On Feynman's paradox

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Received 19 December 2022; accepted 11 January 2023

We review the Feynman paradox related to the conservation of the angular momentum in systems with an electromagnetic field. We show that the angular momentum is conserved if it is adequately defined.

Keywords: Feynman's paradox; angular momentum; electromagnetic field; magnetic flux.

DOI: https://doi.org/10.31349/RevMexFisE.20.020201

1. Introduction

The famous book [1] presents an interesting example which is now widely known as the Feynman paradox. In the original version, one considers an insulating disc that can rotate freely about an axis (perpendicular to the plane of the disc). Near the edge of the disc there are several charged spheres, uniformly spaced. At the center of the disc there is a coil whose axis coincides with that of the disc, carrying a steady current, and the disc is initially at rest (see Fig. 1). If the current is interrupted there will be a variation of the magnetic flux and, according to Faraday's law of induction, there will be a tangential electric field that will produce a force on the charges and a torque on the disc. In this way, the disc must acquire some angular velocity and some angular momentum, in spite of the fact that the initial state is stationary and, therefore, it would seem that the angular momentum is not conserved.

The paradox can be solved by realizing that there is angular momentum in the static electromagnetic field initially present [2, 3] (which turns out to be equal to the final angular momentum of the disc). As is well known, we can assign a linear momentum density to the electromagnetic field (see, *e.g.*, Ref. [4]). Specifically, the density of linear momentum is given by $\mathbf{E}\times\mathbf{B}/4\pi c$ (in cgs units) and, therefore, one can calculate the angular momentum of the electromagnetic field by integrating the density $\mathbf{r}\times(\mathbf{E}\times\mathbf{B})/4\pi c$ over all the space (see, however, Ref. [5]). In this example, the integral can be simplified making use of the vector potential in the Coulomb gauge (*i.e.*, $\nabla\cdot\mathbf{A}=0$), the Maxwell equations, and the Gauss theorem [2, 3].

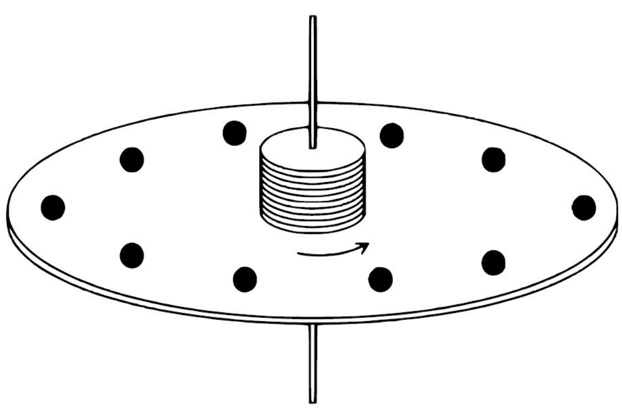

FIGURE 1. The disc can rotate without friction about the axis passing through its center. The charges, represented by the black dots, are attached to the disc.

Padmanabhan's book [3] contains another calculation of the initial angular momentum, making use of the expression for the canonical momentum of a charged particle in an electromagnetic field. The standard Lagrangian for a charged particle in a magnetic field is

$$L_{\mathrm{m}}=\frac{m}{2}\mathbf{v}^2+\frac{q}{c}\mathbf{A}\cdot\mathbf{v}$$

and therefore, employing Cartesian coordinates, their conjugate momenta are the components of

$$\mathbf{P}=m\mathbf{v}+\frac{q}{c}\mathbf{A}.$$

Assuming that $q\mathbf{A}/c$ is a linear momentum of the charge in a magnetic field and using a gauge in which $\mathbf{A}$ only has tangential component, Padmanabhan calculates an initial angular momentum, which coincides exactly with the final angular momentum. The main objection to Padmanabhan's calculation (pointed out by Padmanabhan himself) is that $q\mathbf{A}/c$ is gauge-dependent.

The aim of this paper is to show that making use of the adequate definition of the angular momentum, the angular momentum of the disc at the beginning and at the end coincide.

In Sec. 2 we review the usual computation of the final angular momentum of the disc and in Sec. 3 we review the appropriate definition of the angular momentum for a charged particle in a magnetic field, showing that, with this definition, the angular momentum is conserved in Feynman's example, without having to compute the angular momentum of the electromagnetic field. Throughout this paper we make use of cgs units.

2. The final angular momentum

As shown, *e.g.*, in Ref. [3], the final angular momentum of the system can be readily computed. If L denotes the magnitude of the usual angular momentum (that is, $L = |\mathbf{r} \times m\mathbf{v}|$, for a particle of mass m, with position vector $\mathbf{r}$ and velocity $\mathbf{v}$), then, taking the origin at the center of the disc and denoting by a the radius of the circle passing through the charged spheres, we have

$$\frac{\mathrm{d}L}{\mathrm{d}t} = aQE = \frac{Q}{2\pi} \oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{Q}{2\pi c} \frac{\mathrm{d}\Phi}{\mathrm{d}t}, \qquad (1)$$

where Q is the total charge of the spheres, $\mathbf{E}$ is the electric field acting on the spheres, Φ is the magnetic flux through the circle, and c is the speed of light in vacuum. The last equality comes from the application of Faraday's law. Hence, taking into account that the initial value of L is zero and the final magnetic flux is also equal to zero, from Eq. (1) one finds that the final value of the usual angular momentum of the disc must be

$$L = \frac{Q}{2\pi c} \Phi_{\mathrm{initial}}. \qquad (2)$$

3. The initial angular momentum

Quantities like energy, linear momentum and angular momentum are of interest in physics owing to their conservation. The presence of a magnetic field requires a special treatment (see, *e.g.*, Refs. [6,7]). Making use of the circular cylindrical coordinates (ρ, ϕ, z), a straightforward computation shows that

$$L_z \equiv m\rho^2 \dot{\phi} - G, \qquad (3)$$

where G is a function of ρ and z only such that

$$\mathrm{d}G = \frac{q}{c} \rho \, (-B_z \mathrm{d}\rho + B_\rho \mathrm{d}z), \qquad (4)$$

is a constant of motion for a particle of mass m and charge q moving in a static, axially symmetric, magnetic field $\mathbf{B}$ [6,7]. (Hence, G is defined up to an additive constant.) It may be noticed that in the absence of a magnetic field, the right-hand side of Eq. (3) reduces to the z-component of the usual angular momentum; we use the customary symbol, L_z, in order to emphasize that Eq. (3) is the appropriate definition of the angular momentum when there is a magnetic field. This abuse is similar to the usage of, *e.g.*, $\mathbf{p}$ to denote the linear momentum both in newtonian mechanics and in relativistic mechanics, despite the difference in their definitions.

Indeed, using the fact that the velocity of a particle is given by $\mathbf{v} = \dot{\rho}\,\hat{\rho} + \rho\dot{\phi}\,\hat{\phi} + \dot{z}\,\hat{z}$, where $\{\hat{\rho}, \hat{\phi}, \hat{z}\}$ is the *orthonormal* basis induced by the cylindrical coordinates, and noting that

$$\begin{aligned} \hat{\phi} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(\hat{\phi} \cdot \mathbf{v}) - \mathbf{v} \cdot \frac{\mathrm{d}\hat{\phi}}{\mathrm{d}t} = \frac{\mathrm{d}(\rho\dot{\phi})}{\mathrm{d}t} + \mathbf{v} \cdot \dot{\phi}\hat{\rho} \\ &= \frac{\mathrm{d}(\rho\dot{\phi})}{\mathrm{d}t} + \dot{\rho}\dot{\phi} = \frac{1}{\rho} \frac{\mathrm{d}(\rho^2\dot{\phi})}{\mathrm{d}t}, \end{aligned}$$

multiplying by $\rho\hat{\phi}$ both sides of the equation of motion

$$m \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{q}{c} \mathbf{v} \times \mathbf{B},$$

with $\mathbf{B} = B_\rho \hat{\rho} + B_\phi \hat{\phi} + B_z \hat{z}$, we obtain

$$\frac{\mathrm{d}(m\rho^2\dot{\phi})}{\mathrm{d}t} = \frac{q}{c} \rho \, \hat{\phi} \cdot \mathbf{v} \times \mathbf{B} = \frac{q}{c} \rho \, (\dot{z} B_\rho - \dot{\rho} B_z). \qquad (5)$$

The linear differential form $\rho \, (B_\rho \mathrm{d}z - B_z \mathrm{d}\rho)$ is (locally) exact as a consequence of the fact that $\nabla \cdot \mathbf{B} = 0$ and the assumption of the independence of the components of $\mathbf{B}$ on the angle ϕ. Hence, there exists a function G satisfying Eq. (4) (*i.e.*, $\partial G/\partial z = q\rho B_\rho/c$ and $\partial G/\partial \rho = -q\rho B_z/c$) and the right-hand side of Eq. (5) is equal to $\mathrm{d}G/\mathrm{d}t$, if the magnetic field is static, which amounts to say that $\mathrm{d}L_z/\mathrm{d}t = 0$ (another derivation, making use of the Lagrangian formalism, is given in Ref. [7]).

There are some comments to add about L_z, as defined by Eq. (3). As pointed out already, when the magnetic field is equal to zero, L_z coincides with the z-component of the usual angular momentum; the conservation of L_z is a consequence of the axial symmetry of the magnetic field and, therefore, L_z is the useful definition of angular momentum when there is a magnetic field present, and a very important fact is that L_z is gauge-independent.

It should be remarked that in the problem under consideration there is an additional force on each sphere, produced by the disc, which is holding the sphere at a constant distance from the center of the disc; this force is radial and does not contribute to the right-hand side of (5).

In order to show that the calculation based on (3) coincides with (2), we have to express the total initial angular momentum of the spheres and the disc making use of the definition (3), in terms of the magnetic flux through the circle containing the spheres. To this end we note that the value of G at the point with coordinates $\rho = a$, $z = 0$ is [see Eq. (4)]

$$G(a, 0) = \frac{q}{c} \int_{(0,0)}^{(a,0)} \rho' \, (-B_z \mathrm{d}\rho' + B_\rho \mathrm{d}z').$$

This is a line integral of an exact differential and therefore it can be calculated making use of *any* curve joining the points $(0, 0)$ (which is has been taken arbitrarily as the initial point) and $(a, 0)$. Integrating along a straight line on the plane $z = 0$ we have

$$\begin{aligned} G(a, 0) &= -\frac{q}{c} \int_0^a B_z \, \rho' \mathrm{d}\rho' = -\frac{q}{2\pi c} \int_S B_z \, \rho' \mathrm{d}\rho' \mathrm{d}\phi' \\ &= -\frac{q}{2\pi c} \Phi, \end{aligned}$$

where S is the disc of radius a concentric with the insulating disc and Φ is the magnetic flux through S. Since the initial value of the angular velocity $\dot{\phi}$ is equal to zero, the initial value of L_z for the entire system coincides with (2) [see Eq. (3)]. It may be remarked that the derivation presented in this section is considerably simpler than the calculations based on the angular momentum of the electromagnetic field given, *e.g.*, in Refs. [2, 3].

Finally, it may be remarked that the right-hand side of Eq. (4) is exact even if $\nabla \cdot \mathbf{B}$ is not zero everywhere. In the case of a magnetic monopole, with magnetic charge g, placed at the origin, we would have $\mathbf{B} = g\mathbf{r}/|\mathbf{r}|^3 = g(\rho\hat{\rho} + z\hat{z})/(\rho^2 + z^2)^{3/2}$ and therefore

$$\frac{q}{c}\rho\,(-B_z \mathrm{d}\rho + B_\rho \mathrm{d}z) = \frac{qg}{c}\frac{\big(-\rho z \mathrm{d}\rho + \rho^2 \mathrm{d}z\big)}{(\rho^2 + z^2)^{3/2}} = \mathrm{d}\left[\frac{qg}{c}\frac{z}{(\rho^2 + z^2)^{1/2}}\right].$$

Thus, if the system formed by the electric and magnetic charges is static, there is a non-zero angular momentum whose z-component is $-(qg/c)(z/|\mathbf{r}|)$ [see Eq. (3)]. Hence, the angular momentum of the system is equal to $-(qg/c)(\mathbf{r}/|\mathbf{r}|)$, and its magnitude has the constant value qg/c (independent of the separation between the charges) (see the enlightening discussion presented in Ref. [8]).

In summary, the supposed contradiction in Feynman's paradox comes from the use of an inadequate definition of the angular momentum. As shown above, in order to have a useful definition of the angular momentum one has to take into account its conservation under the appropriate symmetry conditions. Similar modifications are also necessary regarding the linear momentum in classical [6, 7] as well as in quantum mechanics [6].

Acknowledgement

The author wishes to thank the referee for useful suggestions.

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