On Wien's peaks

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Most Modern Physics books contain a chapter on the laws of black body radiation when introducing the principles of Quantum Mechanics. These laws govern many phenomena that we encounter in daily life, technological developments, and scientific research. For that, this old subject is still of great importance, and even now some issues require our attention. This work addresses one of these topics. We describe why it makes no sense to think that the wavelength at which Planck's black-body spectral radiance distribution plotted as a function of the wavelength has its maximum value, must be the same that the wavelength calculated from the peak value obtained when the distribution is plotted as a function of another variable, such as the energy of the photons. We will show how the issue lies in using the correct form to calculate this wavelength from measured quantities.

Keywords: Thermal radiation; Planck's Law; Jacobian's transformation; Wien's peak.

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Wien's displacement law, named after the German physicist Wilhelm Wien, tells us that the emission spectra of objects at different temperatures peak at different wavelengths. These wavelengths are shorter for hotter objects than for cooler ones. The emission spectrum of a hot body can be represented as a graph of the spectral intensity, $I(\lambda)$, as a function of wavelength, λ. The spectral intensity, or spectral radiance, is the amount of light energy incident on a detector per time unit, per unit area, and per wavelength increment interval, $d\lambda$. In other words, it is intensity per wavelength increment interval [1]. It is expressed in Wm^{-3} units.

For a black body at a temperature T, the spectral radiance is given by Planck's law of thermal radiation [2]

$$I(\lambda) = \left(\frac{c}{4}\right)\left(\frac{8\pi hc}{\lambda^5}\right)\frac{1}{(\exp\left[hc/\lambda k_B T\right] - 1)}, \qquad (1)$$

where h is Planck's constant, c is the speed of light in vacuum, and k_B is the Boltzmann constant. The wavelength $(\lambda_{\max})$ at which this function has its maximum value at a given temperature is described by Wien's displacement law:

$$\lambda_{\max} = hc/(4.951 k_B T) = b/T, \qquad (2)$$

where $b = 2.897756 \times 10^{-3}$ mK is Wien's constant. Accordingly, for a body like our Sun with a surface temperature $T \approx 5800$ K, the peak of the radiance spectrum appears in its visible part at a wavelength $\lambda_{max} \approx 500$ nm [Fig. 1a)] [3].

The spectral radiance can also be shown as a function of other quantities, such as the energy $(E = hc/\lambda)$ or frequency $(\nu = c/\lambda)$ of the photons. For example, the radiance, expressed in terms of the energy, is the intensity per energy increment interval, dE. To convert a graph of $I(\lambda)$ vs λ in one of $I\,(E)$ vs E, one must substitute λ by hc/E in Eq. (1) and apply the Jacobian's transformation to scale the ordinates' axis by the factor hc/E^2 [4]. Then, the units of $I\,(E)$ $(\mathrm{m}^{-2}\mathrm{s}^{-1})$ will differ from those of $I(\lambda)$. Energy conservation implies that the values of the radiated energy fluxes, given by the areas under the spectral radiance spectra, must be the same, whether they are represented as a function of λ or E, that is

$$I\,(\lambda)\,d\lambda = I\,(E)\,dE. \qquad (3)$$

Consequently,

$$I\,(E) = I\left(\lambda = \frac{hc}{E}\right)\frac{d\lambda}{dE} = I\left(\lambda = \frac{hc}{E}\right)\frac{d}{dE}\left(\frac{hc}{E}\right)$$
$$= -I\left(\lambda = \frac{hc}{E}\right)\frac{hc}{E^2}. \qquad (4)$$

The minus sign in the above equation only reflects the different directions of integration in energy and wavelength. In the same way, if we want to represent the radiance as a function of frequency, the corresponding Jacobian's transformation is: $I(\nu) = -I(\lambda = c/\nu)c/\nu^2$.

The value of $\lambda_{\max} = 500$ nm for which $I(\lambda)$ is maximum is different from that calculated from the energy at which the $I(E)$ function has a maximum. Fig. 1b) shows that $E_{\max} \approx 1.41$ eV, from which one obtains $\lambda_{\max} = hc/E_{\max} \approx 800$ nm [3]. This value is in the near-infrared part of the spectrum.

The question about where the Wien peaks really are has been the subject of discussion in several works in which it is accepted that the maxima of the spectra occur at different wavelengths when plotted in different scales [4–12]. However, this fact is somewhat paradoxical since the same physical phenomenon is represented as a function of different magnitudes but straightforwardly related to one another. In other words, the spectral radiance spectra must peak at the same wavelength, whether shown as a function of λ or of any other variable.

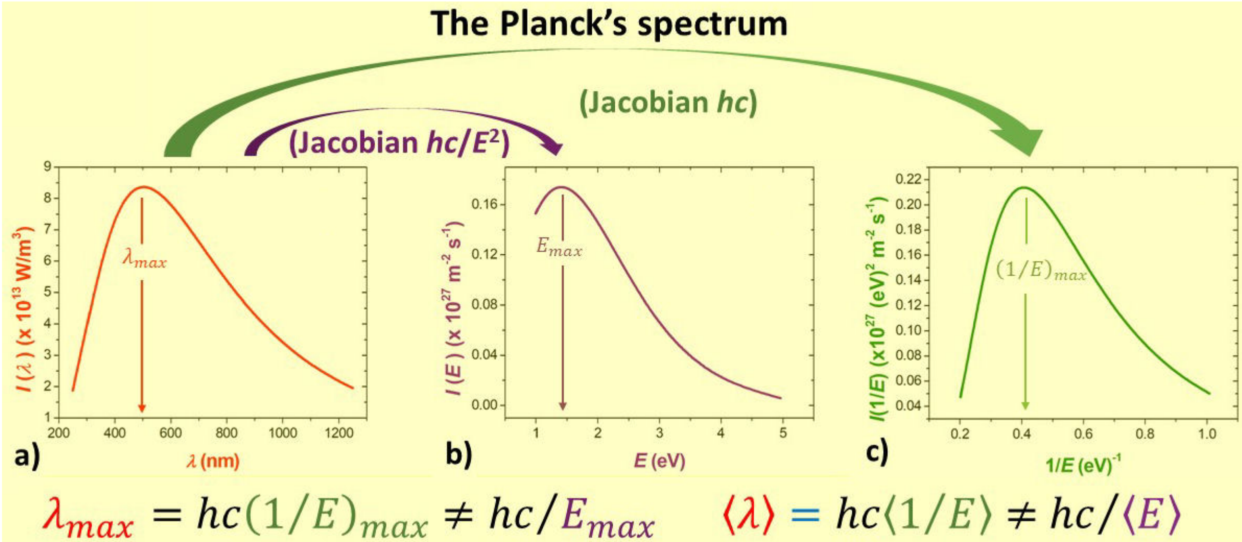

FIGURE 1. Plank's spectral radiation distribution function at $T = 5800$ K as a function of wavelength a), energy b), and energy inverse c). The purple and green arrows show the Jacobian factor when spectrum a) is transformed in b) and c), respectively. Note that the solar spectrum contains light of all colors from the ultraviolet to the mid-infrared, so that the real color of our Sun is white, as it would look when viewed from space. Instead, when the Sun's rays pass through Earth's atmosphere, some molecules distort the shorter-wave photons, causing longer-wave photons to reach us sooner.

For the solution of this paradox, we must consider that the spectral radiance is a distribution function, for which $\lambda_{\max} \neq hc/E_{\max}$ [12], except for the case of a very narrow distribution [3,13–15]. The explanation is as follows.

The average wavelength of the $I(\lambda)$ distribution is defined as

$$\langle\lambda\rangle = \frac{\int \lambda I(\lambda) d\lambda}{\int I(\lambda) d\lambda}. \tag{5}$$

Substituting Eq. (3) and $\lambda = hc/E$ in Eq. (5) leads to

$$\langle\lambda\rangle = \frac{\int \frac{hc}{E} I(E) dE}{\int I(E) dE} = hc\langle 1/E\rangle, \tag{6}$$

where $\langle 1/E\rangle$ represents the average of the energy inverse. In the above equations the integrations are performed over the entire wavelength (energy) range. In analogy with Eq. (3), the energy conservation in terms of λ and E^{-1} variables can be written as

$$I(\lambda) d\lambda = I(1/E) d(1/E). \tag{7}$$

Consequently, the Jacobian $d\lambda/d(1/E) = hc$ applied to $I(\lambda)$ transforms it into $I(1/E)$. As the Jacobian of this transformation is a constant, the spectrum of I preserves the shape after the transformation, except for a scaling factor of hc, so that the wavelength for which $I(\lambda)$ is maximum corresponds to the value of the energy inverse for which $I(1/E)$ has the maximum, as:

$$\lambda_{\max} = hc(1/E)_{\max}. \tag{8}$$

Since in arriving at Eqs. (6) and (8) one has made no assumption about the shape of the spectral distribution function, then these can be generalized for spectral distributions with any shape, including that given by Planck's formula.

In Fig. 1c) Planck's spectral radiance distribution is plotted as a function of $1/E$. The maximum appears at $(1/E)_{\max} \approx 0.4$ eV^{-1}. Substituting this value into Eq. (8) leads to $\lambda_{\max} \approx 500$ nm, which is the same value at which the $I(\lambda)$ versus λ distribution peaks. Note that the reciprocal of $(1/E)_{\max}$ is 2.5 eV, quite different from that at which $I(E)$ has its maximum.

Analogously, one can demonstrate that $\lambda_{\max} \neq c/\nu_{\max}$ but $\lambda_{\max} = c(1/\nu)_{\max}$.

Substituting Eq. (8) into Eq. (2), it can be straightforwardly demonstrated that Wien's law in the $(1/E)$ – domain becomes:

$$(1/E)_{\max} = \lambda_{\max}/hc = 1/(4.9651 k_B T). \tag{9}$$

In the same way, if we plot the Planck's distribution as a function of $1/\nu$ we will have:

$$(1/\nu)_{\max} = \lambda_{\max}/c = h/(4.9651 k_B T). \tag{10}$$

In conclusion, the wavelength at which Planck's blackbody spectral radiance distribution has its maximum value is the same, and is independent of the quantity chosen to plot this function. The issue lies in using the correct form to calculate this wavelength from the measured quantities.

Note that the above results can be generalized for optical emission distribution spectra of any kind, such as fluorescence spectra.

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15. One trivial example of a narrow distribution function is the Dirac Delta, $\delta(\lambda - \lambda_0)$, which is centered at $\lambda{=}\lambda_0$. For this function, the average wavelength is $\langle\lambda\rangle = \int \lambda\delta(\lambda-\lambda_0)d\lambda / \int \delta(\lambda-\lambda_0)d\lambda = \lambda_0 = \int (hc/E)\delta(E - E_0)dE / \int \delta(E-E_0)dE = hc\langle 1/E\rangle = hc/E_0 = hc/\langle E\rangle$. It is obvious to see that, for $\delta(\lambda-\lambda_0)$, $\lambda_{\max} = hc/E_{\max}$ is also fulfilled.