = \int_{-\infty}^{\infty} S_X(2^{-j}f)|\Psi(f)|^2\, df, \quad (3)$$

where $\Psi(f) = \int \psi(t)\mathrm{e}^{-j2\pi ft}\, dt$ is the Fourier integral of the mother wavelet $\psi(t)$, $S_X(.)$ is the power spectral density (PSD) of the process X_t, $\mathbb{E}$, the expectation operator and $d_X(j,k)$ is the DWT of the process X_t at time k and wavelet scale j. Recall that for fractal signals of parameter α, $S_x(f) \sim c|f|^{-\alpha}$ and thus by substituting this well-known PSD into Eq. (3) results in the wavelet spectrum of fractals signals of parameter α, *i.e.*,

$$\mathbb{E}d_X^2(j,k) = 2^{j\alpha}\, C(\alpha,\psi), \quad (4)$$

where $C(\alpha,\psi)$ is a constant that depends on α and the mother wavelet $\psi(t)$ [20]. The wavelet spectrum of fractal processes as given by (4) has been used in the literature for the estimation of the parameter α [19] and for the computation of the so-called wavelet energy [21]. The wavelet energy at scale j is defined as [22],

$$p_j = \frac{\mathbb{E}d_X^2(j,k)}{\sum_j \mathbb{E}d_X^2(j,k)}. \quad (5)$$

Wavelet energy p_j satisfies the axioms of probability, *i.e.*, $0 \le p_j \le 1$, $\sum_j p_j = 1$ and is therefore a probability function or probability mass funcion (pmf) which represents the probability that the energy of a signal to be located at wavelet scale j. From the wavelet energy p_j, many statistical quantities can be computed including entropies, eXtropies, and other information theory quantifiers such as Fisher-Shannon information planes, etc. For fractal signals, their wavelet energy, p_j, is given by,

$$p_j = 2^{(j-1)\alpha} \times \frac{1-2^{\alpha}}{1-2^{\alpha M}}, \quad (6)$$

where α is the fractality parameter and M is the number of scales considered within a signal.

3. Wavelet eXtropy

The wavelet entropy of a fractal process measures the information content or complexity of a random signal or system in the time-scale or wavelet domain. Wavelet entropy, $\mathcal{H}(p)$, is defined as [21,22],

$$\mathcal{H}(p) = -\sum_j p_j \log(p_j), \quad (7)$$

for a given pmf p_j. For fractal processes, the wavelet entropy is computed by substituting Eq. (6) into (7) which results in the wavelet entropy of fractal processes given by the following relation,

$$\mathcal{H}(p) = \alpha\left\{\frac{1}{1-2^{-\alpha}} - \frac{M}{1-2^{-\alpha M}}\right\} - \log 2\left\{\frac{1-2^{\alpha}}{1-2^{\alpha M}}\right\}. \quad (8)$$

From Eq. (8), many important results have been obtained in the literature for fractal Processes. For instance, wavelet entropy is maximum ($H \to 1$) when $\alpha \to 0$ and for stationary short-memory processes it is increasing with increasing α and for long-memory signals, decreasing for increasing α. For nonstationary fractal signals it is completely nonincreasing [22-25]. Wavelet entropy has found extensive applications in many areas of science and technology. Recently, the concept of eXtropy has been introduced as a complementary dual of Shannon entropy [15]. As a matter of fact, the eXtropy concept is currently a research hotspot in physics and statistics and many results such as maximum entropy, belief entropy, etc., are being extended to their eXtropy counterparts giving rise to maximum eXtropy [26], belief eXtropy [27,28], etc. The main purpose and result of the present article is to extend the eXtropy concept to the time-scale domain a give a didactic presentation of their behaviour for fractal processes of parameter α. In order to extend the eXtropy concept to the time-scale domain, Eq. (6) is applied to the standard definition of eXtropy given by the following equation:

$$\mathcal{J}(p) = -\sum_j (1-p_j)\log_2(1-p_j), \quad (9)$$

which results in the following:

$$J(p) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(\frac{2^{\alpha}-1}{1-2^{\alpha M}}\right)^n \times \left\{\frac{1-2^{\alpha}}{1-2^{\alpha M}}\frac{1-2^{(\alpha+\alpha n)M}}{1-2^{(\alpha+\alpha n)}} - \frac{1-2^{\alpha n M}}{1-2^{\alpha n}}\right\}. \quad (10)$$

Equation (10) therefore represents the wavelet eXtropy of fractal signals with parameter α from which many important results can be obtained for the different regions of the parameter α. Recall that for fractal processes, the values of α dictate their statistical behaviour. For instance, it is well known that $\alpha = 0$ represents a completely disorder process, white noise, for $-1 < \alpha < 1$, the process is stationary and for $\alpha > 1$ it is nonstationary. Moreover for $\alpha \in (-1,0)$, the process is short-memory and stationary and for $\alpha \in (0,1)$ it is stationary long-memory. Therefore, the purpose of the article is to completely characterize wavelet eXtropy within these regions in a similar way wavelet entropy was characterized previously in the literature In the following a normalized version of eXtropy and entropy is used in order to provide a comparison of their behaviour.

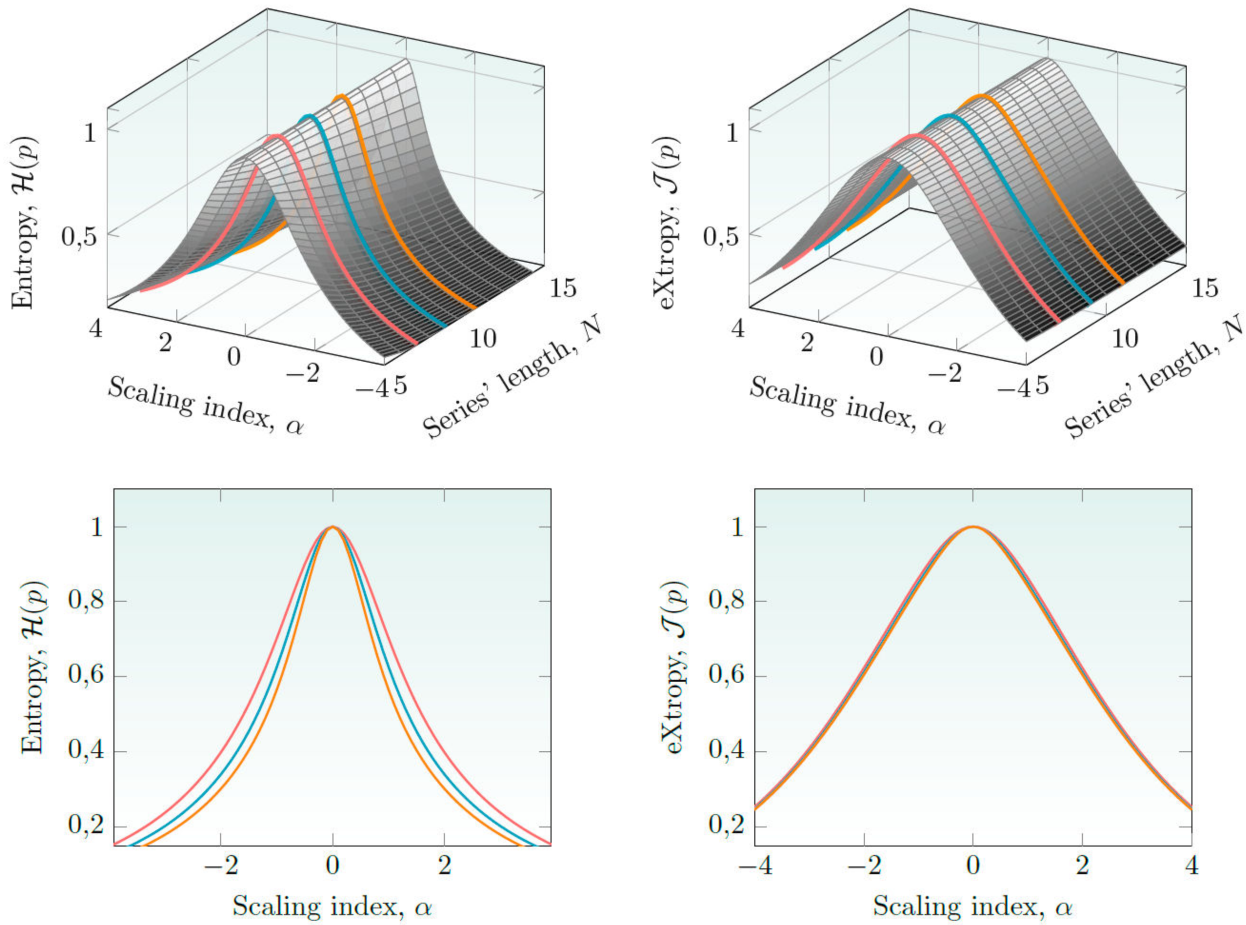

FIGURE 1. Wavelet information planes for fractal signals of parameter α. Top-left plot is the wavelet entropy plane within the range $-4 < \alpha < 4$ and varying time series' length. Top-right plot is the corresponding plane for the wavelet eXtropy within the same fractality index and time series' length range. 2D cross sections for lengths 2^6, 2^8 and 2^{11} for both quantifiers are illustrated in bottom plots.

Figure 1 displays the information planes for the wavelet entropy and wavelet eXtropy of fractal processes of parameter α. Note from the top plots that the behaviour of both information theory quantifiers is similar; maximum at $\alpha = 0$, increasing for $\alpha < 0$ and decreasing for $\alpha > 0$. Although similar behaviour is observed in both information planes, small differences between wavelet entropy and eXtropy are clear. First, a 2D cross section of the 3D information eXtropy plane shows that wavelet eXtropy is independent of time series' length (see bottom plot of Fig. 1), in addition, wavelet eXtropy appears to be wider than wavelet entropy meaning that an identical entropy/eXtropy range is covered by a wider scaling range. The latter result, in principle, indicates that wavelet entropy is a 'better' information quantifier than wavelet eXtropy, however, as we will see next this is not the case. Figure 2, shows a version of the 2D plots given in Fig. 2 within the fractality range $-2 < \alpha < 3$. Note from this figure that both information theory quantifiers are shown within the same plot which allows to better inspect the similarities and differences between entropy and eXtropy values. From this plot, and considering first stationary $(-1 < \alpha < 1)$ and nonstationary $(1 < \alpha < 3)$ fractal signals, it is clear

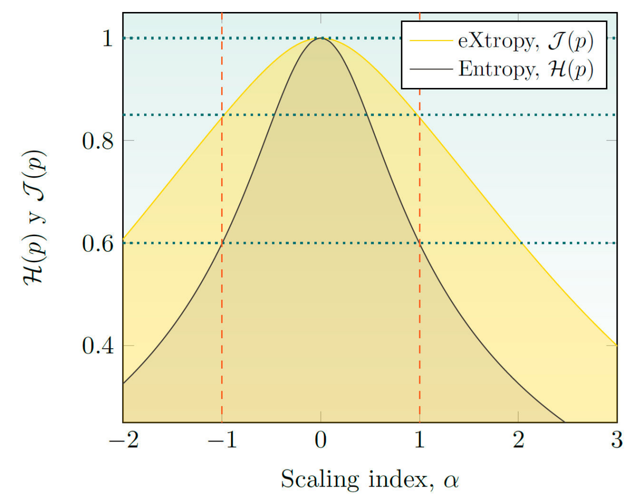

FIGURE 2. Wavelet entropy/eXtropy for fractal signals of parameter α. Wavelet eXtropy displays lower variability values than those observed for wavelet entropy for stationary signals, however, the variability for wavelet eXtropy is higher when considering nonstationary fractal signals ($\alpha > 1$).

that wavelet eXtropy presents lower variability than wavelet entropy for stationary fractal signals. In fact, wavelet eXtropy values will vary from 0.85 to 1 and wavelet entropy ones from 0.6 to 1. For nonstationary fractal signals, however, wavelet eXtropy values are more variable than those observed for wavelet entropy. Wavelet eXtropy, therefore is a valuable tool for characterizing the complexities associated to fractal signals. Wavelet eXtropy is maximum for completely random o disordered fractals such as white noise ($\alpha = 0$), is increasing for $\alpha < 0$ and decreasing for $\alpha > 0$. In addition, presents low variability for stationary fractal signals and high variability for nonstationary ones.

Based on the above, a classification scheme for stationary/nonstationary fractality can be derived from this variability behaviour.

4. Conclusions

In this contribution, an extension of the time-domain eXtropy concept to the wavelet domain was first presented. A closed-form expression for this wavelet eXtropy was then obtained for fractal signals of parameter α and later wavelet eXtropy information planes were computed in a range of both the fractality index and time series' length. Wavelet eXtropy is maximum for white noise, increasing for stationary fractals and decreasing for nonstationary ones. Wavelet eXtropy presents similar behaviour as that of wavelet entropy, however, has lower variability for stationary fractals and higher variability from nonstationary ones. Finally, from this behaviour, a classification scheme based on variability may be derived for fractal processes.

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