Lagrangian analysis of the Feynman paradox

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It is shown that the direct use of the Lagrange equations allows us to analyze the entire process involved in the Feynman paradox, without having to speak of the angular momentum of the electromagnetic field.

Keywords: Feynman's paradox; Lagrangian formalism; angular momentum; electromagnetic field.

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1. Introduction

The Feynman paradox deals with the apparent violation of the law of conservation of angular momentum in a system formed by a coil and a set of electric charges (see, *e.g.*, Refs. [1, 2]). One begins by considering an insulating disc that can rotate without friction about the axis passing through its center, perpendicular to the plane of the disc. Mounted on the disc there is a coil, whose axis coincides with the rotation axis, and some electric charges evenly spaced on a circle concentric with the disc. Initially there is some current circulating in the coil and the disc is at rest; later, the current is stopped. The application of Faraday's law of induction leads to the conclusion that, as the current ceases, there appears an electric field pushing the charges and the disc acquires angular momentum, giving the impression that the angular momentum is not conserved.

The paradox can be solved by considering that the initial electromagnetic field, though static, possesses angular momentum (see, *e.g.*, Refs. [2,3]). Alternatively, we can see that the angular momentum of the charges is conserved if it is appropriately defined [4], without the need of speaking of the angular momentum of the electromagnetic field.

In this paper we give a simplified analysis of the entire process based on the direct application of the Lagrange equations. We show that the Lagrange equations lead to a conserved quantity that reduces to the component of the usual angular momentum along the axis of the disc when the electromagnetic field is absent. The procedure presented here stresses the fact that one can have constants of motion that may depend explicitly on the time (but their total derivative with respect to the time is equal to zero).

It is assumed that the reader is acquainted with the elementary notions of Lagrangian mechanics and electrodynamics.

2. The Lagrangian and the constant of motion

The standard Lagrangian for a charged particle in a given electromagnetic field (in the framework of non-relativistic mechanics), in terms of the circular cylindrical coordinates, (ρ, ϕ, z), is given by

$$L = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) + \frac{q}{c}(A_\rho\dot{\rho} + \rho A_\phi\dot{\phi} + A_z\dot{z}) - q\varphi, \quad (1)$$

where m and q are the mass and the electric charge of the particle, respectively, A_ρ, A_ϕ, A_z are the components of the vector potential of the electromagnetic field with respect to the orthonormal basis $\{\hat{\rho}, \hat{\phi}, \hat{z}\}$ defined by the cylindrical coordinates, and φ is the scalar potential. (The electromagnetic potentials appearing in the Lagrangian (1) correspond to the given electromagnetic field, only excluding the field produced by the charge q itself, which, as usual, is considered as a test charge. Hence, the fields corresponding to these potentials must obey the Maxwell equations $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} = -(1/c)\,\partial\mathbf{B}/\partial t$.)

We shall restrict ourselves to axially symmetric magnetic fields that may depend on the time. Taking the z-axis as the symmetry axis, this means that the components B_ρ, B_ϕ, B_z of the magnetic field are independent of the angle ϕ. Since

$$0 = \nabla \cdot \mathbf{B} = \frac{1}{\rho}\frac{\partial(\rho B_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z},$$

the condition $\partial B_\phi/\partial\phi = 0$, implies that

$$\frac{\partial(\rho B_z)}{\partial z} = -\frac{\partial(\rho B_\rho)}{\partial \rho}, \quad (2)$$

which in turn implies the existence of a function $\Lambda(\rho, z, t)$ such that

$$\rho B_z = \frac{\partial \Lambda}{\partial \rho}, \qquad \rho B_\rho = -\frac{\partial \Lambda}{\partial z}. \quad (3)$$

According to Faraday's law, a time-dependent magnetic field must be accompanied by an electric field, and we shall assume that there is an electric field present which is also invariant under the rotations about the z-axis. Hence, the components E_ρ, E_ϕ, E_z of the electric field are independent of ϕ and the components along $\hat{\rho}$ and $\hat{z}$ of the equation $\nabla \times \mathbf{E} = -(1/c)\,\partial\mathbf{B}/\partial t$ give

$$-\frac{\partial E_\phi}{\partial z} = -\frac{1}{c}\frac{\partial B_\rho}{\partial t} \quad \text{and} \quad \frac{1}{\rho}\frac{\partial(\rho E_\phi)}{\partial \rho} = -\frac{1}{c}\frac{\partial B_z}{\partial t},$$

respectively. These equations together with (3) lead to

$$\frac{\partial}{\partial z}\left(\rho E_\phi + \frac{1}{c}\frac{\partial \Lambda}{\partial t}\right) = 0, \qquad \frac{\partial}{\partial \rho}\left(\rho E_\phi + \frac{1}{c}\frac{\partial \Lambda}{\partial t}\right) = 0.$$

Since E_ϕ and Λ do not depend on ϕ this means that $\rho E_\phi + \frac{1}{c}\frac{\partial \Lambda}{\partial t}$ is a function of t only which can be absorbed into Λ. Thus, Λ is defined up to an additive constant by Eqs. (3) and

$$\rho E_\phi = -\frac{1}{c}\frac{\partial \Lambda}{\partial t}. \tag{4}$$

(It should be remarked that the function Λ is defined up to an additive constant, which is consistent with the fact that it is part of the constant of motion (9).)

The ϕ-component of Faraday's law can be written in the form

$$\frac{1}{c}\frac{\partial B_\phi}{\partial t} + \frac{\partial E_\rho}{\partial z} + \frac{\partial(-E_z)}{\partial \rho} = 0,$$

which has the form of the divergence of a vector field in Cartesian coordinates. Hence, this equation is locally equivalent to the existence of three functions, f, g, h, of (ρ, z, t) only such that

$$B_\phi = \frac{\partial h}{\partial z} - \frac{\partial g}{\partial \rho}, \quad E_\rho = \frac{\partial f}{\partial \rho} - \frac{1}{c}\frac{\partial h}{\partial t},$$
$$-E_z = \frac{1}{c}\frac{\partial g}{\partial t} - \frac{\partial f}{\partial z}. \tag{5}$$

(The functions f, g, h are defined up to the "gradient" of a scalar function of (ρ, z, t) only, but these functions do not appear in the constant of motion (9).)

Taking into account that the expression $\mathbf{B} = \nabla \times \mathbf{A}$ amounts to

$$B_\rho = \frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}, \quad B_\phi = \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho},$$
$$B_z = \frac{1}{\rho}\left[\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right],$$

comparison with Eqs. (3) and (5) shows that we can choose the vector potential in the form

$$A_\rho = h, \qquad \rho A_\phi = \Lambda, \qquad A_z = g \tag{6}$$

and

$$\varphi = -f. \tag{7}$$

Hence, the Lagrangian (1) becomes

$$L = \frac{m}{2}(\dot\rho^2 + \rho^2\dot\phi^2 + \dot z^2) + \frac{q}{c}(h\dot\rho + \Lambda\dot\phi + g\dot z) + qf, \tag{8}$$

where we have made use of Eqs. (6) and (7). Since f, g, h and Λ are functions of (ρ, z, t) only, ϕ is an ignorable coordinate and the momentum conjugate to ϕ must be a constant of motion:

$$\frac{\partial L}{\partial \dot\phi} = m\rho^2\dot\phi + \frac{q}{c}\Lambda = \text{const.} \tag{9}$$

Thus, as pointed out at the Introduction, we arrive at a constant of motion that may depend explicitly on the time, due to the presence of Λ.

It might seem reasonable that if the electric and magnetic fields are invariant under rotations about, *e.g.*, the z-axis, then it should be possible to find potentials sharing this symmetry, but the things are not so trivial. For instance, in the case of a uniform electromagnetic field (that is, the Cartesian components of $\mathbf{E}$ and $\mathbf{B}$ are constant) there do not exist potentials sharing this symmetry (if the scalar potential and the Cartesian components of $\mathbf{A}$ are constant, then the fields $\mathbf{E}$ and $\mathbf{B}$ would be zero). The symmetry of the equations of motion need not be shared by the Lagrangian (see, *e.g.*, Ref. [5]).

It is easy to see that $\Lambda(\rho, z, t)$ is essentially the magnetic flux through an imaginary disc centered at $\rho = 0$ on a plane $z = \text{const}$. In fact, this flux is given by

$$\Phi(\rho, z, t) \equiv \int_0^{2\pi}\left[\int_0^\rho B_z(\rho', z, t)\,\rho'\,\mathrm{d}\rho'\right]\mathrm{d}\phi'$$

then we readily find that

$$\frac{\partial \Phi}{\partial \rho} = \int_0^{2\pi} B_z(\rho, z, t)\,\rho\,\mathrm{d}\phi' = 2\pi\rho B_z(\rho, z, t)$$

and, making use of Eq. (2),

$$\frac{\partial \Phi}{\partial z} = -\int_0^{2\pi}\left[\int_0^\rho \frac{\partial\big(\rho' B_\rho(\rho', z, t)\big)}{\partial \rho'}\,\mathrm{d}\rho'\right]\mathrm{d}\phi'$$
$$= -2\pi\rho B_\rho(\rho, z, t).$$

Finally, from the Faraday law in integral form, we have

$$\frac{\partial \Phi}{\partial t} = -c\int_0^{2\pi} E_\phi\,\rho\,\mathrm{d}\phi = -2\pi c\rho E_\phi.$$

Comparing with Eqs. (3) and (4) it follows that we can take $\Lambda = \Phi/2\pi$ and, from Eq. (9), we have

$$m\rho^2\dot\phi + \frac{q}{2\pi c}\Phi(\rho, z, t) = \text{const} \tag{10}$$

This last equation shows that an arbitrary variation of the magnetic flux Φ must be compensated by a variation of $m\rho^2\dot\phi$, which can be recognized as the z-component of the usual angular momentum and, therefore, the usual angular momentum is not conserved (*cf.* Ref. [4]).

The conclusions obtained up to this point are applicable for a test charge in an arbitrary electromagnetic field, with the only condition that the electric and the magnetic fields be invariant under rotations about the z-axis. Now we shall make use of the constant of motion (10) in connection with the Feynman paradox: As described at the Introduction, at the beginning we have $\dot\phi = 0$, and the initial value of the magnetic flux, Φ_{initial}, is different from zero, while the final value of Φ is zero. Thus, from equation (10), we have

$$0 + \frac{q}{2\pi c}\Phi_{\text{initial}} = m\rho^2\dot\phi_{\text{final}} + 0$$

and, therefore, the final value of the z-component of the usual angular momentum of the entire system is $Q\,\Phi_{\text{initial}}/2\pi c$, where Q is the total electric charge (*cf.* Ref. [2]).

In the standard methods employed to solve the Feynman paradox, the initial and final states are separately considered. As we have shown, such a separation is not necessary; we can follow the evolution of the system, despite the fact that the magnetic field is not static.

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