Non-relativistic representation of the Jackiw-Rebbi soliton

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We consider the Foldy-Whouthuysen (FW) transformation of the Dirac equation coupled to a background soliton field which is equivalent to a position-dependent mass $m(x)$ such that at each limit $x \rightarrow \pm\infty$, the mass to the left and to the right tends to a (possibly different) constant, with a sign difference at each side. We then build-up a third order unitarily transformed Schrödinger-like Hamiltonian as a counterpart of the corresponding to the well known Jackiw-Rebbi model. By further FW-transforming the Dirac spinor, we establish the relation between the non-relativistic and relativistic wave functions up to this order of approximation for generic position dependent mass profiles. For the economic choice $m(x) = m_0 x/|x|$, we find that these spinors are the same up to an overall constant.

Keywords: Jackiw-Rebbi model; Foldy-Whothouysen transformation; topological insulators.

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1. Introduction

The Jackiw-Rebbi (JR) model [1] consists in a one-dimensional Dirac equation coupled to a soliton field $\phi(x)$ which can be written in terms of an effective Dirac equation with a position dependent mass $m(x) \propto \phi(x)$ such that as $x \rightarrow -\infty$, $m(-x) \rightarrow -m_0$ whereas as $x \rightarrow \infty$, $m(x) \rightarrow m_1$, where $m_{0,1} \geq 0$ are constants which might in general be different. Assuming that the soliton field has no dynamics, one usually starts from the Hamiltonian

$$H_0 \equiv \alpha p_x + \beta m\left(x\right), \tag{1}$$

with the Dirac matrices obeying $\alpha^2 = \beta^2 = I_{2\times 2}$, where $I_{2\times 2}$ stands for the identity matrix. In solving

$$H_0\psi = E\psi, \tag{2}$$

we first consider the possibility of zero modes, $E = 0$. Thus,

$$\left(\alpha p_x + \beta m\left(x\right)\right)\psi_0 = 0. \tag{3}$$

By letting

$$\psi_0 = g\left(x\right)\chi \tag{4}$$

with $\alpha\beta\chi = -i\chi$, we straightforwardly find that

$$\psi_0\left(x\right) = e^{-\int_{x_0}^{x} dy\ m(y)}\chi, \tag{5}$$

which exhibits a kink behavior around the point where $\phi(x) = m(x) = 0$, namely, when energy bands show no gap. Far from this point, energy bands develop such a gap. Moreover, a truly remarkable property of this kink is that it describes a fractionally charged excitation [2], a phenomenon that was discussed even before the Fractional Quantum Hall Effect was discivered [3]. JR model has been realized experimentally in polyacetylene [4] and emerges naturally as the continuum limit of the Su-Schrieffer-Heeger [5] model for the electron-phonon interaction in these materials. It is also worth mentioning that it also has renewed interest in connection with the physics of topological insulators [6, 7], since the gap closing resembles the emergence of topologically protected surface mode on these materials. Optical analogs of the JR model have been proposed [8–11], whereas electrostatic and magnetostatic analogs were discussed in Refs. [12, 13], respectively. The cylindrical generalization of the model, namely, the so-called Dirac wires were first introduced in Ref. [14]. The JR model has also been found as a non relativistic limit of some topological superconductors [15]. These are enough reasons to further consider other theoretical aspects of the model that we address in this article. For this purpose, in the next section we address the problem of the Foldy-Wouthuysen (FW) transformation [16] of the Dirac equation with a position dependent mass term. We discuss our findings and conclude in Sec. 3.

2. Foldy-Wouthuysen transformation

We focus our attention in the non-relativistic representation of the JR kink as derived from the FW transformation [16] of the Dirac equation with a position dependent mass term [17, 18]. We commence by diagonalizing the Hamiltonian in Eq. (2) with the corresponding unitary transformation for the FW representation,

$$U(x) \equiv e^{iS}, \tag{6}$$

with

$$S \equiv -\frac{i}{2}\frac{1}{\sqrt{m}}\beta\alpha p_x\frac{1}{\sqrt{m}}. \tag{7}$$

By explicitly representing the Dirac matrices as

$$\alpha \equiv \sigma_x, \qquad \beta \equiv \sigma_z, \tag{8}$$

we write the Dirac Hamiltonian

$$H_0 \equiv \sigma_x p_x + \sigma_z m(x), \tag{9}$$

and the transformation in Eq. (7) as

$$S \equiv \frac{1}{2}\frac{1}{\sqrt{m}}\sigma_y p_x\frac{1}{\sqrt{m}}. \tag{10}$$

In the spirit of the FW transformation, we seek to write the stationary Eq. (2) in the form

$$H\Psi = E\Psi, \tag{11}$$

with the transformed Hamiltonian and wavefuction given by

$$\begin{aligned} H &\equiv U(x) H_0 U^{\dagger}(x) = e^{iS} H_0 e^{-iS}, \\ \Psi &\equiv U(x)\psi = e^{iS}\psi. \end{aligned} \tag{12}$$

Then, by exploiting the Baker–Campbell–Hausdorff formula, we approximate upto third order

$$H \approx H_0 + i[S, H_0] - \frac{1}{2}[S,[S,H_0]] + \ldots \tag{13}$$

Working out explicitly the first commutator, we find

$$\begin{aligned} [S, H_0] &= i\left(H_0 - m\sigma_z - \frac{1}{2}\sigma_z\left\{\frac{1}{\sqrt{m}}p_x\frac{1}{\sqrt{m}}, p_x\right\}\right) \\ &= \left[\frac{1}{2}\frac{1}{\sqrt{m}}p_x\frac{1}{\sqrt{m}}\sigma_y, p_x\sigma_x + m\sigma_z\right] \\ &= [S, p_x\sigma_x] + [S, m\sigma_z], \end{aligned} \tag{14}$$

from where we derive the following useful relations,

$$\begin{aligned} [S, p_x\sigma_x] &= -\frac{i}{2}\sigma_z\left\{\frac{1}{\sqrt{m}}p_x\frac{1}{\sqrt{m}}, p_x\right\} \\ [S, m\sigma_z] &= i\sigma_x p_x. \end{aligned} \tag{15}$$

Next, the commutator

$$\begin{aligned} [S,[S,H_0]] &= \left[S, i\left(\sigma_x p_x - \frac{1}{2}\sigma_z\left\{\frac{1}{\sqrt{m}}p_x\frac{1}{\sqrt{m}}, p_x\right\}\right)\right] \\ &\approx i[S, \sigma_x p_x] + \ldots = \frac{1}{2}\sigma_z\left\{\frac{1}{\sqrt{m}}p_x\frac{1}{\sqrt{m}}, p_x\right\}, \end{aligned} \tag{16}$$

from which we directly find that at the third order,

$$H \approx \left(\frac{1}{4}\left\{\frac{1}{\sqrt{m}}p_x\frac{1}{\sqrt{m}}, p_x\right\} + m\right)\sigma_z. \tag{17}$$

Notice that this Schrödinger-like Hamiltonian is quadratic in the momentum with a non-trivial dependence of the position dependent mass, as demanded by hermiticity of H. A similar form of the Hamiltonian was proposed in Ref. [18] precisely as the effective Schrödinger equation for position dependent mass of charge carriers.

For the wave function, we have

$$\begin{aligned} \Psi_0 &= e^{\frac{i}{2}\frac{1}{\sqrt{m}}p_x\frac{1}{\sqrt{m}}\sigma_y} e^{-\int_{x_0}^{x} dy\ m(y)}\chi \\ &= \left(\cos\left(\frac{1}{2}\frac{1}{\sqrt{m}}p_x\frac{1}{\sqrt{m}}\right)\right. \\ &\quad \left. + i\sin\left(\frac{1}{2}\frac{1}{\sqrt{m}}p_x\frac{1}{\sqrt{m}}\right)\sigma_y\right) e^{-\int_{x_0}^{x} dy\ m(y)}\chi, \end{aligned} \tag{18}$$

but since χ is an eigenvector of σ_y with eigenvalue +1,

$$\begin{aligned} \Psi_0 &= \left(\cos\left(\frac{1}{2}\frac{1}{\sqrt{m}}p_x\frac{1}{\sqrt{m}}\right) + i\sin\left(\frac{1}{2}\frac{1}{\sqrt{m}}p_x\frac{1}{\sqrt{m}}\right)\right) \\ &\quad \times e^{-\int_{x_0}^{x} dy\ m(y)}\chi = e^{\frac{i}{2}\frac{1}{\sqrt{m}}p_x\frac{1}{\sqrt{m}}} e^{-\int_{x_0}^{x} dy\ m(y)}\chi. \end{aligned} \tag{19}$$

Writting formally that

$$e^{\frac{i}{2}\frac{1}{\sqrt{m}}p_x\frac{1}{\sqrt{m}}} = \sum_{n=0}^{\infty}\frac{1}{2^n n!}\left(\frac{1}{\sqrt{m}}\frac{d}{dx}\frac{1}{\sqrt{m}}\right)^n, \tag{20}$$

we cast the wavefunction in the form

$$\Psi_0 = \sum_{n=0}^{\infty}\frac{1}{2^n n!}\left(\frac{1}{\sqrt{m}}\frac{d}{dx}\frac{1}{\sqrt{m}}\right)^n e^{-\int_{x_0}^{x} dy\ m(y)}\chi. \tag{21}$$

Let us observe that

$$\begin{aligned} &\left(\frac{1}{\sqrt{m}}\frac{d}{dx}\frac{1}{\sqrt{m}}\right) e^{-\int_{x_0}^{x} dy\ m(y)} \\ &\qquad = \left(\frac{1}{2}\frac{d}{dx}\left(\frac{1}{m}\right) - 1\right) e^{-\int_{x_0}^{x} dy m(y)} \\ &\qquad \equiv f_1(m)\, e^{-\int_{x_0}^{x} dy\ m(y)}. \end{aligned} \tag{22}$$

Similarly,

$$\begin{aligned} &\left(\frac{1}{\sqrt{m}}\frac{d}{dx}\frac{1}{\sqrt{m}}\right)^2 e^{-\int_{x_0}^{x} dy\ m(y)} \\ &\qquad = \left[\frac{1}{m}\frac{d}{dx}f_1(m) + (f_1(m))^2\right] e^{-\int_{x_0}^{x} dy\ m(y)} \\ &\qquad \equiv f_2(m)\, e^{-\int_{x_0}^{x} dy\ m(y)}. \end{aligned} \tag{23}$$

From here, we define the recurrence relation

$$\begin{aligned}\left(\frac{1}{\sqrt{m}}\frac{d}{dx}\frac{1}{\sqrt{m}}\right)^{n+1} & e^{-\int_{x_0}^{x} dy\; m(y)} \\ &= \left[\frac{1}{m}\frac{d}{dx}f_n\left(m\right) + f_1\left(m\right)f_n\left(m\right)\right]e^{-\int_{x_0}^{x} dy\; m(y)} \\ &= f_{n+1}\left(m\right)e^{-\int_{x_0}^{x} dy\; m(y)}. \end{aligned} \tag{24}$$

Let us further notice that

$$\begin{aligned} f_{n+1}\left(m\right) &= \left[\frac{1}{m}\frac{d}{dx} + f_1\left(m\right)\right]f_n\left(m\right) \\ &\equiv Df_n\left(m\right), \end{aligned} \tag{25}$$

from which we can write

$$f_{n+1}\left(m\right) = D^n f_1\left(m\right). \tag{26}$$

Then

$$\begin{aligned}\Psi_0 &= \sum_{n=0}^{\infty}\frac{1}{2^n n!}\left(\frac{1}{\sqrt{m}}\frac{d}{dx}\frac{1}{\sqrt{m}}\right)^{n} e^{-\int_{x_0}^{x} dy\; m(y)}\chi \\ &= \psi_0 + \sum_{n=1}^{\infty}\frac{1}{2^n n!}\left(\frac{1}{\sqrt{m}}\frac{d}{dx}\frac{1}{\sqrt{m}}\right)^{n} e^{-\int_{x_0}^{x} dy\; m(y)}\chi \\ &= \left(1 + \sum_{n=1}^{\infty}\frac{D^n}{2^n n!}f_1\left(m\right)\right)\psi_0 \\ &\equiv \left(1 + \left(e^{\frac{D}{2}} - 1\right)\left[f_1\left(m\right)\right]\right)\psi_0. \end{aligned} \tag{27}$$

This expression determines the FW representation of the JR kink in terms of the original zero mode for arbitrary shape of the mass profile.

For illustration, let us consider the profile [7,12,19]

$$m(x) = m_0\frac{x}{|x|}. \tag{28}$$

It is straightforward to check that in this case,

$$D^n f_1(m) = (-1)^n, \tag{29}$$

such that, from (27), we directly obtain

$$\Psi_0 = \frac{1}{\sqrt{e}}\psi_0, \tag{30}$$

namely, both the spinors differ by an overall constant (Fig. 1).

Another example is a mass profile given by

$$m(x) = m_0\tanh(x). \tag{31}$$

In this case, the 3^{rd}-order FW zero mode wave function is given by

$$\Psi_0 = \frac{1}{\sqrt{e}}e^{\frac{m_0}{4}\mathrm{csch}^2(x)}\psi_0. \tag{32}$$

Both, the relativistic and the FW non-relativistic approximation wave functions are plotted in Fig. 2, where it can be noted that the functions differ not only in magnitude but in shape as well.

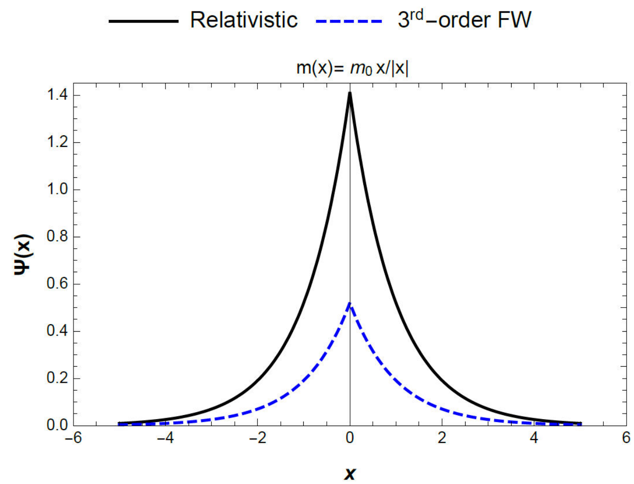

FIGURE 1. Zero mode wave functions for the relativistic case and the 3^{rd}-order FW non-relativistic approximation for the mass profile $m(x) = m_0(x/|x|)$.

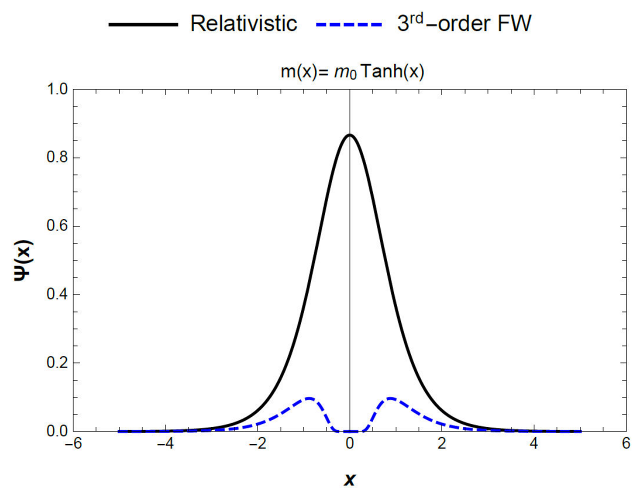

FIGURE 2. Zero mode wave functions for the relativistic case and the 3^{rd}-order FW non-relativistic approximation for the mass profile $m(x) = m_0\tanh(x)$.

3. Discussion and conclusions

In this article we have carried out the FW transformation of the Dirac Hamiltonian (9) with a position dependent mass term. We have focused out attention to the zero mode, which exhibits a kink behavior for different mass profiles. This zero mode transforms according to Eq. (27) for a general $m(x)$. Explicit examples are depicted in Figs. 1 and 2. Although the form of the zero mode depends upon the explicit representation of the Dirac matrices α and β, it is well-known that the Pauli matrices form a basis for 2D Dirac Hamiltonians, and it is possible to use different selections of these basis elements (similarity transformation of basis) in the Hamiltonian, *e.g.*

$$H = \sigma_y p + \sigma_x m(x), \tag{33}$$

so that the systems describes the same physics.

A final word of caution is required. It should be clear that ψ_0 is not a solution of

$$U^\dagger(x)HU(x)\,\psi_0 = 0. \tag{34}$$

This is because the transformation (27) is valid up to third order and thus is not exact. Nevertheless, we observe that up to this approximation, the kink character of the relativistic and non-relativistic spinors holds.

Further non-relativistic FW-like representations of Dirac Hamiltonians of this type are being considered, like the non-minimally coupled electric-Moshinsky oscillator [12]. These ideas are currently under consideration and results shall be presented elsewhere.

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