Image charges from boundary value problems

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The examples usually solved by means of the method of images are revisited solving directly the Laplace equation. We also give a simple derivation of the axially symmetric solutions of the Laplace equation in spherical coordinates and of the translationally symmetric solutions of the Laplace equation in cylindrical coordinates.

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1. Introduction

As is well known, in some few problems of electrostatics it is possible to reproduce the effect of the charge induced on a conductor by means of a few point charges or lines of charge, called image charges (see, *e.g.*, Refs. [1–9]). In the standard procedure followed in the textbooks, the appropriate magnitudes and positions of the image charges or lines of charge are proposed from the start, without justification, and then it is verified that the hypothesis does indeed provide the right answer (invoking the uniqueness theorem).

The method of images can be presented in the introductory courses on electromagnetism since it only requires elementary mathematics. However, at an intermediate level, when the student is becoming familiar with the solution of boundary value problems, the derivation of the standard results of the method of images is an illustrative example of the solution of the problems of electrostatics by a direct approach (see also Refs. [10, 11]).

The aim of this paper is to solve the problems usually treated by means of the method of images, with the straightforward use of the solutions of the Laplace equation. We begin by finding the standard expressions for the axially symmetric solutions of the Laplace equation in spherical coordinates and for the translationally invariant solutions of the Laplace equation in cylindrical coordinates, without even using the corresponding expression of the Laplacian and without explicitly solving ordinary differential equations.

In Sec. 2, starting from the potential of a point charge, we derive the generating function of the Legendre polynomials and the form of the axially symmetric solutions of the Laplace equation in spherical coordinates, which is then employed in the solution of the problem of a point charge and a conducting sphere. In Sec. 3 we give a similar treatment starting from the potential of a uniformly charged infinite wire; we derive the form of the translationally invariant solutions of the Laplace equation in cylindrical coordinates and apply it to the problem of a line of charge and an infinitely long conducting cylinder parallel to the line.

Throughout this paper it is assumed that the reader is acquainted with the basic notions of electrostatics at an intermediate level as presented, *e.g.*, in Refs. [2–4, 6–9].

2. The solution of the Laplace equation in spherical coordinates

In this section we shall derive the generating function of the Legendre polynomials, finding the general solution of the Laplace equation in spherical coordinates for problems with axial symmetry. Then, we make use of this result in the solution of the usual problem of a conducting sphere and a point charge.

We begin by considering the electrostatic potential produced by a static point charge, q (static with respect to some inertial frame), and we choose Cartesian axes in such a way that the charge is at the point with coordinates $(0, 0, a)$. Then, at an arbitrary point, different from $(0, 0, a)$, with spherical coordinates (r, θ, ϕ) the potential is given by

$$\varphi(r,\theta,\phi) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}} \tag{1}$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\left(1+\frac{a^2}{r^2}-2\frac{a}{r}\cos\theta\right)^{-1/2}, \tag{2}$$

assuming that the potential vanishes at infinity. With the aid of the binomial formula we can expand (2) as a power series in $1/r$. Leaving aside the constant factor $q/(4\pi\varepsilon_0)$, we have

$$
\begin{aligned}
\frac{1}{r}\left(1+\frac{a^2}{r^2}-2\frac{a}{r}\cos\theta\right)^{-1/2} &= \frac{1}{r}\left[1+\frac{\left(-\frac{1}{2}\right)}{1!}\left(\frac{a^2}{r^2}-2\frac{a}{r}\cos\theta\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{a^2}{r^2}-2\frac{a}{r}\cos\theta\right)^2+\cdots\right] \\
&= \frac{1}{r}\left[1+\frac{a}{r}\cos\theta+\frac{a^2}{r^2}\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)+\frac{a^3}{r^3}\left(\frac{5}{2}\cos^3\theta-\frac{3}{2}\cos\theta\right)+\cdots\right] \\
&= \frac{1}{r}+\frac{a}{r^2}\cos\theta+\frac{a^2}{r^3}\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)+\frac{a^3}{r^4}\left(\frac{5}{2}\cos^3\theta-\frac{3}{2}\cos\theta\right)+\cdots, \qquad (3)
\end{aligned}
$$

which is an infinite series such that the coefficient of a^n/r^{n+1} is a polynomial of degree n in the variable $\cos\theta$, known as the Legendre polynomial of order n and denoted by $P_n(\cos\theta)$. Thus, the potential (1) has the series expansion

$$
\frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}} = \frac{q}{4\pi\varepsilon_0}\sum_{n=0}^{\infty} a^n \frac{P_n(\cos\theta)}{r^{n+1}}, \qquad (4)
$$

which converges for $r > |a|$. Taking $r = 1$, the last equation is the standard generating function of the Legendre polynomials.

The potential (4) must satisfy the Laplace equation everywhere except at the point where the point charge is located. Thus, assuming that the Laplacian of the series (4) is the series of the Laplacians we have

$$
0 = \frac{q}{4\pi\varepsilon_0}\sum_{n=0}^{\infty} a^n \nabla^2\left(\frac{P_n(\cos\theta)}{r^{n+1}}\right), \qquad (5)
$$

which must be valid for all values of a such that $|a| < r$. This means that the coefficient of each power of a must be equal to zero, that is $\nabla^2[r^{-n-1}P_n(\cos\theta)] = 0$. (In other words, Eq. (5) is a power series in a and therefore, owing to the uniqueness of the series expansions, the coefficient of each power of a must be equal to zero.)

Noting that the left-hand side of (4) is invariant under the interchange of r and a, we conclude that, if $a > 0$,

$$
\frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}} = \frac{q}{4\pi\varepsilon_0} \times \sum_{n=0}^{\infty}\frac{1}{a^{n+1}} r^n P_n(\cos\theta), \qquad (6)
$$

which converges for $r < a$. Again, except at the point where the point charge is located, the last expression must be a solution of the Laplace equation and, using the fact that a is arbitrary, it follows that $\nabla^2[r^n P_n(\cos\theta)] = 0$.

Equations (4) and (6) look like the standard generating function of the Legendre polynomials (see, *e.g.*, Refs. [2, 12, 13]), but there is an important difference. The “generating functions” (4) and (6) do not generate the Legendre polynomials alone, they generate the separable *solutions* $r^{-n-1}P_n(\cos\theta)$ and $r^n P_n(\cos\theta)$ of the Laplace equation (which depend on two variables). While the standard generating function generates solutions of an ordinary differential equation, the generating functions (4) and (6) generate solutions of a partial differential equation (see also the discussion in Sec. 4). Another important fact is that, in some textbooks, the Legendre polynomials are defined through their generating function and then it is shown that they are solutions of the Legendre equation (which arises in the solution of the Laplace equation by separation of variables). In our approach we have no need of writing down the Legendre equation or to talk about the method of separation of variables. As we have shown, it is not even necessary to know the expression of the Laplacian in spherical coordinates in order to conclude that $r^{-n-1}P_n(\cos\theta)$ and $r^n P_n(\cos\theta)$ are solutions of the Laplace equation.

By superposing the solutions $r^n P_n(\cos\theta)$ and $r^{-n-1}P_n(\cos\theta)$ we arrive at the standard expression for the most general axially symmetric solution of the Laplace equation in spherical coordinates

$$
\sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n(\cos\theta), \qquad (7)
$$

where the constants A_n and B_n are determined by the boundary conditions. In order to show that (7) is indeed the most general axially symmetric solution of the Laplace equation we consider any solution of this class, φ, in a neighborhood of the origin, and expand it in a Taylor series:

$$
\begin{aligned}
\varphi(x,y,z) &= \varphi(0,0,0) + a_1 x + a_2 y + a_3 z + a_{11}x^2 + a_{22}y^2 \\
&\quad + a_{33}z^2 + 2a_{12}xy + 2a_{23}yz + 2a_{13}xz + \cdots .
\end{aligned}
$$

Owing to the independence of the coordinates x, y, z, the set of terms of each possible degree must satisfy separately the Laplace equation. Recalling the relation between the Cartesian and the spherical coordinates we see that φ is independent of the azimuthal angle ϕ if and only if the Cartesian coordinates x and y appear in this expansion only through the combination $x^2 + y^2$. This implies, for instance, that $a_1 = a_2 = 0$, leaving just one term of first degree in x, y, z, containing only one arbitrary constant, a_3. Similarly, the allowed terms of second degree in this expansion are of the form $\alpha(x^2+y^2)+\beta z^2$, where α and β are two constants. The condition $0 = \nabla^2[\alpha(x^2+y^2)+\beta z^2] = 2\alpha+2\alpha+2\beta$ implies that $\beta = -2\alpha$ and therefore, also in this case there is only one arbitrary coefficient. The third-degree terms allowed by the symmetry are of the form $\gamma(x^2+y^2)z+\delta z^3$ and the condition $0 = \nabla^2[\gamma(x^2+y^2)z+\delta z^3] = 2\gamma z+2\gamma z+6\delta z$ gives $\gamma = -3\delta/2$, showing that there is only one arbitrary constant. The result is that there is exactly one arbitrary constant for each degree, which is precisely what we have in (7).

It may be remarked that the solution (7) has been derived by calculating the effect of translating a point charge along the z-axis by an arbitrary distance. The more involved problem of finding the effect of translating the charge from the origin by an arbitrary distance in an arbitrary direction must produce the general solution containing the spherical harmonics.

2.1. A conducting sphere and a point charge

We now consider the standard example of a conducting sphere in the electric field produced by a point charge, q, placed at a distance d from the center of the sphere. In order to apply the results derived above, it is convenient to choose the origin of a set of Cartesian axes at the center of the sphere and the z-axis in such a way that the point charge is at the point with Cartesian coordinates $(0,0,d)$. In that way, the system is invariant under rotations about the z-axis or, equivalently, the potential cannot depend on the azimuthal angle ϕ.

Assuming that d is greater than the radius of the sphere, denoted by a, at the points of the region with $a<r<d$ (that is, outside the sphere), the potential must be of the form

$$\varphi = \frac{q}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{d^{n+1}}r^n P_n(\cos\theta)+\sum_{n=0}^{\infty}A_n\frac{P_n(\cos\theta)}{r^{n+1}}, \quad (8)$$

where $A_0, A_1, A_2, \ldots$ are constants to be determined. The first series in (8) is the potential produced by the point charge [see Eq. (6)], while the second series is the potential produced by the charge on the surface of the conducting sphere (including the induced charge, which is not known by now). Note that the second series does not include terms of the form $r^n P_n(\cos\theta)$ since they would tend to infinity when r tends to infinity. The first series in (8) does not have this problem because it is not applicable for $r>d$.

Since the electric field in a conductor must be equal to zero, at all points of the surface of the sphere the potential must have the same value (unspecified by now). In other words, if we evaluate the right-hand side of (8) at $r=a$ we must obtain a constant value, φ_0 say, that cannot depend on θ:

$$\frac{q}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{d^{n+1}}a^n P_n(\cos\theta)+\sum_{n=0}^{\infty}A_n\frac{P_n(\cos\theta)}{a^{n+1}} = \varphi_0 = \varphi_0\, P_0(\cos\theta), \quad (9)$$

where we have made use of the fact that $P_0(\cos\theta)=1$ [see Eq. (3)]. Since Eq. (9) must hold for all values of θ and the set of all the Legendre polynomials is linearly independent (the Legendre polynomial $P_n(x)$, being a polynomial in x of degree n, cannot be expressed as a linear combination of the Legendre polynomials of lower orders), the coefficients of $P_n(\cos\theta)$ on each side of the equation must coincide for $n=0,1,2,\ldots$. Thus,

$$\frac{q}{4\pi\varepsilon_0}\frac{a^n}{d^{n+1}}+\frac{A_n}{a^{n+1}} = 0, \quad \text{for } n=1,2,\ldots \quad (10)$$

and, taking into account separately the coefficients of $P_0(\cos\theta)$ on both sides of (9),

$$\frac{q}{4\pi\varepsilon_0}\frac{1}{d}+\frac{A_0}{a} = \varphi_0. \quad (11)$$

Equations (10) trivially give

$$A_n = -\frac{q}{4\pi\varepsilon_0}\frac{a^{2n+1}}{d^{n+1}} = \frac{(-qa/d)}{4\pi\varepsilon_0}\left(\frac{a^2}{d}\right)^n, \quad \text{for } n=1,2,\ldots, \quad (12)$$

while, from (11),

$$A_0 = -\frac{q}{4\pi\varepsilon_0}\frac{a}{d}+a\varphi_0 = \frac{(-qa/d)}{4\pi\varepsilon_0}+a\varphi_0. \quad (13)$$

Substituting (12) and (13) into Eq. (8) we obtain

$$\varphi = \frac{q}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{d^{n+1}}r^n P_n(\cos\theta)+\frac{(-qa/d)}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\left(\frac{a^2}{d}\right)^n\frac{P_n(\cos\theta)}{r^{n+1}}+\frac{a\varphi_0}{r}. \quad (14)$$

By comparing with (4) we see that the second series in (14) *is* the potential produced by a point charge of magnitude $-qa/d$ at the point with Cartesian coordinates $(0,0,a^2/d)$, which is inside the conducting sphere. Similarly, the last term in (14) *is* the potential produced by a point charge of magnitude $4\pi\varepsilon_0\, a\varphi_0$ at the center of the sphere.

By comparison with Eqs. (4) and (6) we can write down the sums of the series in Eq. (14), namely

$$\varphi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2+d^2-2dr\cos\theta}}-\frac{1}{\sqrt{a^2+(dr/a)^2-2dr\cos\theta}}\right]+\frac{a\varphi_0}{r}.$$

This expression is valid at all points with $r \geqslant a$ and clearly shows that $r = a$ is an equipotential surface.

Thus, the electric field (or the potential) produced by the charge on the conductor (including the induced one), at the points outside the sphere, is *exactly* that of two point charges: one of magnitude $-qa/d$ at the point with Cartesian coordinates $(0, 0, a^2/d)$, and one of magnitude $4\pi\varepsilon_0\, a\varphi_0$ at the center of the sphere.

2.2. An infinite conducting plane and a point charge

The case of an infinite conducting plane, which is usually the first example of the method of images considered in the textbooks, can be viewed as a limiting case of the problem of the sphere treated above, when the radius of the sphere tends to infinity. To this end, it is convenient to use the distance of the point charge to the surface of the sphere, $d - a$, which will be denoted by s, instead of the parameter d, that is

$$s \equiv d - a.$$

The distance of the image charge to the surface of the sphere is then

$$a - \frac{a^2}{d} = a\frac{d-a}{d} = \frac{as}{s+a},$$

which tends to s when a tends to infinity. Similarly, the magnitude of the image charge,

$$-q\frac{a}{d} = -q\frac{a}{s+a},$$

tends to $-q$ when a tends to infinity.

Thus, assuming that the potential of the plane is zero, only one image charge, of magnitude $-q$, is necessary, which is situated symmetrically with respect to the plane (that is, the real and the image charges are at the same distance from the plane on a line perpendicular to the plane).

3. The solution of the Laplace equation in cylindrical coordinates

In this section, following a procedure similar to that employed in the previous section, among other things, we shall obtain the form of the solutions of the Laplace equation in cylindrical coordinates that do not depend on the z coordinate.

The electrostatic potential produced by an infinitely long uniformly charged wire with a longitudinal charge density λ, expressed in the circular cylindrical coordinates (r, θ, z), can be taken as

$$\varphi(r, \theta, z) = -\frac{\lambda}{2\pi\varepsilon_0}\ln r, \tag{15}$$

if the z-axis coincides with the charged wire (see, *e.g.*, Refs. [2, 14]). Hence, if the wire is parallel to the z-axis and passes through the point of cylindrical coordinates $(d, \phi, 0)$, the potential (15) becomes

$$\varphi(r, \theta, z) = -\frac{\lambda}{2\pi\varepsilon_0}\ln\sqrt{r^2 + d^2 - 2rd\cos(\theta - \phi)} \tag{16}$$

[*cf.* Eq. (1)]. Taking advantage of the fact that (16) does not depend on z, we can suppress the coordinate z in what follows and note that (15) is the real part of the complex-valued function

$$f(z) = -\frac{\lambda}{2\pi\varepsilon_0}(\ln r + \mathrm{i}\theta) = -\frac{\lambda}{2\pi\varepsilon_0}\ln(r\mathrm{e}^{\mathrm{i}\theta}) = -\frac{\lambda}{2\pi\varepsilon_0}\ln z,$$

where now z is the complex variable $x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\theta}$. (Note that the modulus, r, and the argument, θ, of z coincide with the coordinates r and θ of the cylindrical coordinates, respectively.)

Thus, if the charged wire is translated to the point of the complex plane $d\mathrm{e}^{\mathrm{i}\phi}$, with the aid of the well-known series expansion

$$\ln(1 - z) = -\sum_{n=1}^{\infty}\frac{z^n}{n}, \qquad |z| < 1$$

(which can be readily verified by differentiating both sides with respect to z, using the fact that the sum of the geometric series, $\sum_{n=0}^{\infty} z^n$, is $1/(1 - z)$ and that $\ln 1 = 0$), the electrostatic potential (16) must be the real part of

$$\begin{aligned}
f(z) &= -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(z - d\mathrm{e}^{\mathrm{i}\phi}\right) = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(r\mathrm{e}^{\mathrm{i}\theta} - d\mathrm{e}^{\mathrm{i}\phi}\right) = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left\{r\mathrm{e}^{\mathrm{i}\theta}\left[1 - \frac{d}{r}\mathrm{e}^{\mathrm{i}(\phi-\theta)}\right]\right\} \\
&= -\frac{\lambda}{2\pi\varepsilon_0}\left\{\ln\left(r\mathrm{e}^{\mathrm{i}\theta}\right) + \ln\left[1 - \frac{d}{r}\mathrm{e}^{\mathrm{i}(\phi-\theta)}\right]\right\} = -\frac{\lambda}{2\pi\varepsilon_0}\left[\ln\left(r\mathrm{e}^{\mathrm{i}\theta}\right) - \sum_{n=1}^{\infty}\frac{1}{n}\frac{d^n}{r^n}\mathrm{e}^{\mathrm{i}n(\phi-\theta)}\right] \\
&= -\frac{\lambda}{2\pi\varepsilon_0}\left\{\ln r + \mathrm{i}\theta - \sum_{n=1}^{\infty}\frac{1}{n}\frac{d^n}{r^n}\left[\cos n(\theta - \phi) - \mathrm{i}\sin n(\theta - \phi)\right]\right\},
\end{aligned}$$

that is, (16) has the series expansion

$$-\frac{\lambda}{2\pi\varepsilon_0}\ln\sqrt{r^2 + d^2 - 2rd\cos(\theta - \phi)} = -\frac{\lambda}{2\pi\varepsilon_0}\left\{\ln r - \sum_{n=1}^{\infty}\frac{1}{n}\frac{d^n}{r^n}\left[\cos n(\theta - \phi)\right]\right\}, \tag{17}$$

which converges for $r > d$ (*cf.* Ref. [15]).

Since the expression (16) is invariant under the interchange of r and d (if $d > 0$), from (17) we obtain the series

$$-\frac{\lambda}{2\pi\varepsilon_0}\ln\sqrt{r^2+d^2-2rd\cos(\theta-\phi)} = -\frac{\lambda}{2\pi\varepsilon_0}\left\{\ln d - \sum_{n=1}^{\infty}\frac{1}{n}\frac{r^n}{d^n}\big[\cos n(\theta-\phi)\big]\right\}, \tag{18}$$

which converges for $r < d$. The functions (17) and (18) must be solutions of the Laplace equation (except at the points on the wire) and they are series formed with the separable functions $r^{\pm n}\cos n\theta$, $r^{\pm n}\sin n\theta$ and $\ln r$ (recall that $\cos n(\theta-\phi) = \cos n\phi\cos n\theta + \sin n\phi\sin n\theta$) which, making use of the same argument as in Sec. 2, owing to the fact that d and ϕ are arbitrary, must be separately solutions of the Laplace equation.

By superposing the solutions $r^{\pm n}\cos n\theta$, $r^{\pm n}\sin n\theta$ and $\ln r$ we obtain the most general translationally symmetric solution of the Laplace equation in cylindrical coordinates:

$$\sum_{n=1}^{\infty}\big(A_n r^n\cos n\theta + B_n r^n\sin n\theta + C_n r^{-n}\cos n\theta + D_n r^{-n}\sin n\theta\big) + E\ln r + F, \tag{19}$$

where A_n, B_n, C_n, D_n, E, F are constants determined by the boundary conditions. One can convince oneself, on physical grounds, that (19) is the most general solution of the Laplace equation independent of z: the charge producing an electrostatic potential independent of z must be a collection of infinite uniformly charged lines parallel to the z-axis, as the one considered above (in much the same way as an arbitrary charge distribution can be viewed as a collection of point charges), hence, the potential is a superposition of expressions of the form (17) and (18), which is of the form (19).

The form of the solution (19) has been obtained thanks to the availability of the two arbitrary parameters d and ϕ appearing in Eqs. (17) and (18); the general solution of the Laplace equation in cylindrical coordinates would be obtained by including rotations of the wire by an arbitrary angle about an axis perpendicular to the wire.

3.1. An infinitely long conducting cylinder and a line of charge

Now we shall apply the foregoing results to the standard problem of an infinitely long conducting cylinder parallel to a line of charge. As we shall see, it is convenient to place the origin of a set of Cartesian axes at some point of the axis of the cylinder and the line of charge passing through the point with Cartesian coordinates $(d,0,0)$, that is, we are setting $\phi = 0$. Denoting by a the radius of the cylinder, for points satisfying $a < r < d$, the potential must have the form [see Eq. (18)]

$$\varphi = -\frac{\lambda}{2\pi\varepsilon_0}\left(\ln d - \sum_{n=1}^{\infty}\frac{1}{n}\frac{r^n}{d^n}\cos n\theta\right) + \sum_{n=1}^{\infty}A_n\frac{1}{r^n}\cos n\theta + B\ln r + C, \tag{20}$$

where $B, C, A_1, A_2, \ldots$ are unknown constants [*cf.* Eq. (8)]. The sum of the last three terms is the potential produced by the charge on the cylinder (including the induced one, which is unknown at the start):

$$\varphi_{\rm cyl} \equiv \sum_{n=1}^{\infty}A_n\frac{1}{r^n}\cos n\theta + B\ln r + C. \tag{21}$$

The potential at the surface of the cylinder must be some constant, φ_0 say (independent of θ), that is, evaluating (20) at $r = a$,

$$-\frac{\lambda}{2\pi\varepsilon_0}\left(\ln d - \sum_{n=1}^{\infty}\frac{1}{n}\frac{a^n}{d^n}\cos n\theta\right) + \sum_{n=1}^{\infty}A_n\frac{1}{a^n}\cos n\theta + B\ln a + C = \varphi_0$$

and, making use of the linear independence of the set $\{1, \cos\theta, \cos 2\theta, \ldots\}$, it follows that

$$\frac{\lambda}{2\pi\varepsilon_0}\frac{1}{n}\frac{a^n}{d^n} + A_n\frac{1}{a^n} = 0, \quad \text{for } n = 1, 2, \ldots,$$

and

$$-\frac{\lambda}{2\pi\varepsilon_0}\ln d + B\ln a + C = \varphi_0.$$

Hence,

$$A_n = -\frac{\lambda}{2\pi\varepsilon_0}\frac{1}{n}\frac{a^{2n}}{d^n} = \frac{-\lambda}{2\pi\varepsilon_0}\frac{1}{n}\left(\frac{a^2}{d}\right)^n, \quad \text{for } n = 1, 2, \ldots, \tag{22}$$

and

$$C = \varphi_0 + \frac{\lambda}{2\pi\varepsilon_0} \ln d - B \ln a. \tag{23}$$

Substituting (22) and (23) into Eq. (21) we obtain

$$\begin{aligned}\varphi_{\rm cyl} &= \sum_{n=1}^{\infty} \frac{(-\lambda)}{2\pi\varepsilon_0} \frac{1}{n} \left(\frac{a^2}{d}\right)^n \frac{1}{r^n} \cos n\theta + B \ln r + \varphi_0 + \frac{\lambda}{2\pi\varepsilon_0} \ln d - B \ln a \\ &= -\frac{(-\lambda)}{2\pi\varepsilon_0} \left[\ln r - \sum_{n=1}^{\infty} \frac{1}{n} \frac{(a^2/d)^n}{r^n} \cos n\theta\right] - \frac{\lambda - 2\pi\varepsilon_0 B}{2\pi\varepsilon_0} \ln r + \varphi_0 + \frac{\lambda}{2\pi\varepsilon_0} \ln d - B \ln a,\end{aligned} \tag{24}$$

which *is* the potential produced by a line of charge with linear density $-\lambda$ parallel to the z-axis passing through the point with Cartesian coordinates $(a^2/d, 0, 0)$ [see Eq. (17)] and a line of charge with linear density $\lambda - 2\pi\varepsilon_0 B$ passing through the origin (that is, coincident with the z-axis) [see Eq. (15)].

Hence, the potential (and the electric field) produced by the charge on the conducting cylinder is exactly that of a pair of lines of charge inside the region enclosed by the cylinder, one at a distance a^2/d from the axis of the cylinder and, possibly, one on this axis).

There are two interesting particular cases: if $B = 0$ then the net charge on the cylinder is equal to zero, and if $B = \lambda/2\pi\varepsilon_0$ then the net linear density of charge on the cylinder is equal to λ.

4. Concluding remarks

The procedure followed in Secs. 2 and 3 to find the well-known solutions of the Laplace equation in spherical and cylindrical coordinates can be applied with other coordinate systems. The presence of a free parameter [such as the parameters a, d and ϕ contained in (1) and (16)] allows us to get an infinite set of solutions of the Laplace equation but, by contrast with the results derived above, such solutions need not be separable, which, however, does not change the fact that they are solutions of the Laplace equation; the separability is only the condition usually imposed in order to find the solutions of the Laplace equation through the solution of ordinary differential equations.

It should be pointed out that the reason behind the fact that we are getting solutions of the Laplace equation starting from simple solutions is that this equation is invariant under the rigid translations in the three-dimensional Euclidean space. More complicated solutions are obtained with the aid of rotations, which would lead to the spherical harmonics and the Bessel functions.

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1. L. Eyges, *The Classical Electromagnetic Field* (Addison-Wesley, Reading, Mass., 1972, reprinted by Dover, New York, 1980). Sec. 4.4.
2. J.R. Reitz, F.J. Milford, and R.W. Christy, *Foundations of Electromagnetic Theory*, 4th ed. (Addison-Wesley, Reading, Mass., 1979). Chap. 3.
3. R.K. Wangsness, *Electromagnetic Fields*, 2nd ed. (Wiley, New York, 1986). Sec. 11-2.
4. P. Lorrain, D.R. Corson, and F. Lorrain, *Electromagnetic Fields and Waves*, 3rd ed. (Freeman, New York, 1988). Sec. 11.2.
5. J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1998). Chap. 2.
6. D.J. Griffiths, *Introduction to Electrodynamics*, 4th ed. (Pearson, Boston, 2013). Sec. 3.2. https://doi.org/10.1017/9781108333511.
7. A. Zangwill, *Modern Electrodynamics* (Cambridge University Press, Cambridge, 2013). Sec. 8.3. https://doi.org/10.1017/CBO9781139034777.
8. T. Matsushita, *Electricity and Magnetism* (Springer, Tokyo, 2014). Sec. 2.2. https://doi.org/10.1007/978-4-431-54526-2.
9. F. Lacava, *Classical Electrodynamics, From Image Charges to the Photon Mass and Magnetic Monopoles* (Springer, 2016). Chap. 3. https://doi.org/10.1007/978-3-319-39474-9.
10. L. Eyges, *op. cit.*, Sec. 4.5.
11. A. Zangwill, *op. cit.*, Sec. 8.3.2.
12. R.K. Wangsness, *op. cit.*, Sec. 8-1.
13. A. Zangwill, *op. cit.*, Sec. 4.5.1.
14. L. Eyges, *op. cit.*, Sec. 3.4.
15. L. Eyges, *op. cit.*, Eq. (3.48).