A catenary-like cable confined in a circular cylinder

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The problem of obtaining the curve of a cable suspended between two points, supporting only its own weight, was solved simultaneously, in the 17th century, by Johann Bernoulli, Leibniz, and Huygens. This curve is called the catenary. This article solves a modified problem in which the suspended cable is confined in a vertical cylinder. For this, an functional is formulated to describe the potential energy of a fixed-length confined cable in any possible arrangement. Then, the variational problem of extremizing this functional is presented and the Euler-Lagrange differential equation is deduced. The analytical solution of this equation is obtained for the cable suspended by two points at equal and different heights. Furthermore, the tensile force acting on the cable is determined. Numerical results are presented comparing the effect of confinement on the tensile force in relation to the traditional catenary.

Keywords: Catenary; flexible cable; variational calculus.

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1. Introduction

A flexible cable suspended between two points, supporting only its own weight (*i.e.*, in static equilibrium), forms a curve called catenary. This word comes from the Latin word catena which means chain. Examples of catenaries are the cables used in transmission lines and telephone lines.

The problem of determining the equation of the curve of flexible cable, of fixed length, between the two points and in static equilibrium, was proposed by Galileo Galilei. In this problem it is assumed that the cable is homogeneous, inelastic and flexible in the sense that any resistance offered to bending is negligible [1]. Galileo conjectured that the solution to the problem was a parable. But Christiaan Huygens, aged just 17, in 1647, and Joachim Jungius, in 1669, showed that Galileo's conjecture was false, without, however, presenting the analytical solution of the catenary [2, 3]. Only in 1691, after a public challenge by Jakob Bernoulli, three different solutions appeared to obtain the catenary equation given by Leibniz, Huygens and Johann Bernoulli. Huygens presented a solution using the classical euclidean mathematics. Whereas Leibniz and Bernoulli used the newly invented Calculus and this resulted was the first public success of Calculus [2].

The catenary equation is given by:

$$y(x) = a\cosh(x/a) = \frac{1}{2}a\left(e^{x/a} + e^{-x/a}\right). \qquad (1)$$

The parameter a determines the shape of the catenary (*i.e.*, how fast the catenary opens).

Another way to solve the problem of finding the catenary equation is through variational calculus. Variational calculus is a powerful mathematical tool that is applied to formulate modern physical theories of both particles and physical fields. A known problem that variational calculus manages to solve in an elegant way is the brachistochrone problem, and at present, there are articles still discussing this issue [4]. One way to solve the catenary problem using variational calculus is briefly described below.

First, it is considered that, given two fixed points A and B, there are infinite possible configurations of a suspended cable with a fixed length between A and B (although, experimentally, there is only one stable cable configuration). Next, the gravitational potential energy of the cable is calculated considering that the cable has a certain mass and linear density. The variational principle is then used with a Lagrange multiplier [5] to minimize the functional relative to the gravitational potential energy of the cable, subject to the constraint that the total length of the suspended cable is fixed. The solution to this variational problem is the physical cable configuration that solves the catenary problem.

In the classic catenary problem presented so far, the cable is localized on a flat vertical surface parallel to the gravitational field. In this article, the previous problem is formulated in a different situation. The suspended cable is now confined to a cylindrical vertical surface. Then we use variational calculus to determine the second-order Euler-Lagrange differential equation, whose solution results in the static equilibrium geometric configuration of the suspended cable[i].

The solution of the Euler-Lagrange equation is obtained for two cases: (1) points A and B that suspend the cable are at the same height. (2) A and B can be at different heights.

Next, the tensile force in the cable confined to the cylinder is calculated. Results that are important from an engineering point of view to analyze the cable tensile strength. Finally, numerical experiments will be shown. As far as we know no other research papers exist describing this specific problem. Because of this, we do not provide a dedicated related work section.

2. The flexible cable model suspended on a vertical cylindrical surface

Consider a flexible, inelastic, homogeneous cable of linear density μ and length L suspended by two points A and B fixed in a vertical cylinder with a circular cross section. The cable is subject only to the action of its own weight and is completely confined in the cylindrical surface of radius R and sufficiently large in height. The A and B points can be at different heights. Determine the equation of the curve describing the suspended cable. The gravitational field is uniform.

The surface of the cylinder is parameterized by the vector function:

$$\Phi(z,\theta) = (R\cos\theta, R\sin\theta, z), \tag{2}$$

where $z \in \mathbb{R}$ e $\theta \in [0, 2\pi]$. So there are two ways to represent a point on the cylinder surface. First, with respect to the coordinates of the space $\mathbb{R}^3$ (extrinsic coordinates) with the triple (x, y, z) such that $x^2 + y^2 = R^2$, $z \in \mathbb{R}$, and then by the pair (z, θ) which is known as the intrinsic coordinates of the cylindrical surface. The non-compact coordinate z, which varies along the generator of the cylinder, and the compact coordinate θ, which varies along the cross section of the cylinder.

In the intrinsic coordinates of the cylindrical surface, the curve confined in the cylinder is written as

$$z = z(\theta).$$

Consider h as the difference in height between point B and point A. h may assume positive or negative values. For mathematical convenience, the cable starts at point $A = (-\theta_0, 0)$ and ends at point $B = (\theta_0, h)$.

The Fig. 1 shows that the arc length element ds along an arbitrary cable is calculated approximately with a right triangle, whose legs are: the differential element dz (vertical) and the arc length element in the direction of the cross section $Rd\theta$. Then by the Pythagorean theorem:

$$ds = \sqrt{(Rd\theta)^2 + dz^2}. \tag{3}$$

As $z = z(\theta)$, then $dz = z'd\theta$ and $z' = dz/d\theta$. Hence

$$ds = \sqrt{(Rd\theta)^2 + z'd\theta^2} = \sqrt{R^2 + z'^2}\, d\theta. \tag{4}$$

The previous approach is practical and intuitive because it uses specific cylinder properties. However, we can also study using more general concepts, valid for arbitrary surfaces defined by differentiable functions. Thus, the element

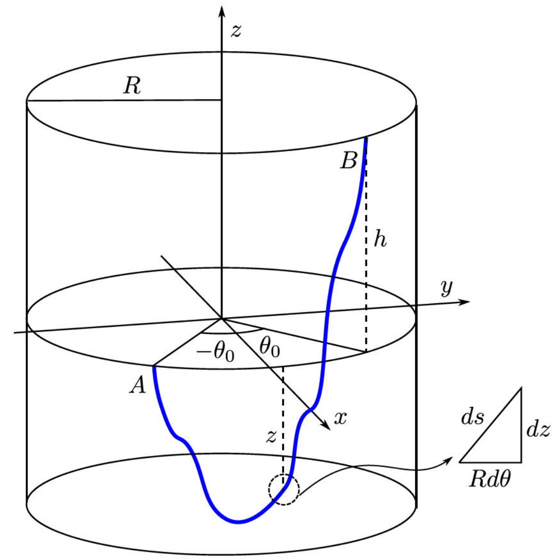

FIGURE 1. Cable of arbitrary shape in the vertical cylinder of radius R.

of arc length ds between two points of a curve, totally confined in an arbitrary surface S, can be calculated using the first differential forms of the differential geometry. Therefore, we have the parametric form of S, Equation (2), and with the following identification $u = z, v = \theta$, we obtain $\Phi_z = (0, 0, 1), \Phi_\theta = (-R\sin\theta, R\cos\theta, 0)$ which is substituted in the first fundamental forms

$$E =< \Phi_z, \quad \Phi_z >= 1;$$
$$F =< \Phi_z, \quad \Phi_\theta >= 0;$$
$$G =< \Phi_\theta, \quad \Phi_\theta >= R^2;$$

and from

$$ds = \int_A^B \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\, dt$$

(see Ref. [5]) obtain

$$ds = \sqrt{\dot{z}^2 + R^2\dot{\theta}^2}\, dt$$

or

$$ds = \sqrt{(Rd\theta)^2 + dz^2}$$

because $dz = \dot{z}dt, d\theta = \dot{\theta}dt$ and as $dz = (dz/d\theta)\, dz$ then $\dot{z} = (dz/d\theta)\dot{\theta}$, hence we obtain the same expression of (4) for the element of arc length ds in the cylinder, demonstrated previously. So the cable length is given by

$$L = \int_{-\theta_0}^{\theta_0} ds = \int_{-\theta_0}^{\theta_0} \sqrt{R^2 + z'^2}\, d\theta. \tag{5}$$

According to classical mechanics, the lower the potential energy of a mechanical system, the more stable that system will be. Then, we find the total potential energy of the cable in an arbitrary geometric configuration and minimize it by the variational calculus. It is understood that there is a functional called *Potential Energy* $E_P : M \to \mathbb{R}$, where M is the set of all smooth functions (curves) that represent cables with length of L and fixed by the ends at points A and B. The functional E_P associates each arbitrary curve $z = z(\theta) \in M$ to a real number $E_P(z)$ which is the physical potential energy of the curve $z = z(\theta)$ in the cylinder.

We can define the potential energy of an element ds (of mass dm) of the cable, located at the coordinate (θ, z), as being

$$dE_p = zgdm = zg\mu ds,\ z < 0,$$

where g is the acceleration of gravity. It was assumed that the xy-plane is the reference level for calculating the potential energy of any particle (see Fig. 2). The total potential energy E_p of the cable is given by

$$E_p = \int_{-\theta_0}^{\theta_0} dE_p = \int_{-\theta_0}^{\theta_0} zg\mu ds.$$

Applying (4), we obtain

$$E_p = \int_{-\theta_0}^{\theta_0} g\mu z\sqrt{R^2 + z'^2}\, d\theta. \tag{6}$$

From the point of view of variational calculus, we have just built a functional because the numerical value of the definite integral (the image of the functional) depends on the cable configuration with fixed ends A and B. Therefore, the problem of determining the curve equation such that the potential energy of the cable is minimal means finding the extreme of the functional

$$J[z] = E_p = \int_{-\theta_0}^{\theta_0} g\mu z\sqrt{R^2 + z'^2}\, d\theta, \tag{7}$$

subject to the constraint of the fixed length of the cable, the equation (5).

We can convert this constrained problem into an equivalent unconstrained problem using the Lagrange multipliers method [5]. The new functional to be minimized is as follows:

$$K[z]{=}\int -\theta_0\theta_0 \left(g\mu z\sqrt{R^2 + z'^2}{+}\lambda\sqrt{R^2 + z'^2}\right)\, d\theta, \tag{8}$$

where λ is the Lagrange multiplier. The function being integrated (Lagrangian function) is given by

$$\mathcal{L}(z, z') = g\mu z\sqrt{R^2 + z'^2} + \lambda\sqrt{R^2 + z'^2}. \tag{9}$$

Because this function depends only on z and z', then the Euler-Lagrange equation can be simplified to the following equation (see the proof in A):

$$\mathcal{L} - z'\frac{\partial\mathcal{L}}{\partial z'} = C_1. \tag{10}$$

Applying this equation to (9), we obtain,

$$(g\mu z + \lambda)\sqrt{R^2 + z'^2} - z'\left[\frac{z'(g\mu z + \lambda)}{\sqrt{R^2 + z'^2}}\right] = C_1,$$

$$(g\mu z + \lambda)\left(\frac{R^2 + z'^2}{\sqrt{R^2 + z'^2}} - \frac{z'^2}{\sqrt{R^2 + z'^2}}\right) = C_1.$$

By considering $C_2 = C_1/R^2, R \neq 0$, we obtain

$$g\mu z + \lambda = C_2\sqrt{R^2 + z'^2}, \tag{11}$$

which is the differential equation for the cable confined in the cylinder described at the beginning of this section. The solution to this equation will be presented in Sec. 4, along with the constraint (5).

3. An alternative solution

In the equilibrium configuration, the coordinate abscissa of the center of gravity of the cable is minimal. It is given by

$$\bar{z} = \frac{\int \mu z ds}{\int \mu ds}. \tag{12}$$

So in the variational process, we could minimize the functional associated with the abscissa. Then we would get the same result as the previous functional (7) as shown below.

By applying $\int \mu ds = \mu L$ and (4) to (12), we obtain the functional to be minimized

$$J[z] = \bar{z} = \frac{\int_{-\theta_0}^{\theta_0} z\mu\sqrt{R^2 + z'^2}\, d\theta}{\int_{-\theta_0}^{\theta_0} \mu ds} = \int_{-\theta_0}^{\theta_0} \frac{z\sqrt{R^2 + z'^2}}{L}\, d\theta,$$

subject to the constraint of the fixed length of the cable.

As before, we can convert this constrained problem into an equivalent unconstrained problem:

$$K[z] = \int_{-\theta_0}^{\theta_0} (z/L + \lambda)\sqrt{R^2 + z'^2} d\theta. \tag{13}$$

Applying the Euler-Lagrange Equation (10) to Lagrangian from (13), we obtain the differential equation

$$z/L + \lambda = C\sqrt{R^2 + z'^2}, \tag{14}$$

which is identical to the Eq. (11) except for the constants that will be eliminated during the development of the solution in Sec. 4.

4. Differential equation solution and boundary conditions

In this section we will find the solution to (11).

Case $C_2 \neq 0$

By squared both sides of (11), and rearranging the terms, we obtain

$$\pm \frac{z'}{\sqrt{(g\mu z + \lambda)^2/C_2^2 - R^2}} = 1.$$

Because $dz = z'\,d\theta$, this equation can be written as:

$$\pm \frac{dz}{\sqrt{(g\mu z + \lambda)^2/C_2^2 - R^2}} = d\theta.$$

Integrating both sides and replacing $u = g\mu z + \lambda$ and $dz = du/(g\mu)$, we get

$$\pm \frac{C_2}{g\mu} \int \frac{du}{\sqrt{u^2 - R^2 C_2^2}} = \theta + C_3.$$

The signal $\pm$ can be absorbed into the constant C_2 and by replacing $u = RC_2 \cosh(v)$ and $du = RC_2 \sinh(v) dv$, we get

$$\frac{C_2}{g\mu} \int \frac{RC_2 \sinh(v) dv}{\sqrt{R^2 C_2^2 \cosh^2(v) - R^2 C_2^2}} = \frac{C_2}{g\mu} \int \frac{\sinh(v) dv}{\sqrt{\cosh^2(v) - 1}} = \frac{C_2}{g\mu} v = \theta + C_3.$$

If we now substitute $v = \cosh^{-1}(u/(RC_2))$, we get

$$\frac{C_2}{g\mu} \cosh^{-1}\left(\frac{u}{RC_2}\right) = \theta + C_3.$$

Hence

$$u = RC_2 \cosh\left(\frac{g\mu}{C_2}\theta + \frac{g\mu}{C_2} C_3\right).$$

By substituting $u = g\mu z + \lambda$, we get

$$z = aR \cosh\left(\frac{\theta + b}{a}\right) + c, \tag{15}$$

where

$$a = \frac{C_2}{g\mu}, \qquad b = C_3, \qquad c = -\frac{\lambda}{g\mu}.$$

Equation (15) is the solution of the differential equation (11) and therefore is the equation of the cable confined in the cylinder. Note that it is the catenary Eq. (1) multiplied by R in the θz-plane. The parameters a, b and c depend on the boundary conditions and the constraint of the fixed length of the cable Eq. (5). The parameters b and c translate the cable horizontally and vertically, respectively, to fit the points A and B. Firstly, for pedagogical purposes, we will determine the parameters for the particular case in which points A and B have the same height. Next, we will study the general case in which A and B can also be at different heights.

4.1. Particular case ($h = 0$): cable suspended at points with the same height

In this case, the boundary conditions are $z(-\theta_0) = z(\theta_0) = 0$. Under these conditions, the cable is symmetrical with respect to the z axis. This implies that $b = 0$ in Eq. (15), because values of $b \neq 0$ would shift the cable horizontally causing the loss of this symmetry. Therefore, the cable equation for Case 1 is given by

$$z = aR \cosh(\theta/a) + c. \tag{16}$$

Now we calculate the parameter a. A cable has a fixed length equal to L given by (5). Solving it, we get

$$L = \int_{-\theta_0}^{\theta_0} \sqrt{R^2 + z'^2}\, d\theta = \int_{-\theta_0}^{\theta_0} \sqrt{R^2 + [R\sinh(\theta/a)]^2} d\theta$$
$$= R \int_{-\theta_0}^{\theta_0} \cosh(\theta/a) d\theta.$$

Thus,

$$2aR \sinh(\theta_0/a) = L. \tag{17}$$

Equation (17) relates a to L, but it is a transcendental equation. Because of this, to get the a parameter, we must resort to numerical methods such as the Newton-Raphson method [6].

Applying $z(\theta_0) = 0$ to (15), we obtain the parameter c directly,

$$c = -aR \cosh(\theta_0/a). \tag{18}$$

We can obtain an alternative form for this equation as follows. Using (17) and the relation $\cosh^2(\theta_0/a) - \sinh^2(\theta_0/a) = 1$, we obtain

$$\cosh(\theta_0/a) = \sqrt{\left(\frac{L}{2aR}\right)^2 + 1}.$$

Applying it to (18), we get

$$c = -\sqrt{(L/2)^2 + (aR)^2}. \tag{19}$$

Finally, we replace the parameters a and c with their values calculated in Eq. (16) to obtain the final solution to the problem.

Although (17) is not solved analytically, we can still study geometrically what kind of solution it can provide. To do this, we introduce two auxiliary functions:

$$g(x) = \sinh(\theta_0 x), \tag{20}$$

$$f(x) = \frac{L}{2R} x, \tag{21}$$

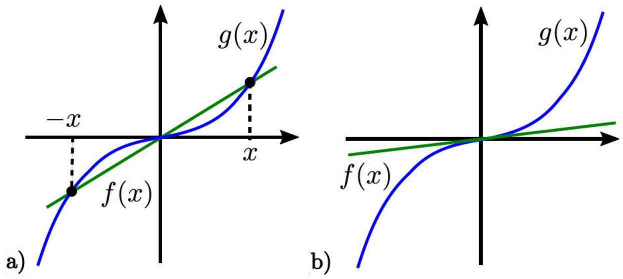

FIGURE 2. Auxiliary functions $f(x)$ e $g(x)$. a) The intersections between $f(x)$ and $g(x)$ generate two candidate solutions. b) For $x \neq 0$, there is no intersection between $f(x)$ and $g(x)$ (no candidate solution).

where $x \equiv 1/a$. Thus, (17) has the following form $g(x) = f(x)$ with x being the unknown to be determined. With the exception of the point $x = 0$, the points of intersection between $f(x)$ and $g(x)$ are candidate solutions for the problem [see Fig. 2a)]. However, if the slope of $f(x)$ is small, as shown in the Fig. 2b), then there will be no solution.

In the case of Fig. 2a), there must be two solutions in Eq. (17), which are symmetric with respect to the origin of the x-axis ($x = 1/a$). However, we observe that the negative solution ($a < 0$) is invalid, because when it is substituted in Eq. (15), it results in a curve with concavity facing downwards and therefore would not minimize the functional (8). Then, the positive solution $(a > 0)$ would be the only solution to the problem. In the case of Fig. 2b), there is no candidate solution because the slope of $f(x)$ is lesser than the slope of $g(x)$ at point $x = 0$. Hence the condition for a solution to exist is:

$$f'(0) = \frac{L}{2R} > g'(0) = \theta_0. \tag{22}$$

Rearranging the terms,

$$L > R(2\theta_0). \tag{23}$$

This solubility condition of (17) is confirmed geometrically. Because according to Fig. 1, the length L of cable must be greater than the circular distance between the A and B ends of the cable, located here at the same height, for the problem to make some physical sense. That is, $L > R(2\theta_0)$, which is exactly the condition of the algebraic solubility (23) arising from the transcendental Eq. (17).

4.2. General case ($h \in \mathbb{R}$): cable can be suspended at points of different heights

In this case, the boundary conditions are $z(-\theta_0) = 0$ and $z(\theta_0) = h$. Applying them to (15), we obtain

$$0 = aR\cosh((-\theta_0 + b)/a) + c, \tag{24}$$

$$h = aR\cosh((\theta_0 + b)/a) + c. \tag{25}$$

Subtracting the first equation from the second, we have

$$\cosh\left(\frac{\theta_0 + b}{a}\right) - \cosh\left(\frac{-\theta_0 + b}{a}\right) = \frac{h}{aR}.$$

Putting $\cosh$ in the exponential form and rearranging the terms, we get the relationship

$$\frac{e^{b/a} - e^{-b/a}}{2} = \sinh(b/a) = \frac{h}{2aR\sinh(\theta_0/a)}. \tag{26}$$

Now we will use the fixed length constraint of the cable given by (5) to get another relationship. Solving we get

$$\begin{aligned} L &= \int_{-\theta_0}^{\theta_0} \sqrt{R^2 + z'^2}\, d\theta \\ &= \int_{-\theta_0}^{\theta_0} \sqrt{R^2 + \left[R\sinh\left(\frac{\theta + b}{a}\right)\right]^2}\, d\theta \\ &= R\int_{-\theta_0}^{\theta_0} \cosh\left(\frac{\theta + b}{a}\right) d\theta. \end{aligned}$$

Hence

$$L = aR\sinh\left(\frac{\theta_0 + b}{a}\right) - aR\sinh\left(\frac{-\theta_0 + b}{a}\right). \tag{27}$$

Putting $\sinh$ in the exponential form and rearranging the terms, we get the relationship

$$\frac{e^{b/a} + e^{-b/a}}{2} = \cosh(b/a) = \frac{L}{2aR\sinh(\theta_0/a)}. \tag{28}$$

We apply the relationships (26) and (28) in the formula $\cosh^2(b/a) - \sinh^2(b/a) = 1$ to obtain

$$2aR\sinh(\theta_0/a) = \sqrt{L^2 - h^2}. \tag{29}$$

The parameter a can be determined by (29) which relates a to L. However, it is also a transcendental equation that must be solved by numerical methods.

Using the relationships (26) and (28) again, we have

$$\tanh(b/a) = \frac{\sinh(b/a)}{\cosh(b/a)} = \frac{h}{L}. \tag{30}$$

So we get the parameter b,

$$b = a\tanh^{-1}(h/L). \tag{31}$$

Now we will describe another alternative for determining the parameter b. Putting the $\tanh$ in the exponential form in (30) we have,

$$\frac{e^{b/a} - e^{-b/a}}{e^{b/a} + e^{-b/a}} = \frac{h}{L}. \tag{32}$$

By rearranging the terms, we get

$$(e^{b/a})^2 = \frac{L + h}{L - h}.$$

So we have another formula for the parameter b,

$$b = \frac{a}{2}\ln\left(\frac{L + h}{L - h}\right). \tag{33}$$

Finally, we can get the parameter c directly from (24):

$$c = -aR\cosh\left(\frac{\theta_0 - b}{a}\right). \tag{34}$$

5. The tensile force on cable

The cable, by hypothesis, is totally flexible. That is, it does not resist any bending. Because of this, the internal force acting on the cable is always in the direction of the cable. At the general position θ, this force or tension in the cable is denoted by

$$\mathbf{T}(\theta) = T_x(\theta)\mathbf{i} + T_y(\theta)\mathbf{j} + T_z(\theta)\mathbf{k}.$$

The cable equation is given in (15). Its minimum point is at $\theta = -b$ (see left side in Fig. 3). For mathematical convenience, we will study the cable segment after its minimum point ($\theta \geq -b$), that is, the segment in interval $[-b, \theta]$. The length of this segment is given by

$$s(\theta) = aR\sinh((\theta + b)/a). \tag{35}$$

This equation can be obtained in the same way as (27), just changing the integration interval to $[-b, \theta]$.

The rectangular coordinates of the cable curve are obtained by the equations

$$x = R\cos(\theta),$$
$$y = R\sin(\theta),$$
$$z = aR\cosh((\theta + b)/a) + c.$$

In other words, these equations are the parametric components of the vector function that we associate with the curve that physically represents the cable. By deriving this vector function we will obtain the tangent vector to the curve. Because the tension $\mathbf{T}$ is in the direction of the cable curve, then the direction cosines of $\mathbf{T}$, (*i.e.*, $\cos\theta_x$, $\cos\theta_y$ and $\cos\theta_z$), are equal to the direction cosines of the tangent vector to the cable.

The tangent vector to the cable curve is

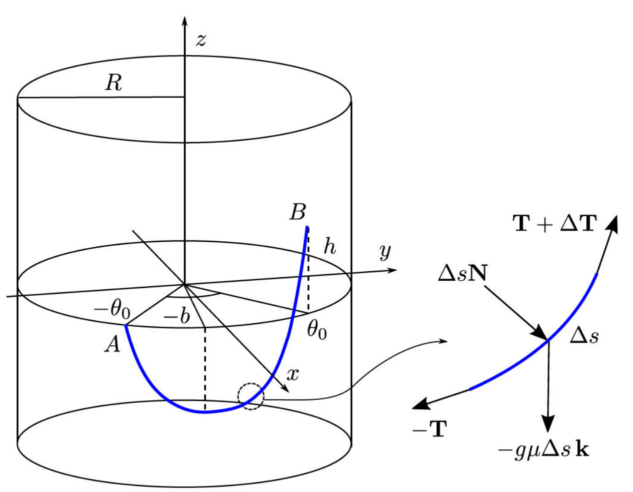

FIGURE 3. The forces acting on a small segment of cable Δs.

$$\mathbf{v} = v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k} = \frac{dx}{d\theta}\,\mathbf{i} + \frac{dy}{d\theta}\,\mathbf{j} + \frac{dz}{d\theta}\,\mathbf{k}$$
$$= -R\sin\theta\,\mathbf{i} + R\cos\theta\,\mathbf{j} + R\sinh((\theta + b)/a)\,\mathbf{k},$$

and its length is

$$\|\mathbf{v}\| = \sqrt{(-R\sin\theta)^2 + (R\cos\theta)^2 + (R\sinh((\theta + b)/a))^2}$$
$$= R\cosh((\theta+b)/a).$$

So, the direction cosines of $\mathbf{T}$ are given by

$$\cos\theta_x = \frac{v_x}{\|\mathbf{v}\|} = -\frac{\sin\theta}{\cosh((\theta + b)/a)}, \tag{36}$$

$$\cos\theta_y = \frac{v_y}{\|\mathbf{v}\|} = \frac{\cos\theta}{\cosh((\theta + b)/a)}, \tag{37}$$

$$\cos\theta_z = \frac{v_z}{\|\mathbf{v}\|} = \tanh((\theta + b)/a). \tag{38}$$

The external forces acting on the cable are the cable's own weight and the reaction force, normal to the cylinder surface, resulting from the reaction of the cylinder wall to the contact of cable. This reaction force is distributed along the cable. Therefore, it is expressed in unit of force per unit of length and denoted by

$$\mathbf{N} = N\cos\theta\,\mathbf{i} + N\sin\theta\,\mathbf{j}.$$

The right side of the Fig. 3 shows the forces acting on a small section Δs of the cable, where $\Delta s = s(\theta) - s(\theta + \Delta\theta)$ and $\Delta\mathbf{T} = \mathbf{T}(\theta + \Delta\theta) - \mathbf{T}(\theta)$. The weight force acting on the section Δs is always a vector in the vertical direction that points downwards, that is, $-g\mu\Delta s\,\mathbf{k}$. Since Δs is small, the resultant normal force can be approximated by $\Delta s\,\mathbf{N}$. In this way, we can write the three static equilibrium equations as

$$\Delta T_x + N\cos\theta\Delta s \approxeq 0,$$
$$\Delta T_y + N\sin\theta\Delta s \approxeq 0,$$
$$\Delta T_z - g\mu\Delta s = 0.$$

Dividing the equations by $\Delta\theta$ and making $\Delta\theta \longrightarrow 0$, we get

$$\frac{dT_x}{d\theta} + N\cos\theta\frac{ds}{d\theta} = 0, \tag{39}$$

$$\frac{dT_y}{d\theta} + N\sin\theta\frac{ds}{d\theta} = 0, \tag{40}$$

$$\frac{dT_z}{d\theta} - g\mu\frac{ds}{d\theta} = 0. \tag{41}$$

We obtain the vertical component T_z of tension $\mathbf{T}$ by integrating (41):

$$T_z(\theta) = T_z(-b) + \int_{-b}^{\theta} g\mu\frac{ds}{d\theta'}d\theta' \tag{42}$$

$$= T_z(-b) + \int_{-b}^{\theta} g\mu\frac{d}{d\theta'}\left(aR\sinh((\theta' + b)/a)\right)d\theta'. \tag{43}$$

At the minimum point ($\theta = -b$), the direction of tension $\mathbf{T}$ is horizontal. Therefore $T_z(-b) = 0$. So the result of (43) is

$$T_z(\theta) = g\mu aR\sinh((\theta+b)/a). \tag{44}$$

Substituting (44) and (38) in the relationship $T_z = T\cos\theta_z$, we obtain the magnitude of the cable tension,

$$T(\theta) = g\mu aR\cosh((\theta+b)/a). \tag{45}$$

Using the relationships $T_x = T\cos\theta_x$ and $T_y = T\cos\theta_y$ and (36) and (37), we obtain the other components of T,

$$T_x(\theta) = -g\mu aR\sin\theta, \tag{46}$$

$$T_y(\theta) = g\mu aR\cos\theta. \tag{47}$$

Substituting (46) in (39), we get

$$\frac{d}{d\theta}(-g\mu aR\sin\theta)+N\cos\theta\frac{d}{d\theta}\left(aR\sinh((\theta+b)/a)\right)d\theta = 0.$$

Solving the above equation for N, we have the magnitude of the normal force per unit of length.

$$N(\theta) = \frac{g\mu a}{\cosh((\theta+b)/a)}. \tag{48}$$

Summarizing the results so far, we have the cable tension

$$\mathbf{T} = g\mu aR\left[-\sin\theta\,\mathbf{i}+\cos\theta\mathbf{j}+\sinh((\theta+b)/a)\,\mathbf{k}\right], \tag{49}$$

and the reaction of the cylinder wall to the cable contact

$$\mathbf{N} = \frac{g\mu a}{\cosh((\theta+b)/a)}\left(\cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j}\right). \tag{50}$$

Converting these two results to cylindrical coordinates, we obtain,

$$\mathbf{T} = g\mu aR\left[\hat{\mathbf{e}}_\theta+\sinh((\theta+b)/a)\,\hat{\mathbf{e}}_z\right], \tag{51}$$

$$\mathbf{N} = \frac{g\mu a}{\cosh((\theta+b)/a)}\,\hat{\mathbf{e}}_r. \tag{52}$$

where $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$ e $\hat{\mathbf{e}}_z$ are the unit vectors for cylindrical coordinates.

6. Load on the cylinder wall

It should be noted that the cable forms a distributed load on the cylinder wall. This load, denoted by $\mathbf{D}$, has a diagonal direction and is composed of two components: the reaction force in reverse (*i.e.*, $-\mathbf{N}$) and the force weight per unit length. That is,

$$\mathbf{D} = -\frac{g\mu a}{\cosh((\theta+b)/a)}\,\hat{\mathbf{e}}_r - g\mu\,\hat{\mathbf{e}}_z. \tag{53}$$

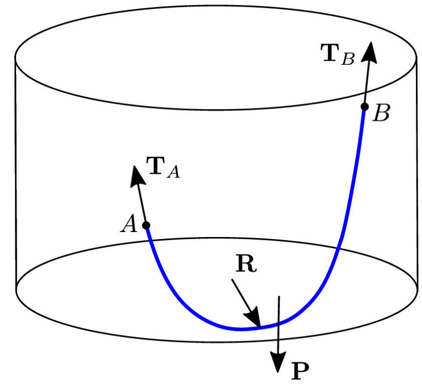

FIGURE 4. External forces acting on the cable.

7. External forces

Now we can directly determine the external forces acting on the cable (see Fig. 4). The forces that suspend the cable at points A and B are, respectively

$$\mathbf{T}_A = -\mathbf{T}(-\theta_0) = g\mu aR\big[-\sin\theta_0\,\mathbf{i} - \cos\theta_0\mathbf{j} - \sinh((-\theta_0+b)/a)\,\mathbf{k},\big], \tag{54}$$

$$\mathbf{T}_B = \mathbf{T}(\theta_0) = g\mu aR\big[-\sin\theta_0\,\mathbf{i}+\cos\theta_0\mathbf{j} + \sinh((\theta_0+b)/a)\,\mathbf{k}.\big]. \tag{55}$$

The resultant force $\mathbf{R}$ of the reaction of the wall on the cable is given by

$$\mathbf{R} = \int_{-\theta_0}^{\theta_0}\mathbf{N}ds = \int_{-\theta_0}^{\theta_0}\frac{g\mu a}{\cosh((\theta+b)/a)} \times(\cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j})ds. \tag{56}$$

From (35), we obtain $ds = R\cosh((\theta+b)/a)d\theta$. Substituting it in the equation above, we get

$$\mathbf{R} = 2g\mu aR\sin\theta_0\,\mathbf{i}. \tag{57}$$

Substituting (27) in the resultant weight force $\mathbf{P} = g\mu L\,\mathbf{k}$, we can rewrite it as

$$\mathbf{P} = g\mu aR\big[\sinh\left((\theta_0+b)/a\right) - \sinh\left((-\theta_0+b)/a\right)\big]\,\mathbf{k}. \tag{58}$$

Finally, we can easily check the balance of forces. That is, $\mathbf{T}_A+\mathbf{T}_B+\mathbf{R}+\mathbf{P} = 0$.

8. Numerical Experiments

8.1. Cable suspended at points with the same height

Consider a flexible cable with a length of $L = 3.0$ m, density $\mu = 3$ kg/m and $g = 9.81$ m/s^2, confined in a cylinder of radius $R = 0.5$ m. The cable is suspended at points

$A = (-\pi/6.0)$ and $B = (\pi/6.0)$ in cylindrical coordinates. The cable equation is described by (15). We will determine the parameters a, b and c of the cable equation.

Because the cable is suspended at points of the same height ($h = 0$), so $b = 0$. We get the parameter a from (17). Then

$$2a \sinh\left(\frac{\pi}{6a}\right) = 3,0,$$

whose solution is $a = 0.139066$ (obtained with numerical computing software). We get the parameter c from (19) or (18):

$$c = -\sqrt{(3,0/2)^2 + (0,139066 \times 0,5)^2} = -1,50161.$$

Substituting parameter values in (15), we obtain the cable equation,

$$z_1(\theta) = 0.0695332 \cosh(\theta/0.139066) - 1.50161.$$

The maximum tension in the cable occurs at points A and B because they are at the same height. Substituting the point A (or B) in (45), we obtain the maximum tension:

$$T_{\max} = 9,81 \times 3 \times 0,139066 \times 0.5 \cosh(\pi/(6 \times 0,139066)) = 44.1924\,\mathrm{N}. \quad (59)$$

8.2. Cable suspended at different height points

Consider the same cable and cylinder as the previous example, with the difference that in this section the cable is suspended at the points $A = (-\pi/6, 0)$ e $B = (\pi/6, 1)$. So $h = 1$ m. Parameter a is determined by (29)

$$2a \times 0.5 \sinh\left(\frac{\pi}{6a}\right) = \sqrt{6^2 - 1^2}, \quad (60)$$

whose solution is $a = 0,142100$ (obtained by numerical computing software).

Parameter b is obtained by (33)

$$b = \frac{0,142100}{2} \ln\left(\frac{3+1}{3-1}\right) = 0,049248. \quad (61)$$

Parameter c is obtained by (34)

$$c = -0,142100 \times 0,5 \cosh\left(\frac{-\pi/6 - 0,049248}{0,142100}\right) = -1,001892.$$

Substituting these values in (15), we obtain the cable equation,

$$z_2(\theta) = 0,0710499 \times \cosh\left(\frac{\theta + 0,0492481}{0,1421}\right) - 1,001892. \quad (62)$$

The maximum tension on the cable is located at point B because it is higher. Substituting the point B in Eq. (45), we obtain the maximum tension $T_{\max} = 58,9157$ N. Note that raising the B point by 1 m, compared to the previous example, increased the maximum tension significantly (see Eq. (59)). Finally, the graphs of the cables z_1 and z_2 are shown in Fig. 5.

8.3. Cable forces on cylinders with different radius.

This experiment compares the tensions and reaction forces between cables confined in different cylinders and also with the catenary (unconfined cable).

Consider a cable of length $L = 330$ m, shown in Fig. 6, with density $\mu = 12$ kg/m and $g = 9.81$ m/s^2, suspended between points A and B of the same height ($h = 0$) and separated by a fixed distance of $d = 300$ m.

Figure 7 shows the tension on this cable versus its partial length s for cylinders with radius 300, 350, 500, and 900 m. In this graph, the tension T was determined as follows. Equation (35), with $b = 0$, gives the partial length of the cable from the half of the cable. So to determine the length of the cable, from the beginning, we add $L/2$, that is

$$s = L/2 + aR \sinh(\theta/a),$$

hence

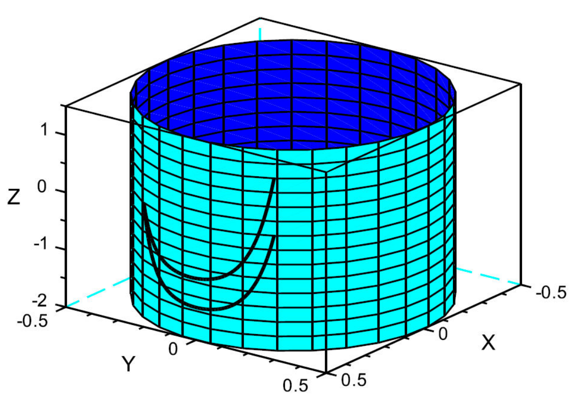

FIGURE 5. Flexible cables confined in a circular cylinder. Cables are the same length but suspended at different points.

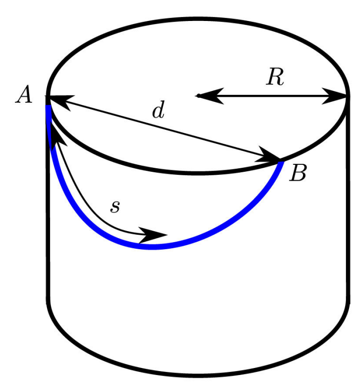

FIGURE 6. Cable suspended at points A and B separated by a distance d.

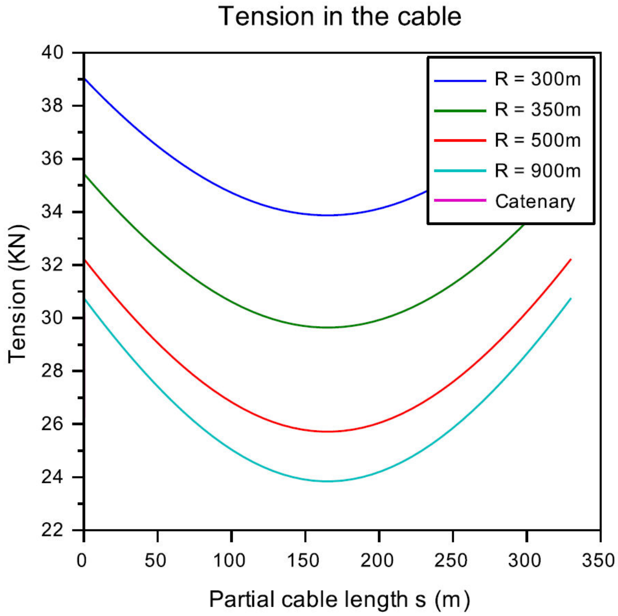

FIGURE 7. The graph shows the tension in the cable, along its length, when confined to different cylinders and without confinement (catenary).

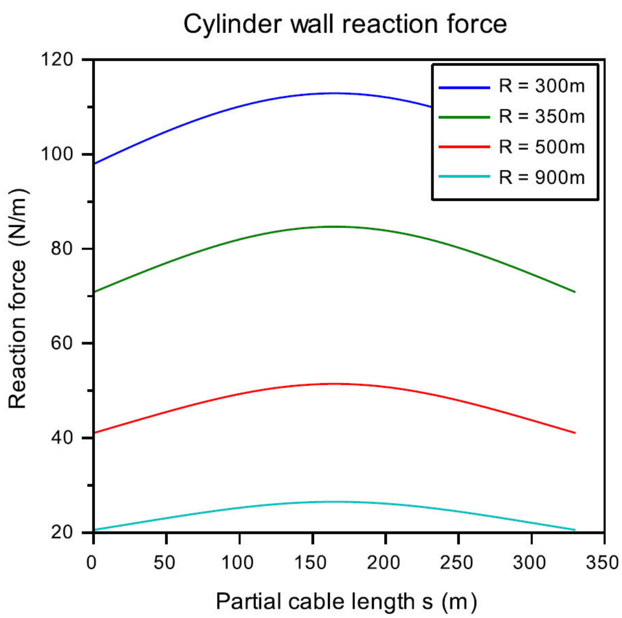

FIGURE 8. The graph shows the wall reaction force per unit length.

$$\theta = a \sinh^{-1}\left(\frac{s - L/2}{aR}\right).$$

Then we can determine the traction, given by (45):

$$T = g\mu aR \cosh(\theta/a).$$

Likewise, Fig. 8 shows the cylinder reaction force per unit length (48), for cylinders with different radius. According to Figs. 7 and 8, the tension of the cable and the reaction forces increase significantly as the cylinder radius decreases. On the other hand, when the radius increases, the tensile and reaction forces decrease and approach the behavior of a catenary.

9. Conclusion

This work solved a modified version of the classical catenary problem in which the cable is confined in a cylinder. The article addressed several aspects of the problem ranging from cable equation deduction to cable tensile analysis. We believe this approach is useful for physics students and engineers interested in flexible cable analysis. The cable equation deduction was based on the principle that the cable equilibrium configuration is the minimum gravitational potential energy configuration. Variational calculus was used to obtain the curve that minimizes the potential energy of the cable. We obtain formulas for the cases of cables suspended at points of the same height and different heights. In addition, we also obtain the formula for the tension at each point of the cable. The analysis of tension in cables confined to surfaces is an aspect that differs from the classical catenary problem. The classical catenary has no contact with surfaces and surface reaction forces do not exist. This article presents examples to illustrate applications of the formulas obtained. And numerical experiments have shown that confinement in the cylinder significantly increases cable tension compared to classical catenary. Numerical results also showed that increasing the radius of the cylinder reduces the tension in the cable, as long as the distance between the support points is maintained. Finally, this work serves as an introduction to future work with cables confined to a cylinders, spheres and other surfaces.

Appendix

A. Particular form of the Euler-Lagrange equation in which x does not appear explicitly in the Lagrangian, *i.e.*, $\mathcal{L} = \mathcal{L}(y, y\prime)$

Suppose that in the functional of an elementary problem of variational calculus

$$S[y] = \int_{x_0}^{x_f} \mathcal{L}(x, y, y')\, dx,$$

we have $\mathcal{L} = \mathcal{L}(y, y\prime)$. So $(\partial\mathcal{L}/\partial y\prime) = (\partial\mathcal{L}/\partial y\prime)(y, y\prime)$, because if the Lagrangian depends only on y and $y\prime$, then its partial derivative $\partial\mathcal{L}/\partial y\prime$ must also depend only on y and $y\prime$. This implies that the Euler-Lagrange equation becomes:

$$\frac{\partial\mathcal{L}}{\partial y} = \frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial y\prime}(y, y\prime)\right)$$

Since $y = y(x)$ and $y\prime = y\prime(x)$, then differentiating the right-hand side with respect to x and using the chain rule, we get:

$$\frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial y\prime}(y, y\prime)\right) = \frac{\partial}{\partial y}\left(\frac{\partial\mathcal{L}}{\partial y\prime}\right)\frac{dy}{dx} + \frac{\partial}{\partial y\prime}\left(\frac{\partial\mathcal{L}}{\partial y\prime}\right)\frac{dy\prime}{dx}.$$

Therefore, the Euler-Lagrange equation becomes:

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial}{\partial y}\Big(\frac{\partial \mathcal{L}}{\partial y\prime}\Big)\frac{dy}{dx} + \frac{\partial}{\partial y\prime}\Big(\frac{\partial \mathcal{L}}{\partial y\prime}\Big)\frac{dy\prime}{dx}.$$

Passing everything to the left-hand side and multiplying by y', we get

$$y'\Big(\frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial y}\Big(\frac{\partial \mathcal{L}}{\partial y\prime}\Big)\frac{dy}{dx} - \frac{\partial}{\partial y\prime}\Big(\frac{\partial \mathcal{L}}{\partial y\prime}\Big)\frac{dy\prime}{dx}\Big) = 0.$$

By rearranging the terms, we obtain

$$y'\frac{\partial \mathcal{L}}{\partial y} + \frac{\partial \mathcal{L}}{\partial y\prime}\frac{dy\prime}{dx} - \frac{\partial \mathcal{L}}{\partial y\prime}\frac{dy\prime}{dx} - y'\frac{\partial}{\partial y}\Big(\frac{\partial \mathcal{L}}{\partial y\prime}\Big)\frac{dy}{dx} - y'\frac{\partial}{\partial y\prime}\Big(\frac{\partial \mathcal{L}}{\partial y\prime}\Big)\frac{dy\prime}{dx} = 0,$$

Because the first two terms of the previous equation are $(d/dx)(\mathcal{L})$ and the last three terms are

$$\frac{d}{dx}\Big(y'\frac{\partial \mathcal{L}}{\partial y\prime}(y, y\prime)\Big),$$

then

$$\frac{d}{dx}\Big(\mathcal{L} - y\prime\frac{\partial \mathcal{L}}{\partial y\prime}\Big) = 0.$$

Hence

$$\mathcal{L} - y\prime\frac{\partial \mathcal{L}}{\partial y\prime} = C \qquad \text{(A.1)}$$

where C is a constant. Equation (A.1) is easier to solve than the Euler-Lagrange equation because it has already been partially solved.

Acknowledgement

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i. It is known in mechanics that by minimizing the potential energy of the cable, we are finding the shape of the cable in static equilibrium.

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6. It is known in mechanics that by minimizing the potential energy of the cable, we are finding the shape of the cable in static equilibrium.
7. There is another approach to finding the static equilibrium configuration of the cable. Determine the coordinates of the center of mass ($\bar{X}$,$\bar{Y}$,$\bar{Z}$) of the cable. Then minimize the coordinate $\bar{Z}$, because in, a situation of static equilibrium, greater stability means less value for $\bar{Z}$.
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