The propagator of the inverted Caldirola-Kanai oscillator

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In this paper, we present three methods to calculate the propagator for the inverted Caldirola-Kanai oscillator. The first method is the Feynman path integral. The second method was formulated by Schwinger for deriving the relativistic Green function but has rarely been applied to calculate the non-relativistic propagator. The third method is the application of the integrals of the motion of a quantum system in evaluating the propagator. The comparison of advantages and difficulties of each method is also discussed.

1. Introduction

The wave mechanics of E. Schrödinger is the accepted method for comprehending the nature of quantum physics [1]. This approach aims to solve Schrödinger's equation and produce the wave function $\psi(x, t)$, which may be understood as the probability amplitude of finding a particle at the coordinates x and t. Propagator methods are a substitute strategy. The propagator is the amplitude of the transition probability for a particle moving from the starting location (x', t') to the destination (x'', t''). The three approaches for computing the non-relativistic propagator are presented in this study. The Feynman path integral [2] is the first technique. R. P. Feynman [3] first proposed this approach, and it was subsequently used to address issues in both quantum mechanics and quantum field theory [4, 5]. The Feynman path integral, which is discussed in an undergraduate quantum mechanics course, has recently become the most widely used method for calculating the non-relativistic propagator. The Schwinger approach is the second technique. When estimating the relativistic Green function for a charged particle with a spin of 1/2 in constant and plane wave external electromagnetic fields, J. Schwinger developed this technique [6]. Later, the propagator in both relativistic and non-relativistic quantum mechanics was calculated using the Schwinger method [7-12]. This approach can only be used in quantum field theory, though. In 1975, V. V. Dodonov, I. A. Malkin, and V. I. Man'ko developed the third approach [13]. This approach is based on the relationship between a quantum system's Green function or propagator and the integrals of the motion. The initial position and momentum operator can be used to express the propagator, which is the eigenfunction of the integrals of the motion. This approach for computing the propagator has numerous applications in both relativistic and non-relativistic quantum mechanics, according to a research study [14-16], although it is not covered in textbooks on quantum mechanics and quantum field theory.

This essay also compares the benefits and drawbacks of each teaching approach for students and teachers. Inverted Caldilora-Kanai oscillators are the systems used to illustrate the methods in this paper [17]. This oscillator was chosen because it provides an accurate expression for the propagator and has numerous applications in researching the physics of black hole horizon events and quantum Hall lowest Landau levels [18]. We will talk about how these techniques relate to both classical and quantum mechanics in the end. The structure of this essay is as follows. We will compute the propagator for an inverted Caldirola-Kanai oscillator using the Feynman path integral method in Sec. 2. In Sec. 3, the propagator of the same system as Sec. 2 was derived using the Schwinger approach. Section 4 explains how to use the integrals of the motion to evaluate the propagator. Finally, Sec. 5 provides the conclusion.

2. Feynman path integral for an inverted Caldirola-Kanai oscillator

In this section, the propagator for an inverted Caldirola-Kanai oscillator will be obtained by applying the Feynman path integral. S. Baskoutas and A. Jannussis [17] propose the following as the Hamiltonian of this oscillator:

$$H(t) = \frac{p^2}{2m}e^{-rt} - \frac{m}{2}\omega^2 x^2 e^{rt}, \tag{1}$$

where r is the damping constant coefficient, and ω is the constant frequency. The Hamiltonian's equivalent Lagrangian in Eq. (1) is

$$L(t) = \frac{1}{2}me^{rt}\dot{x}^2 + \frac{1}{2}me^{rt}\omega^2 x^2. \tag{2}$$

When the Lagrangian in Eq. (2) is subjected to the Euler-Lagrange equation, the resulting equation of motion has the following form:

$$\ddot{x} + r\dot{x} - \omega^2 x = 0. \tag{3}$$

The answer to (3) can be expressed as

$$x(t) = e^{\frac{-rt}{2}}[A\cosh\Omega t + B\sinh\Omega t], \tag{4}$$

where $\Omega^2 = \omega^2 + (r^2/4)$ where A and B are constants.

We define the classical paths between the points of (x', t') and (x'', t'') as

$$x_{cl}(t) = \frac{e^{-rt}}{\sinh \Omega(t'' - t')} \left[e^{\frac{rt'}{2}} \sinh \Omega(t'' - t)x' + e^{\frac{rt''}{2}} \sinh \Omega(t - t')x'' \right]. \tag{5}$$

We enforce the boundary conditions of $x(t') = x'$ and $x(t'') = x''$. This gives us the constants A and B in Eq. (4). The official definition of the action is provided by

$$S(x'', t''; x', t') = \int_{t'}^{t''} L(\dot{x}, x, t) dt. \tag{6}$$

By substituting the Lagrangian of (2) into (6) and then integrating portions of the first term of (6) with the aid of (3), the classical action can be expressed as follows:

$$S_{cl}(x'', t''; x', t') = \frac{m}{2} x''_{cl} \dot{x}''_{cl} - \frac{m}{2} x'_{cl} \dot{x}'_{cl}. \tag{7}$$

When we insert the classical paths from (5) into (7) to calculate the classical action, the outcome is

$$S_{cl}(x'', t''; x', t') = -\frac{mr}{4}(e^{rt''}x''^2 - e^{rt'}x'^2) + \frac{m\Omega}{2}(e^{rt''}x''^2 + e^{rt'}x'^2)\coth \Omega(t'' - t')$$

$$- m\Omega e^{\frac{r(t''+t')}{2}} \operatorname{csch}\Omega(t'' - t')x'x''. \tag{8}$$

Feynman [2] states that the quantum propagator is denoted as

$$K(x'', t''; x', t') = \int e^{\frac{i}{\hbar} S[x(t)]} D[x(t)], \tag{9}$$

where the measure $D[x(t)]$ is the integration over all paths connecting from (x', t') to (x'', t'').

The propagator for the general quadratic Lagrangian $L(t) = (1/2)a(t)\dot{x}^2 - (1/2)b(t)x^2 + c(t)x$ where $a(t)$, $b(t)$ and $c(t)$, are well-behaved functions of time, which can be computed from the semiclassical approximation of the path integral as [19]

$$K(x'', t''; x', t') \left[\frac{1}{2\pi i \hbar} \left| \frac{\partial^2 S_{cl}(x'', t''; x', t')}{\partial x' \partial x''} \right| \right]^{\frac{1}{2}} \exp \left[\frac{i}{\hbar} S_{cl}(x'', t''; x', t') \right]. \tag{10}$$

The pre-exponential function can be represented by replacing the traditional action of (8) with (10).

$$\left[\frac{1}{2\pi i \hbar} \left| \frac{\partial^2 S_{cl}(x'', t''; x', t')}{\partial x' \partial x''} \right| \right]^{\frac{1}{2}} = \left(\frac{m\Omega e^{r(t''+t')/2}}{2\pi i \hbar \sinh \Omega(t'' - t')} \right)^{\frac{1}{2}}. \tag{11}$$

We can get the appropriate propagator for the inverted Caldirola-Kanai oscillator by substituting (8) and (11) into (10), where

$$K(x'', t''; x', t') = \left(\frac{m\Omega e^{r(t''+t')/2}}{2\pi i \hbar \sinh \Omega(t'' - t')} \right)^{\frac{1}{2}} \exp \left[-\frac{imr}{4\hbar} \left(e^{rt''}x''^2 - e^{rt'}x'^2 \right) \right]$$

$$\times \exp \left[\frac{im\Omega}{2\hbar \sinh \Omega(t'' - t')}, \left(\left[e^{rt''}x''^2 + e^{rt'}x'^2 \right] \cosh \Omega(t'' - t') - 2e^{\frac{r(t''+t')}{2}} x'x'' \right). \tag{12}$$

The propagator of (12) can be rewritten as

$$K(x, x'; t) = \left(\frac{m\Omega e^{rt/2}}{2\pi i \hbar \sinh \Omega t} \right)^{\frac{1}{2}} \exp \left[-\frac{imr}{4\hbar} \left(e^{rt}x^2 - x'^2 \right) \right]$$

$$\times \exp \left(\frac{im\Omega}{2\hbar \sinh \Omega t} \left[\left(e^{rt}x^2 + x'^2 \right) \cosh \Omega t - 2e^{\frac{rt}{2}} xx' \right] \right), \tag{13}$$

where $x'' = x, x' = x', t'' = t$ and $t' = 0$.

3. Schwinger method for an inverted Caldirola-Kanai oscillator

The Heisenberg's equations of the operators $i\bar{h}(d\hat{x}(t)/dt) = [\hat{x}(t), \hat{H}(t)]$ and $i\bar{h}(d\hat{p}(t)/dt) = [\hat{p}(t), \hat{H}(t)]$ are solved in order to get the position operator $\hat{x}(t)$ and the momentum operator $\hat{p}(t)$ as

$$\hat{x}(t) = e^{-\frac{rt}{2}}\left(\cosh\Omega t + \frac{r}{2\Omega}\sinh\Omega t\right)\hat{x}(0) + \frac{e^{-\frac{rt}{2}}}{m\Omega}\sinh\Omega t\hat{p}(0), \tag{14}$$

and

$$\hat{p}(t) = \frac{m\omega^2}{\Omega}e^{\frac{rt}{2}}\sinh\Omega t\hat{x}(0) + e^{\frac{rt}{2}}\left(\cosh\Omega t - \frac{r\sinh\Omega t}{2\Omega}\right)\hat{p}(0), \tag{15}$$

where $\hat{x}(0)=\hat{x}(t=0)$, and $\hat{p}(0)=\hat{p}(t=0)$, and

$$\hat{H}(t) = e^{-rt}\frac{\hat{p}^2(t)}{2m} - \frac{1}{2}me^{rt}\omega^2\hat{x}^2(t). \tag{16}$$

By removing $\hat{p}(0)$ from (15) with the aid of (14), the momentum operator $\hat{p}(t)$ can be expressed in terms of $\hat{x}(t)$ and $\hat{x}(0)$ as:

$$\hat{p}(t) = me^{rt}\left(\Omega\coth\Omega t - \frac{r}{2}\right)\hat{x}(t) - \frac{m\Omega e^{\frac{rt}{2}}}{\sinh\Omega t}\hat{x}(0). \tag{17}$$

The Hamiltonian operator is then rewritten in time ordered by substituting (14) and (17) into (16) in such a way that, for each term of $\hat{H}(t)$, the operator $\hat{x}(t)$ must write on the left and the operator $\hat{x}(0)$ must write on the right, assisted by the commutator $[\hat{x}(0), \hat{x}(t)] = (i\bar{h}\sinh\Omega t/m\Omega)e^{-(rt/2)}$ as

$$\hat{H}_{\text{ord}}(t) = \frac{me^{rt}}{2}\left(\Omega^2\operatorname{csch}^2\Omega t - r\Omega\coth\Omega t + \frac{r^2}{2}\right)\hat{x}^2(t) - m\Omega e^{\frac{rt}{2}}\left(\Omega\operatorname{csch}\Omega t\coth\Omega t - \frac{r}{2}\operatorname{csch}\Omega t\right)$$

$$\times\hat{x}(t)\hat{x}(0) + \frac{1}{2}m\Omega^2\operatorname{csch}^2\Omega t\hat{x}^2(0) - \frac{i\bar{h}}{2}\left(\Omega\coth\Omega t - \frac{r}{2}\right). \tag{18}$$

The propagator can be determined using the equation

$$K(x, x'; t) = C(x, x')\exp\left(-\frac{i}{\bar{h}}\int_0^t \frac{<x(t)|\hat{H}_{ord}(t)|x'(0)>}{<x(t)|x'(0)>}dt\right) = C(x, x')\exp\left(-\frac{i}{\bar{h}}\int_0^t\left[\frac{1}{2}me^{rt}\left(\Omega^2\operatorname{csch}^2\Omega t\right.\right.\right.$$

$$\left.- r\Omega\coth\Omega t + \frac{r^2}{2}\right)x^2 + \frac{1}{2}m\Omega^2\operatorname{csch}^2\Omega tx'^2 - m\Omega e^{\frac{rt}{2}}\left(\Omega\operatorname{csch}\Omega t\coth\Omega t - \frac{r}{2}\operatorname{csch}\Omega t\right)xx'$$

$$\left.- \frac{i\bar{h}}{2}\left(\Omega\coth\Omega t - \frac{r}{2}\right)\right)dt\bigg], \tag{19}$$

where $C(x, x')$ is an arbitrary integration constant. This equation was provided by S. Pepore and B. Sukbot [10].

The following step is to integrate each term of (19). It is simple to determine how to integrate the first term of (19) across time by using the equation

$$-\frac{im}{2\bar{h}}x^2\int_0^t e^{rt}\left(\Omega^2\operatorname{csch}^2\Omega t - r\Omega\coth\Omega t + \frac{r^2}{2}\right)dt = \frac{im\Omega}{2\bar{h}}e^{rt}\coth\Omega tx^2 - \frac{imr}{4\bar{h}}e^{rt}x^2. \tag{20}$$

It is possible to integrate the second term of (19) as

$$-\frac{im\Omega^2}{2\bar{h}}x'^2\int_0^t\operatorname{csch}^2\Omega tdt = \frac{im\Omega}{2\bar{h}}\coth\Omega tx'^2. \tag{21}$$

In order to assess the third term of (19), the following relation can be used:

$$\frac{im\Omega}{\bar{h}}xx'\int_0^t e^{\frac{rt}{2}}\left(\Omega\operatorname{csch}\Omega t\coth\Omega t - \frac{r}{2}\operatorname{csch}\Omega t\right)dt = -\frac{im\Omega}{\bar{h}}e^{\frac{rt}{2}}\operatorname{csch}\Omega txx'. \tag{22}$$

Last but not least, the integration of the final term in Eq. (19) over time can be expressed as

$$-\int_0^t\left(\frac{\Omega}{2}\coth\Omega t - \frac{r}{4}\right)dt = -\frac{1}{2}\ln(\sinh\Omega t) + \frac{rt}{4}. \tag{23}$$

If we insert (20)-(23) into (19), the required propagator has the form:

$$K(x, x^{'}; t) = C(x, x^{'}) \sqrt{\frac{e^{\frac{rt}{2}}}{i\bar{h} \sinh \Omega t}} \exp\left(-\frac{imr}{4} e^{rt} x^2\right) \exp\left[\frac{im\Omega}{2\bar{h} \sin h\Omega t}(e^{rt} \cos h\Omega t x^2 + \cos h\Omega t x^{'2} - 2e^{\frac{rt}{2}} xx^{'})\right]. \quad (24)$$

The application of

$$i\bar{h}\frac{\partial K(x, x^{'}, t)}{\partial x^{'}} = <x(t)|\hat{p}(0)|x^{'}(0)>, \quad (25)$$

with the aid of (14) we must, however, recast the operators $\hat{p}(0)$ in terms of the operators $\hat{x}(t)$ and $\hat{x}(0)$ as

$$\hat{p}(0) = \frac{m\Omega e^{\frac{rt}{2}}}{\sinh \Omega t}\hat{x}(t) - m\Omega\left(\coth \Omega t + \frac{r}{2\Omega}\right)\hat{x}(0). \quad (26)$$

Then, by replacing (24) and (26) into (25), it is simple to demonstrate that

$$i\bar{h}\frac{\partial C(x, x')}{\partial x'} = -\frac{mr}{2}x'C(x, x'). \quad (27)$$

It is possible to solve Eq. (27) to obtain

$$C(x, x') = C(x)\exp\left(\frac{imr}{4\bar{h}}x'^2\right). \quad (28)$$

If we replace (28) with (24), the propagator can be express as

$$K(x, x^{'}; t) = C(x)\sqrt{\frac{e^{\frac{rt}{2}}}{i\bar{h} \sinh \Omega t}} \exp\left(-\frac{imr}{4\bar{h}}(e^{rt} x^2 - x'^2)\right) \exp\left[\frac{im\Omega}{2\bar{h} \sinh \Omega t}(e^{rt} x^2 + x'^2)\cosh \Omega t - 2e^{\frac{rt}{2}} xx')\right]. \quad (29)$$

By applying the relation

$$-i\bar{h}\frac{\partial K(x, x'; t)}{\partial x} = <x(t)|\hat{p}(t)|x'(0))>, \quad (30)$$

one can derive the arbitrary constant of integration $C(x)$.

Similar to this, we found that $\partial C(x)/\partial x = 0$ by inserting (29) into (30), which suggests that $C(x)$ is constant. The propagator's initial condition,

$$\lim_{t \to 0^+} K(x, x^{'}; t) = \delta(x - x^{'}), \quad (31)$$

can be used to determine the constant $C(x)$.

The constant $C(x)$ can be expressed as

$$C = \sqrt{\frac{m\Omega}{2\pi i\bar{h}}}. \quad (32)$$

if we apply (31) to (29).

We can get the propagator for an inverted Caldirola-Kanai oscillator that has the same form as calculated via the Feynman path integral by putting (32) into (29).

4. Integrals of the motion for an inverted Caldirola-Kanai oscillator

In order to start using this method, first determine the integrals of the motion, which were defined as

$$\frac{d\hat{I}}{dt} = \frac{\partial \hat{I}}{\partial t} + \frac{1}{i\bar{h}}[\hat{H}, \hat{I}] = 0, \quad (33)$$

where $\hat{I}$ are integrals of the motion operator.

The integrals of the motion $\hat{x}(0)$ and $\hat{p}(0)$ that matched the Hamiltonian operator in Eq. (16) can therefore be expressed as

$$\hat{x}(0) = \left(e^{rt} \cosh \Omega t - \frac{r}{2\Omega} e^{\frac{rt}{2}} \sinh \Omega t \right) \hat{x}(t) - \left(\frac{e^{-\frac{rt}{2}}}{m\Omega} \sinh \Omega t \right) \hat{p}(t), \tag{34}$$

and

$$\hat{p}(0) = \left(\frac{m\omega^2}{\Omega} e^{\frac{rt}{2}} \sinh \Omega t \right) \hat{x}(t) + \left(e^{-\frac{rt}{2}} \cosh \Omega t + \frac{e^{-\frac{rt}{2}}}{2\Omega} \sinh \Omega t \right) \hat{p}(t). \tag{35}$$

The Green function, also known as the propagator, is the eigenfunction of the integrals of motion $\hat{x}(0)$ and $\hat{p}(0)$, which fulfill the equations

$$\hat{x}(0) K(x, x'; t) = x' K(x, x'; t), \tag{36}$$

and

$$\hat{p}(0) K(x, x'; t) = i\bar{h} \frac{\partial K(x, x'; t)}{\partial x'}. \tag{37}$$

In coordinate representation of numbers (36) and (37) can be written clearly as

$$\left(x \left(e^{\frac{rt}{2}} \cosh \Omega t - \frac{r}{2\Omega} e^{\frac{rt}{2}} \sinh \Omega t \right) + \frac{i\bar{h}}{m\Omega} e^{-\frac{rt}{2}} \sinh \Omega t \frac{\partial}{\partial x} \right) K(x, x'; t) = x' K(x, x'; t), \tag{38}$$

and

$$\left(x \left(\frac{m\omega^2}{\Omega} e^{\frac{rt}{2}} \sinh \Omega t \right) - i\bar{h}(e^{-\frac{rt}{2}} \cosh \Omega t + \frac{re^{-\frac{rt}{2}}}{2\Omega} \sinh \Omega t) \frac{\partial}{\partial x} \right) K(x, x'; t) = i\bar{h} \frac{\partial K(x, x'; t)}{\partial x'}. \tag{39}$$

To make (38) and (39) easier to compute, we rewrite them in terms of

$$\frac{\partial K(x, x'; t)}{\partial x} = -\frac{im\Omega}{\bar{h}} \left[\frac{e^{\frac{rt}{2}}}{\sinh \Omega t} x' - \left(e^{rt} \coth \Omega t - \frac{re^{rt}}{2\Omega} \right) x \right] K(x, x'; t), \tag{40}$$

and

$$\frac{\partial K(x, x'; t)}{\partial x'} = -\frac{i}{\bar{h}} \left[m\Omega \frac{e^{\frac{rt}{2}}}{\sinh \Omega t} x - \left(m\Omega \coth \Omega t + \frac{mr}{2} \right) x' \right] K(x, x'; t). \tag{41}$$

If we integrate (40) with respect to the variable x, the outcome can be expressed as

$$K(x, x'; t) = C(x', t) \exp \left(\frac{i}{\bar{h}} \left[\left\{ \frac{m\Omega}{2} e^{rt} \coth \Omega t - \frac{mr}{4} e^{rt} \right\} x^2 - \frac{m\Omega}{\sinh \Omega t} e^{\frac{rt}{2}} xx' \right] \right), \tag{42}$$

where $C(x', t)$ is an arbitrary constant of integration.

The formula for $C(x', t)$ in Eq. (42), can be obtained substituting in Eq. (41) with the result being:

$$\frac{\partial C(x', t)}{\partial x'} = \frac{i}{\bar{h}} \left(m\Omega \coth \Omega t + \frac{mr}{2} \right) x' C(x', t). \tag{43}$$

The answer to problem (43) can be expressed as

$$C(x', t) = C(t) \exp \left(\frac{i}{\bar{h}} \left[\frac{m\Omega}{2} \coth \Omega t + \frac{mr}{2} \right] x'^2 \right), \tag{44}$$

where $C(t)$ is a freely chosen integration constant.

We get the propagator in the form of

$$K(x, x'; t) = C(t) \exp \left(\frac{i}{\bar{h}} \left[\left(\frac{m\Omega}{2} e^{rt} \coth \Omega t - \frac{mr}{4} e^{rt} \right) x^2 + \left(\frac{m\Omega}{2} \coth \Omega t + \frac{mr}{4} \right) x'^2 - \frac{m\Omega e^{\frac{rt}{2}}}{\sinh \Omega t} xx' \right] \right). \tag{45}$$

The propagator from Eq. (45) is substituted into the Schrödinger equation

$$i\bar{h}\frac{\partial K(x,x';t)}{\partial t} = \frac{\bar{h}^2}{2m}e^{-rt}\frac{\partial^2 K(x,x';t)}{\partial x^2} - \frac{1}{2}me^{rt}\omega^2 x^2 K(x,x';t),$$

(46)

to calculate $C(t)$.

Following the aforementioned procedure, we arrive at

$$\frac{dC(t)}{dt} = C(t)\left(\frac{r}{4} - \frac{\Omega\coth\Omega t}{2}\right).$$

(47)

The answer to the problem (47) can be expressed as

$$C(t) = \frac{C}{\sqrt{\sinh\Omega t}}e^{\frac{rt}{4}},$$

(48)

where C is an arbitrary integration constant.

The constant C can be represented by the expression

$$C = \sqrt{\frac{m\Omega}{2\pi i\bar{h}}},$$

(49)

if we assume the initial condition of propagator

$$K(x,x';t=0) = \delta(x-x').$$

(50)

We derive the propagator that agrees with the outcome determined by the Schwinger method and Feynman path integral by substituting (48) and (49) into (45).

5.   Conclusions

We have successfully calculated the exact propagator for the inverted Caldilora-Kanai oscillator using three different methods. For a quadratic Lagrangian, we suggested here that Feynman path integral is the simplest technique because it only requires substituting the classical action in the semiclassical approximation formula in Eq. (10). However, for non-quadratic Lagrangian, the calculation of the pre-exponential function of the propagator is very difficult even deriving the simple harmonic oscillator propagator [20-22]. The second method formulated by Schwinger is normally applied to quantum field theory but is appropriated for non-relativistic quantum mechanics. However, the calculation of integration with respect to time in Eq. (19) is not easy. The third method has the difficulty in modifying (38)-(39) to (40)-(41).

The connection between classical mechanics and quantum mechanics of Feynman path integral is explicitly shown in (10) by the classical action $S_{cl}(x'',t'';x',t')$. The pre-exponential function in Eq. (10) comes from the sum of all historical paths that present the wave-particle duality and Heisenberg uncertainty principle. In Schwinger's formulation, the pre-exponential factor appears from the commutation relation of $[\hat{x}(0),\hat{x}(t)]$ that is calculated in Eq. (23). This demostrates the different nature between classical mechanics and quantum mechanics. In classical mechanics, the physical observables are complex numbers that can commute. But in quantum mechanics, the physical observables are operators and cannot commute, which shows the importance of the ordering of measurements. The integrals of the motion, which are linear functions of the coordinate and momentum operators, specify the initial point on the classical trajectory. The pre-exponential function of this method can be obtained by substituting (45) into Schrödinger's equation in Eq. (46), which implies that the transition probability amplitude can appear from the wave equation of Schrödinger.

Finally, we conclude here that the presentation of these three methods will help students and teachers compare the advantage and difficulties of each of them. We also recommend here that these three methods are powerful methods in the calculation of the propagator for non-relativistic and relativistic quantum systems.

1. E. Merzbacher, Quantum mechanics, 3rd. ed. (John Wiley and Sons, 1998), pp. 25-75.

2. R.P. Feynman and A.R. Hibbs, Quantum mechanics and path integral, (McGraw-Hill: New York, 1965), pp. 15-37.

3. R.P. Feynman, *Space-time approach to non relativistic quantum mechanics*, Rev. Mod. Phys. **20** (1948) 367, `https://doi.org/10.1103/RevModPhys.20.367`

4. L.S. Schulman, Techniques and application of path integrals, (McGraw-Hill, New York, 1965), pp. 47-57.

5. A. Das, Field Theory: A path integral approach (World Scientific, Singapore, 1993), pp. 20-60.

6. J. Schwinger, *On gauge invariance and vacuum polarization*, Phys. Rev. **82** (1951) 664-679.

7. J.D. Likken, J. Sonnenschein, and N. Weiss, *The theory of anionic superconductivity: A review*, Int. J. Mod. Phys. A. **6** (1991) 5155, `https://doi.org/10.1142/S0217751X91002434`

8. H.B. Filho, C. Farina and A.N. Vaidya, *Schwinger's method for electron propagator in a plane wave field revisited*, Phys. Lett. A. **215** (1996) 109, `https://doi.org/10.1016/0375-9601(96)00228-9`

9. C. Farina and A.S. Santonja, *Schwinger's method for a harmonic oscillator with a time-dependent frequency*, Phys. Lett. A. **184** (1993) 23, `https://doi.org/10.1016/0375-9601(93)90340-6`

10. S. Pepore and B. Sukbot, *Schwinger method and Feynman path integral for a harmonic oscillator with mass growing with time*, Chin. J. Phys. **53** (2015) 060004.

11. S. Pepore and B. Subbot, *Schwinger method for dual damped oscillators*, Chinese. J. Phys. **53** (2015) 100002.

12. S. Pepore and B. Sukbot, *Schwinger method for coupled harmonic oscillators and time-dependent linear potential*, Chinese. J. Phys. **53** (2015) 120004.

13. V.V. Dodonov, I.A. Malkin and V.I. Man'ko, *Integrals of the motion, Green function, and coherent states of dynamical systems*, Int. J. Theor. Phys. **14** (1975) 37, `https://doi.org/10.1007/BF01807990`

14. V.V. Dodonov, I.A. Malkin and V.I. Man'ko, *Coherent states and Green functions of relativistic quadratic systems*, Physcia. A. **82** (1976) 113, `https://doi.org/10.1016/0378-4371(76)90094-7`

15. S. Pepore, *Integrals of the motion and Green functions for time-dependent mass harmonic oscillators*, Rev. Mex. Fis. **64** (2018) 30-35.

16. S. Pepore, *Integrals of the motion and Green function for dual damped oscillators and coupled harmonic oscillators*, Rev. Mex. Fis. **64** (2018) 150, `https://doi.org/10.31349/RevMexFis.64.150`

17. S. Baskoutas and A. Jannussis, *Quantum mechanics of the inverted Caldilora-Kanai oscillator*, J. Phys. A. **27** (1994) 2189, `https://doi.org/10.1007/BF02728488`

18. V. Subramanyan, S.S. Hedge, S. Vishveshwara, and B. Bradlyn, *Physics of the inverted harmonic oscillator: From the lowest Landau to event horizons*, Ann. Phys. **435** (2021) 1684705, `https://doi.org/10.1016/j.aop.2021.168470`

19. D.C. Khandekar, S.V. Lawande, and K.V. Bhagwat, Path integral method and their applications (World Scientific, Singapore, 1993), pp. 29-34.

20. L.Q. English and R.R. Winters, *Continued fractions and the harmonic oscillator using Feynman path integral*, Am. J. Phys. **65** (1997) 390, `https://doi.org/10.1119/1.18545`

21. S.M. Cohen, *Path integral for the quantum harmonic oscillator using elementary methods*, Am. J. Phys. **66** (1997) 537, `https://doi.org/10.1119/1.18896`

22. K. Hira, *Derivation of the harmonic oscillator propagator using the Feynman path integral and recursive relations*, Eur. J. Phys. **34** (2013) 777, `https://dx.doi.org/10.1088/0143-0807/34/3/777`