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# Revisiting Max Planck's idea: Planck's Law, Casimir force and Planck's constant

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Max Planck (1858-1947) is one of the most renowned scientists in physics. He is even a commonplace character in humanities courses since he was one of the European scientists who most influenced the opinion and perspective of European society. This work intends to present a pedagogical introduction to the quantization of energy, Planck's most valuable contribution to science. This is an important topic that is frequently included in physics and engineering curricula but is often presented in a vague and forced manner. This work seeks a more intuitive introduction to the quantization of energy by presenting Planck's law and the Casimir force, along with a practical activity that students can perform to directly measure Planck's constant and achieve a more cohesive understanding of the concepts.

*Keywords:* Planck's law; Casimir force; energy quantization; Planck's constant.

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## 1. Introduction

Undoubtedly, Planck's lifetime was an epoch of deep-reaching changes in the field of science [1]. His findings heralded the birth of quantum mechanics and provided the foundation for understanding phenomena, which later had important technological implications. The Casimir force, for example, became essential to the construction of microelectromechanical and nanoelectromechanical systems [2-4]. Interestingly, his doctoral dissertation focused on the second law of thermodynamics. He sought to clarify its meaning and consequences. It was until the fourth decade of his life that Planck focused on the problem of thermal electromagnetic radiation emitted by an opaque, non-reflective body, *i.e.* a black body. After experiments by Rubens-Kurlbaum and Lummer-Pringsheim [5] where the energy leaving the body and the temperature of this idealized black body was measured as a function of the wavelength for different temperatures, Planck derived the distribution law for black-body radiation, which fitted all measurements perfectly. In his formula, he used the fact that light was emitted and absorbed in energy units. This was the first time that the concept of energy quanta appeared [6,7].

The absorption and emission of radiation are tangible intuitive processes to early learners as they are closely related to the sense of sight, it is also said that we only know the light by the phenomena of its production and destruction. These complementary phenomena of absorption and emission of radiation are presented to students from an early age [8] and science museums make a great effort to describe them in an accessible manner. Pedagogical experiments often show the emission of a gas lamp with a prism revealing unique color lines depending on the gas used. Additionally, passing a spectrally continuous radiation source through a gas cell reveals the disappearance of absorption gas lines that coincidentally match the energies of the emission lines, if the same gas is used. One becomes familiar with the idea of emission and absorption lines for dilute gases and continuous regions in the case of solid materials such as glowing hot metals. An intermediate case that is becoming more familiar is that of LED devices that for the reciprocity between emission and absorption, emit radiation and can also be used as radiation detectors.

The black body refers to the radiation of an ideal body that emits all kinds of electromagnetic waves, as a function of the body's temperature. Of course, at 0 K it is a perfect absorber. On the other hand, the LED is a simple semiconductor device that can emit radiation of a different color for each material; from the infrared, which is used in remote controls, to the ultraviolet, which is used in sterilization processes. The absorption process in a semiconductor is called the internal photoelectric effect or generation of carriers, and the emission process is referred to as recombination. A forward-polarized LED uses recombination in the semiconductor to generate light, a reverse-polarized LED uses generation to convert light in excess current.

This paper presents a pedagogical introduction to energy quantization. We use a framework that outlines scientific concepts from a chronologically-accurate perspective. In other words, it seeks to present the ideas proposed at a given point in time by justifying and supporting them only with concepts that were known at that point in time. This paper is divided into the following sections: first, Planck's law discusses discrete values of energy as essential justification to replace equipartition for Bose-Einstein statistics. Next, Casimir's force extends the need of energy discrete values to explore the zero-point energy whose fluctuation is the precursor of the uncertainty principle, and finally, an estimation of Planck's constant with ubiquitous components guides the interest to evaluate the Planck's constant. It is our hope that this work will provide an alternative pedagogical pathway to introduce students to the concept of energy quantization in a more intuitive and cohesive manner.

## 2. Planck's law

In Professor Steven Weinberg's quantum mechanics classes [9], it is discussed that, at the end of the 19th century, there existed an experimentally measurable function called the volumetric energy density of the electromagnetic field $U$ $[Jm^{-3}]$, whose change over time was defined by the energy flux across its surface and by the energy dissipated per unit volume [10], as seen in Eq. (1).

$$\frac{\partial U}{\partial t} = -\nabla \cdot (\vec{E} \times \vec{H}) - \vec{J} \cdot \vec{E}, \tag{1}$$

where $\vec{J}$, $\vec{E}$, and $\vec{H}$ are the current density, the electric field strength, and the magnetic field strength. In such a way that a cavity of volume $V$ at temperature $T$ emits $du(\nu,T) \equiv \rho(\nu,T)d\nu$ in the frequency range defined by the filter between $(\nu, \nu + d\nu)$. This spectral energy, explicit in Eq. (11) was determined to be a universal function, that is, no matter where or how it was measured the result would be the same, as illustrated in Fig. 1.

Equation (2) presents the energy density of the electromagnetic field $U$ as the sum of the filtered spectral energy density slices $\rho(\nu,T)d\nu$ for every available frequency.

$$U(T) = \int_0^\infty \rho(\nu,T)d\nu. \tag{2}$$

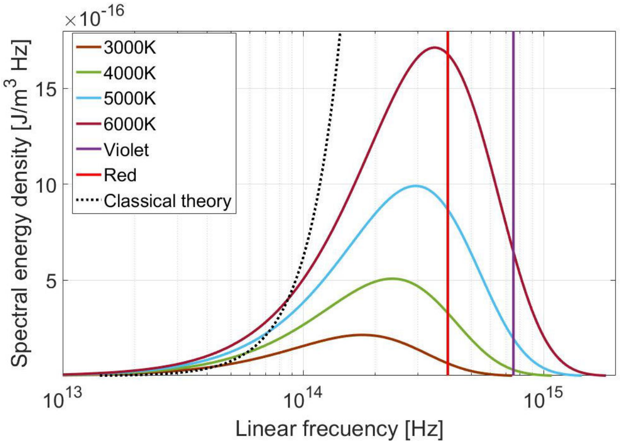

FIGURE 1. The increase in temperature of a black body moves the peak of radiation towards higher frequencies. A black body only appears black at $T = 0$ K.

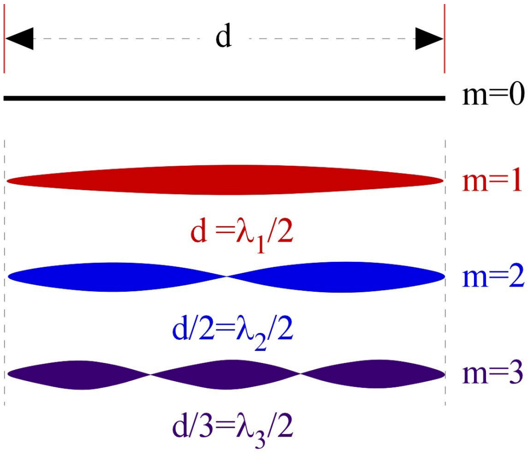

FIGURE 2. Modes in an electromagnetic wave between perfectly conduction boundaries separated by distance $d$.

It is important to remember that visible light is an electromagnetic wave, and therefore it is commonly represented by several variables, such as the wavelength $\lambda$ related to the spatial distance between two peaks of the wave, the wave number $k$ associated with the spatial frequency of the wave, the linear frequency $\nu$ that represents the temporal frequency of the wave, the angular frequency $\omega$ that evokes the circular motion of the linear frequency $\omega = 2\pi\nu$, and the energy $E$ that characterizes the wave. Note that $\lambda$ and $k$ depend on the medium where the electromagnetic wave propagates. To simplify calculations, we will assume that waves exist or propagate in a vacuum, so the propagation relations are:

$$c = \lambda_0 \nu = \frac{2\pi}{k_0}\nu = \lambda_0 \frac{\omega}{2\pi} = \lambda_0 \frac{E}{h}, \tag{3}$$

where $c$ is the speed of light and $h$ is Planck's constant.

It was also known that the spectral composition of waves as described by Fourier may be composed of many harmonic waves. Here, it was recognized that electromagnetic waves, like mechanical waves, could exist in modes defined by boundaries, see Fig. 2 for an illustration and Eq. (4) for its description.

The state of the wave will be defined by an integer $m$, of half wavelengths, as seen in Fig. 2, which can be written as:

$$m_i \frac{\lambda_i}{2} = d_i, \tag{4}$$

where the total length $d_i$ sets the boundaries of the wave, $i$ is any direction and in three dimensions a box is a convenient image for the modes of electromagnetic waves.

Since the electromagnetic modes can be a very large number, it is not convenient to try to count them directly. As with any problem with an intractable number of components, the solution lies in employing statistical methods. In this case, we want to know the energy contributed by the modes. For this, we must determine the distribution of wave modes $\rho(\xi)$ and the energy associated with each $E(\xi)$, both for the same variable of interest $\xi$. With these, it will be possible to obtain the volumetric energy density of the electromagnetic waves in the system, described by:

$$U(T) = \int_0^\infty \rho(\xi,T)E(\xi,T)d\xi \quad \left[\frac{\mathrm{J}}{\mathrm{m}^3}\right]. \tag{5}$$

In the integral, the nature of the variable is not specified because electromagnetic waves can be represented by different variables.

Many authors point to Planck's contribution as a fitting to experimental values, but others emphasize that Planck's contribution went far beyond fitting variables and explaining thermal radiation. It gave rise to one of the pillars of modern science, the quantization of energy.

At that time, it was believed that in nature everything is continuous, that waves and particles were isolated phenomena, and that everything is independent and local unless the influence is pass on by something physical like a wave or a particle. Due to theoretical work of the early twentieth century and modern experimental evidence, today it is accepted that energy is quantized, and that the behavior of waves and particles can be manifested by the same object (see hypothesis of the Duc de Broglie [11], page 49), and most surprising of all, that not everything is independent or local (see the principle of uncertainty and entanglement [11]).

To determine the volumetric energy of electromagnetic waves, it is necessary to know the distribution of the modes, and this is obtained by the derivative of the cavity's number of modes per unit volume as is presented by Eq. (6)

$$\rho(\xi) = \frac{d}{d\xi}\left(\frac{N(\xi)_m}{V_c}\right), \tag{6}$$

where $N(\xi)_m$ and $V_c$ are the modes measured in the volume of $\xi$ and the volume of the cavity, respectively. And the number of modes is determined with Eq. (7) by the modes contained in the volume of the variable $\xi$.

$$N(\xi)_m = 2\left(\frac{1}{8}\right)\left(\frac{4}{3}\pi(m(\xi))^3\right). \tag{7}$$

The number two corresponds to the independent polarization, the eighth is due to the fact that the mode numbers are all positive, so they include only one eighth of the sphere, and the third parenthesis refers to the modes in the complete sphere of variable $\xi$, the function of the different variables that represent the waves which are explicitly included in Eq. (8), where $i$ represents every considered axes.

$$m(\xi) = \frac{2d_i}{\lambda} = \frac{kd_i}{\pi} = \frac{2\nu d_i}{c} = \frac{2Ed_i}{ch} = \frac{\omega d_i}{\pi c}. \tag{8}$$

The cavity volume is $V_c = d_x d_y d_z$, finally, the distribution of states is presented in Eq. (9) for all variables describing the waves.

$$\begin{aligned}\rho(\xi) &= \frac{d}{d\lambda}\left(\frac{8\pi}{3\lambda^3}\right) = \frac{d}{dk}\left(\frac{k^3}{3\pi^2}\right) = \frac{d}{d\nu}\left(\frac{8\pi\nu^3}{3c^3}\right)\\ &= \frac{d}{dE}\left(\frac{8\pi E^3}{3c^3h^3}\right) = \frac{d}{d\omega}\left(\frac{\omega^3}{3\pi^2c^3}\right).\end{aligned} \tag{9}$$

As an example, if we set the variable $\xi$ to represent linear frequency, one obtains that the mode distribution for the cavity in vacuum has the expression in Eq. (10), the expression after that corresponds to the case where the variable $\xi$ describes the wavelength.

$$\begin{aligned}\rho(\xi\to\nu) &= \frac{d}{d\nu}\left(\frac{8\pi\nu^3}{3c^3}\right) = \frac{8\pi\nu^2}{c^3},\\ \rho(\xi\to\lambda) &= \frac{d}{d\lambda}\left(\frac{8\pi}{3\lambda^3}\right) = \frac{8\pi}{\lambda^4}.\end{aligned} \tag{10}$$

To this distribution of states, it is necessary to include the energy associated with the occupation of the states as indicated in Eq. (5), and this will be determined by assigning a thermal distribution of energy, this implies that the occupation of a state of higher energy is less likely than one of lower energy, a property described by the Boltzmann factor $N(E+\Delta E) = N(E)e^{-\Delta E/kT}$. Once again, the explicit expression for energy can be replaced for any of the variables in the Eq. (3). We have that the energy can be divided according to $\langle E\rangle = kT$, in the case of assuming that all values of the energy are allowed (continuous case) or it is distributed as $\langle E\rangle = E/(e^{-E/kT}-1)$, in the case of assuming that only discrete values of the energy are accessible, as Planck proposed. Referral details can be found in Appendix A.

With Eq. (3), $E = h\nu = h\omega/2\pi = hc/\lambda_0 = hck_0/2\pi$ one of the expressions for the representation of the volumetric energy of the electromagnetic waves, written as a function of linear frequency and temperature, called Planck's law is:

$$\begin{aligned}\rho(\nu,T)d\nu &= \left(\frac{8\pi\nu^2}{c^3}\right)\left(\frac{h\nu}{e^{h\nu/kT}-1}\right)d\nu\\ &= \frac{8\pi h\nu^3}{c^3}\frac{d\nu}{e^{h\nu/kT}-1}.\end{aligned} \tag{11}$$

And the integral over all the linear frequencies, the energy density of the electromagnetic field from Eq. (2) is:

$$\begin{aligned}
U(T) &= \int_0^\infty \left(\frac{8\pi\nu^2}{c^3}\right)\left(\frac{h\nu}{e^{h\nu/kT}-1}\right)d\nu \\
&= \frac{8\pi h}{c^3}\int_0^\infty \frac{nu^3 d\nu}{e^{h\nu/kT}-1} \\
&= \frac{8\pi h}{c^3}\left(\frac{kT}{h}\right)^4\int_0^\infty \frac{x^3 d\nu}{e^x-1}, \qquad (12) \\
&= \frac{8\pi k^4T^4}{c^3h^3}\left(\frac{\pi^4}{15}\right) = \frac{8\pi^5k^4}{15c^3h^3}T^4 \\
&= aT^4 = \frac{4\sigma}{c}T^4. \qquad (13)
\end{aligned}$$

This expression is known as the Stefan-Boltzmann law, $a$ is the radiation density constant and $\sigma$ is the Stefan-Boltzmann constant, the factor $c/4$ is explained in the next paragraph.

Planck's law as a function of the wavelength can be derived following the same steps as for the linear frequency:

$$\begin{aligned}
\rho(\lambda,T)d\lambda &= \left(\frac{8\pi}{\lambda^4}\right)\left(\frac{hc/\lambda}{e^{hc/\lambda kT}-1}\right)d\lambda \\
&= \frac{8\pi hc}{\lambda^5}\frac{d\lambda}{e^{hc/\lambda kT}-1}.
\end{aligned}$$

It is important to notice the equations so far are for energy density $U$, if one wishes to know the surface power density $R$ leaving the box from the area $A$, it is enough to multiply the expressions by c/4. Surface power density multiplied by the area and the time unit is equal to the energy density times the volume produced by the time unit ($V = A * ct$), which means $RAt = UctA$ or $R = Uc$. The average of the exit angle $\cos^2(\theta)$ equals one-half, and since we are calculating the power density emitted on one side of the surface, the total surface power is half of that, *i.e.* c/4.

The passage of time makes it impossible to know the details of how the events happened, Planck could well have arrived at his equation as a desperate act to reconcile an equation with the spectral radiation of a black body, but the sequence that was described allows Planck to be presented with a deep physical intuition and confidence in mathematical language to find such an equation. In the next section, we will discuss another important physical conception that is credited to another scientist, but the fundamental idea is also attributable to Planck.

Note: In the Eq. (12), the result of the integral that uses the Euler-Riemann zeta function and the gamma function,

$$\int_0^\infty \frac{x^{s-1}dx}{e^x-1} = \zeta(s)\Gamma(s)$$

was used, the Euler-Riemann zeta function and the gamma function for $s = 4$, and they are respectively: $\zeta(4) = \pi^4/90$; $\Gamma(4) = 3!$. And to illustrate the time of these achievements, Leonhard Euler (1707-1783) was a Swiss polymath, Georg Friedrich Bernhard Riemann (1826-1866) was a German mathematician and the gamma function was derived by Daniel Bernoulli (1700-1782) a German physicist.

## 3. Casimir's force

They say that Hendrik Brugt Gerhard Casimir (1909-2000), a Dutch physicist, in 1947 working at Philips on the theory of Van der Waals forces, after a conversation with Niels Bohr was inspired to study the effect of zero-point energy on the force between perfectly conducting parallel plates. The cavity between such plates cannot withstand all modes of the electromagnetic field. In particular, wavelengths comparable to the plate separation and longer are excluded from the internal region between the plates. This fact leads to the situation that there is a zero-point radiation overpressure outside the plates which acts to push them together. This attractive force between plates can be thought of as analogous to the attraction of clips by surface tension, it is recommended to avoid the myth of using the ship attraction equivalent that appears to be unfounded [12], and the resulting effect is now called the Casimir force. It has the property of increasing in strength with the inverse of the fourth power of the plate separation [13]. The force ceases when the elements of the plates come into contact. The smoothness of the surface of the plates is a limiting factor, or when the plates are so close together that the corresponding wavelengths of zero-point radiation no longer "see" a perfectly conducting surface. The actual discontinuous nature of the plates, as opposed to the true surface and molecular nature of the materials, becomes an important factor for very short distances.

Zero-point energy is a concept that Planck proposed in 1911 and that can be visualized by statistical arguments, in which the lowest value of energy is no longer zero, and its fluctuations are described by the uncertainty principle. These fluctuations in the energy minimum are sufficient to explain the attraction or repulsion between perfectly conducting parallel plates and the emission or absorption of radiation by displacing one of them.

In the previous section, the mean harmonic energy calculated by Planck was $\langle E\rangle = h\nu/(e^{-h\nu/kT}-1)$, the uncertainty principle corrects it to $\langle E\rangle = h\nu/2 + h\nu/(e^{-h\nu/kT}-1)$ and the first term is the so-called zero-point energy.

Therefore, the zero-point energy for harmonic oscillator modes is:

$$H_0 = \sum_m \frac{h\nu_m}{2}. \qquad (14)$$

With what we learned in the previous section for the modes of electromagnetic waves in a box, the energy of the zero-point is defined by the modes $m$ that it contains, see Fig. 3:

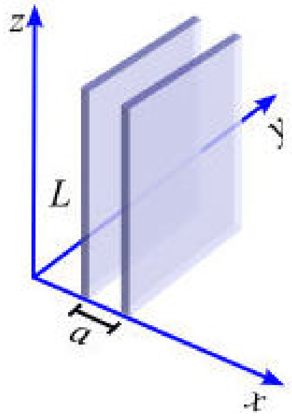

FIGURE 3. Geometry of two conducting plates separated by a distance $a$.

$$\begin{aligned}\nu_m &= \frac{c\,|k|}{2\pi} = \frac{c}{2\pi}\sqrt{k_x^2+k_y^2+k_z^2}\\ &= \frac{c}{2\pi}\sqrt{\left(\frac{m_x\pi}{d_x}\right)^2+\left(\frac{m_y\pi}{d_y}\right)^2+\left(\frac{m_z\pi}{d_z}\right)^2}\\ &= \frac{c}{2}\sqrt{\left(\frac{m_x}{a}\right)^2+\left(\frac{m_y}{L}\right)^2+\left(\frac{m_z}{L}\right)^2},\end{aligned}\tag{15}$$

and

$$\begin{aligned}H_0^{\text{box}} &= \sum_m \frac{h\nu_m}{2}\\ &= \frac{hc}{4}\sum_{pol}\sum_{m_x=0}^{\infty}\sum_{m_y=0}^{\infty}\sum_{m_z=0}^{\infty}\eta(m_x,m_y,m_z),\end{aligned}\tag{16}$$

where

$$\eta(m_x,m_y,m_z) = \sqrt{\left(\frac{m_x}{a}\right)^2+\left(\frac{m_y}{L}\right)^2+\left(\frac{m_z}{L}\right)^2}.$$

For a resonant box, the sum over the polarization ($pol$) accounts for the identical orthogonal modes fixed by the walls, where $a \ll L$, the modes along $x$ will be discretely added and the modes along $y$ and $z$ will be part of a continuous integration.

$$\begin{aligned}H_0^{\text{box}} &= \sum_m \frac{h\nu_m}{2}\\ &= \frac{hc}{4}\sum_{pol}\sum_{m_x=0}^{\infty}\int_0^{\infty}\int_0^{\infty}\eta(m_x,m_y,m_z)dm_z dm_y\\ &= \frac{hc}{4a}\sum_{\text{pol}}\sum_{m_x=0}^{\infty}\int_0^{\infty}\int_0^{\infty}\eta^*(m_x,m_y,m_z)dm_z dm_y,\end{aligned}\tag{17}$$

where

$$\eta*(m_x,m_y,m_z) = \sqrt{(m_x)^2+\left(m_y^2+m_z^2\right)\left(\frac{a}{L}\right)^2}.$$

With the change of variables $\xi_y = m_y a/L$ and $\zeta_z = m_z a/L$, the equation looks cleaner,

$$\begin{aligned}H_0^{\text{box}} &= \frac{hcL^2}{4a^3}\sum_{pol}\sum_{m_x=0}^{\infty}\\ &\times\int_0^{\infty}\int_0^{\infty}\sqrt{m_x^2+\xi_y^2+\zeta_z^2}\,d\zeta_z d\xi_y.\end{aligned}\tag{18}$$

Switching from Cartesian to polar coordinates, with the change: $\xi_y = \sqrt{u}\cos\varphi$ and $\zeta_z = \sqrt{u}\sin\varphi$, and with the result

$$\begin{aligned}\int_0^{\infty}\int_0^{\infty}d\xi_y d\zeta_z &= \int_0^{\infty}\int_0^{\pi/2}d\sqrt{u}\sqrt{u}d\varphi\\ &= \frac{1}{2}\int_0^{\infty}\int_0^{\pi/2}du d\varphi = \frac{\pi}{4}\int_0^{\infty}du,\end{aligned}$$

the equation is transformed to the next expression:

$$H_0^{box} = \frac{hc\pi L^2}{16a^3}\sum_{pol}\sum_{m_x=0}^{\infty}\int_0^{\infty}\sqrt{m_x^2+u}du.\tag{19}$$

With only one polarization for $m = 0$, and two polarization for the rest of the modes, the sum in $x$ can be separated into two groups:

$$\begin{aligned}H_0^{box} &= \frac{hc\pi L^2}{16a^3}\\ &\times\left[\int_0^{\infty}\sqrt{u}du + 2\sum_{m_x=1}^{\infty}\int_0^{\infty}\sqrt{m_x^2+u}du\right].\end{aligned}\tag{20}$$

Contrasted to the zero-point energy of free space ($a \gg L$), where now all modes will be part of a continuous integration,

$$\begin{aligned}H_0^{vac} &= \frac{hc}{4a}\sum_{pol}\\ &\times\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\eta^*(m_x,m_y,m_z)dm_z dm_y dm_x.\end{aligned}\tag{21}$$

With the same simplifications used for the case of a small box, Eq. (21) will be transformed in the following expression:

$$H_0^{vac} = \frac{hc\pi L^2}{16a^3}2\int_0^{\infty}\int_0^{\infty}\sqrt{m_x^2+u}du dm_x.\tag{22}$$

And the difference between the two energies, small box and large box (vacuum) is:

$$V = \frac{1}{L^2}\left[H_0^{\text{box}} - H_0^{\text{vacuum}}\right] = \frac{hc\pi}{8a^3}D, \tag{23}$$

where the variable $D$ explicitly is:

$$D = \frac{1}{2}\int_0^\infty \sqrt{u}du + \sum_{m=1}^{\infty}\int_0^\infty \sqrt{m^2+u}du - \int_0^\infty\int_0^\infty \sqrt{m^2+u}dudm. \tag{24}$$

Defining the divergent function that will simplify the infinite integrals

$$I(m) = \int_0^\infty \sqrt{m^2+u}du = \int_0^\infty \sqrt{\nu}d\nu,$$

we have that,

$$D = \frac{1}{2}I(0) + \sum_{m=1}^{\infty} I(m) - \int_0^\infty I(m)dm. \tag{25}$$

With the Euler-Maclaurin formula [14] used to assess the difference between the discrete addition and continuous integral associated with the same function, independently discovered by Euler and the Scottish mathematician Colin Maclaurin (1698-1746):

$$\sum_{m=i+1}^{j} f(m) - \int_i^j f(m)dm = \frac{f(j)-f(i)}{2} + \sum_{k=1}^{p/2}\frac{B_{2k}}{(2k)!}\left(f^{2k-1}(j) - f^{2k-1}(i)\right) + R_p. \tag{26}$$

With $i = 0$ and $j = \infty$, the explicit expression for $D$ takes the following form:

$$D = \frac{1}{2}I(0) + \frac{I(\infty)-I(0)}{2} + \frac{B_2}{2!}\left(\frac{dI(\infty)}{dm} - \frac{dI(0)}{dm}\right) + \frac{B_4}{4!}\left(\frac{d^3I(\infty)}{dm^3} - \frac{d^3I(0)}{dm^3}\right) + \ldots \tag{27}$$

simplified by removing the first couple of $I(0)$,

$$D = \frac{1}{2}I(\infty) + \frac{B_2}{2!}\left(\frac{dI(\infty)}{dm} - \frac{dI(0)}{dm}\right) + \frac{B_4}{4!}\left(\frac{d^3I(\infty)}{dm^3} - \frac{d^3I(0)}{dm^3}\right) + \ldots. \tag{28}$$

And using the Bernoulli numbers $B_2 = 1/6$, and $B_4 = -1/30$, evaluating the integral $I(m)$ in the limits $m$, $j$ one will have $I(m) = (2j^{2/3}/3) - (2m^3/3)$, moreover with $I(\infty) = 0$. For the derivatives of $I(M)$ only two survive, $dI(m)/dm = -2m^2$, $d^3I(m)/dm^3 = -4$, and finally $d^jI(m)/dm^j = 0$, for any $j \geq 4$, it is possible to see:

$$D = -\frac{1}{2!}\left(\frac{1}{6}\right)(0) - \frac{1}{4!}\left(-\frac{1}{30}\right)(-4) = -\frac{1}{180}. \tag{29}$$

With this value, Eq. (23) will end up as:

$$V = \frac{hc\pi}{8a^3}\left(-\frac{1}{180}\right). \tag{30}$$

We can now compute the attractive force, also known as the Casimir force per unit area:

$$F(a) = -\frac{dV}{da} = -\frac{hc\pi}{480a^4}. \tag{31}$$

The importance of the Casimir force is relevant in the technology of submicroscopic mechanisms. His conception is not particularly intuitive, but the systematic work of the zero point energy and its effect on near boundaries is essential to understand the origin of the force produced by geometry and defined by the quantization of energy. Much work must be done to assess the effects of surfaces that are not flat, nor perfect conductors, but it seems that Planck's hypothesis is sufficient to account for the essence of the real effects.

Until now Planck's constant has appeared as a constant of smallness indicating the subtlety of quantization effects. However, an exploration of photonic devices can provide an approximation to their quantification. In the following section, a guide is presented, which can be simplified with the available experimental resources, to explore how electronic devices work and their underlying fundamental concepts, including Planck's constant.

## 4. Estimation of Planck's constant

One potentially valuable pedagogical approach to discussing Planck's contribution is by having students directly measuring his constant by relating optical wavelength or frequency with photon energy. This can be easily accomplished by performing electrical and optical measurements of various commercial light-emitting diodes (LEDs) of different colors.

A set of LEDs were purchased of a range of colors and semiconductor materials and their spectral emission curves were measured, see Fig. 4, at the shorter wavelengths we have the UV, violet, and blue colors, and at the longest wavelengths are red and IR. Moreover, we do not need the spectral response of the light, only the peak emission of the LED will be related to the voltage used.

Electrically, the curve of voltage vs current of a LED is an exponential function and in forward bias, it is expected that a current of 10 to 20 mA is necessary to shine without being damaged. The voltage is almost constant and unique for each LED color. On the other hand, the emission color is defined by the semiconductor material used. In the mentioned current range, the spectral response of the light can be measured and

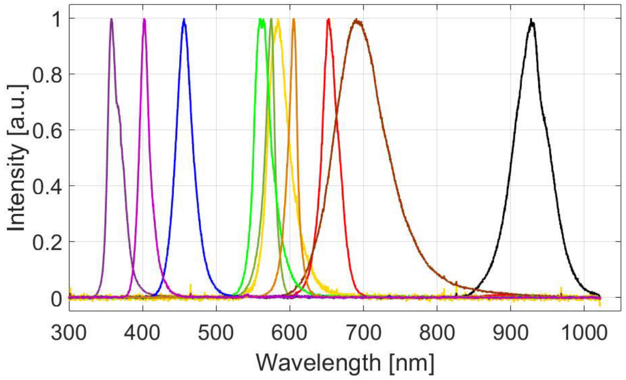

FIGURE 4. Normalized curves of LEDs when biased forward with 10 mA.

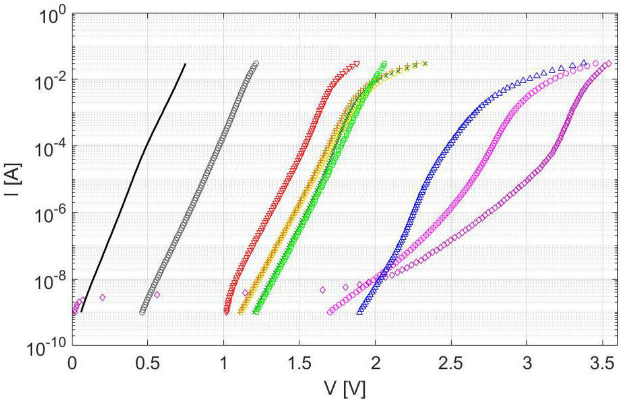

FIGURE 5. Current vs forward voltage for the assortment of LEDs. As a reference, a rectifying IN4007 diode is presented at the extreme left.

the peak emission of the LED can be related to the voltage used.

Figure 5 presents the electrical behavior of the devices, IR is closer to ideal performance (exponential function parallel to a rectifying diode IN4007, presented at the far left). Visible LEDs have mixed results due to the different technologies to produce them and the UV LED at the far right is the most distant from ideal action. The curvature at high currents of some LEDs is due to serial resistance.

It is important to understand the correlation of the current and the voltage in the device, with the optical power and the Planck measurement respectively. The electrical power (current times voltage) and optical power (brightness) in a LED are correlated, and because the voltage changes in the LED are small, the optical power is proportional to the current.

On the other hand, the voltage at some level of current (10 mA), will be used to estimate the Planck constant. Identifying the electrical energy in units of eV as the measured voltage.

Some details are intentionally ignored; the applied voltage in the active device is smaller than the band gap of the semiconductor involved, only a fraction of the electrical current produces radiative recombination or light, the serial resistance is ignored, the technology used to produce the LED and hence the color may include same semiconductor with impurities that change the color. This might explain some of the scatter in the experimental values measured deviating the results from a perfectly straight line in Fig. 6.

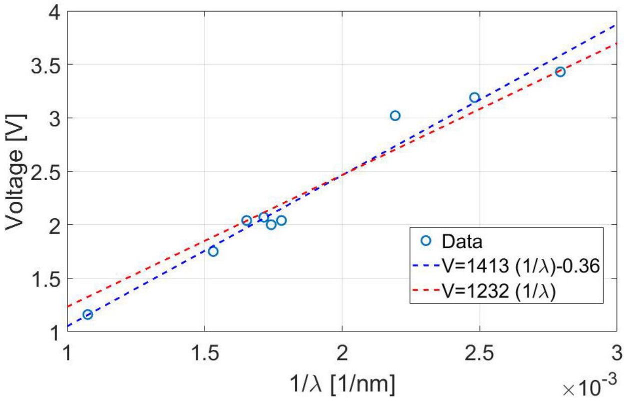

FIGURE 6. Relation between the voltage at 10 mA in the LED and the inverse of the peak emission. Using the light dispersion in vacuum, the Planck's constant was determined from the slope.

Light emission in an LED is inherently different from black-body radiation. Its physical process rather resembles the opposite of the photoelectric effect, where the energy of the electron when recombining is transformed into radiation. The efficiency of this process is not 100 %, consequently, some of the energy of the electric current is transformed into heat resulting in an increase in temperature. Keeping the LED cold with a thermoelectric cooler will improve the results, but even so, it is considered a cold emission source, unlike black-body radiation, where the temperature is the governing parameter for the emission.

The electrical current that passes through the LED is proportional to the electron charge, the efficiency, and the optical power divided by the photon emission energy [15], but if we use constant current and assume the same efficiency, one can assume that the photon energy will be proportional only to the applied voltage and mostly independent of the current. The electrostatic potential energy of the electron is equal to the voltage multiplied by the charge of the electron, and the voltage applied to the LED can be related to the wavelength peak emission:

$$E = eV = \frac{hc}{\lambda}, \quad V = \frac{hc}{e}\frac{1}{\lambda},$$

and from least squares, see Fig. 6, we get a slope of 1232 $\times 10^{-9}$ [Vm], this value is produced forcing the fitting to cross the origin. Finding $h$ from this equation one obtains,

$$h = \frac{e(\text{slope})}{c} = \frac{(1.6 \times 10^{-19})(1232 \times 10^{-9})}{(3 \times 10^{8})}$$
$$= 6.6 \times 10^{-34}[\text{Js}]. \tag{32}$$

The most accepted value for Planck's constant $h = 6.62607004 \times 10^{-34}$ [Js].

It is our hope that placing Planck's concepts in the historical context of the time can lead to a greater appreciation of his contribution to energy quantization. We have outlined Planck's concepts justified by mathematical arguments and supported his work by fellow mathematicians and physicists. Finally, we have presented a simple experiment that can allow learners to directly compute Planck's constant to better understand these concepts.

## Appendix

## A. The expected value of the energy

For the case of assuming that the energy is a continuous function, the mean is determined in a common way and is achieved by doing the complete integral:

$$\begin{aligned}\langle E\rangle &= \frac{\int\limits_0^\infty E e^{-E/kT} dE}{\int\limits_0^\infty e^{-E/kT} dE} \\ &= \frac{\int\limits_0^\infty x e^{-ax} dx}{\int\limits_0^\infty e^{-ax} dx} = \frac{\frac{1}{a^2}}{\frac{1}{a}} = \frac{k^2T^2}{kT} = kT. \qquad \text{(A.1)}\end{aligned}$$

In the case of accepting that the energy is a discrete function, the mean is achieved by making the complete sum:

$$\begin{aligned}\langle E\rangle &= \frac{\sum_{n=0}^{\infty} nE e^{-nE/kT}}{\sum_{n=0}^{\infty} e^{-nE/kT}} \\ &= \frac{0 + Ee^{-E/kT} + 2Ee^{-2E/kT} + \dots}{1 + e^{-E/kT} + 2e^{-2E/kT} + \dots}. \qquad \text{(A.2)}\end{aligned}$$

With the change of variable $x = e^{-E/kT}$, to improve its appearance:

$$\begin{aligned}\langle E\rangle &= \frac{xE + 2x^2E + 3x^3E + \dots}{1 + x + x^2 + x^3 + \dots} \\ &= \frac{E(x + 2x^2 + 3x^3 + \dots)}{1 + x + x^2 + x^3 + \dots}. \qquad \text{(A.3)}\end{aligned}$$

Using the result of the infinite geometric series with an argument less than one:

$$1 + x + x^2 + x^3 + \dots|_{x<1} = \frac{1}{1-x}, \qquad \text{(A.4)}$$

and the auxiliary result

$$\begin{aligned} s &= x + 2x^2 + 3x^3 + 4x^4 + \dots \\ xs &= x^2 + 2x^3 + 3x^4 + 4x^5 + \dots \\ s - xs &= x + x^2 + x^3 + x^4 + \dots \\ s - xs &= x(1 + x + x^2 + x^3 + \dots). \end{aligned}$$

You can see from the last relation the following result:

$$s = \frac{x\left(\frac{1}{1-x}\right)}{1-x} = \frac{x}{(1-x)^2}. \qquad \text{(A.5)}$$

Which yields the Planck result for the case of discrete states for the electromagnetic waves:

$$\begin{aligned}\langle E\rangle &= \frac{E\left(\frac{x}{(1-x)^2}\right)}{\frac{1}{1-x}} = \frac{Ex}{1-x} \\ &= \frac{E}{\frac{1}{x} - 1} = \frac{E}{e^{E/kT} - 1}. \qquad \text{(A.6)}\end{aligned}$$

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