Electromagnetic fields with symmetry

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Received 10 May 2023; accepted 23 June 2023

We show that if an electromagnetic field is invariant under translations or rotations, three of the six components of the field can be expressed in terms of a (gauge-invariant) scalar potential which is also invariant under these transformations. This scalar potential appears in the constant of motion associated with this symmetry for a charged test particle in this field. We also show that the Cartesian components of the electromagnetic field can be combined to form two $\mathrm{SO}(2,1)$ vectors.

Keywords: Electromagnetic fields; symmetries; constants of motion; Lorentz transformations.

DOI: https://doi.org/10.31349/RevMexFisE.21.020201

1. Introduction

If one is interested in particles interacting with an electromagnetic field possessing some symmetry, the standard approach would consist in imposing the symmetry at the level of the Lagrangian or the Hamiltonian. However, the Lagrangian may not possess all the symmetries of the corresponding equations of motion (see, *e.g.*, Refs. [1,2]; an example in the context of continuous systems is that of the free electromagnetic field: the source-free Maxwell equations are invariant under the so-called duality rotations [3], but the standard Lagrangian for the source-free electromagnetic field is not invariant under these transformations) and this is particularly clear in the case of the usual Lagrangian for a charged particle in an electromagnetic field, which is written in terms of the electromagnetic potentials and not of the electromagnetic fields themselves. For instance, the electromagnetic potentials for a uniform static electric or magnetic field cannot be uniform and static (uniform static potentials only yield fields equal to zero); in the case of a magnetic monopole, whose magnetic field would be spherically symmetric, the vector potential cannot be spherically symmetric (see, *e.g.*, Ref. [3], chap. 6).

One of the aims of this paper is to show that if an electromagnetic field is invariant under translations along a fixed direction or rotations about a fixed axis one can find electromagnetic potentials that are also invariant under those transformations. Furthermore, we show that the invariance of the electromagnetic field under translations or rotations implies the existence of a gauge-invariant scalar potential that determines three of the six components of the electromagnetic field; these three components are the only ones involved in the time derivative of the component of the linear or the angular momentum along the symmetry axis of a charged particle interacting with the field.

In Sec. 2 we begin by considering electromagnetic fields invariant under translations or rotations making use of the standard vector formalism, showing that half of the components of the electromagnetic field can be expressed in terms of a gauge-invariant scalar potential, which is also invariant under the corresponding transformations. In Sec. 3 we consider the equations of motion of a test charge in an electromagnetic field possessing one of these symmetries, making use of the elementary vector formalism (without Lagrangians). In Sec. 4 we show that if the electromagnetic field is invariant under translations along a fixed direction or rotations about a fixed axis, then there exist electromagnetic potentials for this field with the same symmetry properties, and we use them, in Sec. 5, to construct the usual Lagrangian for a test charge. In both treatments of the equations of motion for a charged particle we arrive at the same constants of motion, which are made out of the scalar potentials mentioned above. In Sec. 6 we apply the results of Secs. 3 and 5 to the case of a charged particle in the field of a magnetic monopole, finding the constants of motion associated with the invariance under rotations of the field. In Sec. 7 we show that if the electromagnetic field is invariant under translations along the z-axis, the six Cartesian components of the electromagnetic field are grouped into two sets of three components each which transform as vectors under the Lorentz transformations in $2+1$ dimensions, obtained by restricting the usual Lorentz transformations to those leaving invariant the z-axis.

2. Electromagnetic fields invariant under translations or rotations

In this section we show that if an electromagnetic field is invariant under translations or rotations, half of the components of the electromagnetic field can be expressed in terms of a gauge-invariant scalar potential. This result can be employed afterwards in connection with the behavior of charged test particles in the framework of Newtonian mechanics, relativistic mechanics, or quantum mechanics (see Secs. 3 and 5).

2.1. Electromagnetic fields invariant under translations

We start by considering an electromagnetic field that may depend on the time in an arbitrary manner with the only restriction that the Cartesian components of the electric and magnetic fields be independent of z; this condition, employed in the equation $\nabla \cdot \mathbf{B} = 0$, leads to

$$\frac{\partial B_x}{\partial x} = -\frac{\partial B_y}{\partial y}, \tag{1}$$

while the x- and y-components of Faraday's law, $\nabla \times \mathbf{E} = -(1/c)\, \partial\mathbf{B}/\partial t$, yield

$$\frac{\partial E_z}{\partial y} = -\frac{1}{c}\frac{\partial B_x}{\partial t}, \qquad -\frac{\partial E_z}{\partial x} = -\frac{1}{c}\frac{\partial B_y}{\partial t}. \tag{2}$$

Equations (1) and (2) are locally equivalent to the existence of a function $\Pi(x, y, t)$, defined up to an additive constant, such that

$$B_x = \frac{\partial \Pi}{\partial y}, \qquad B_y = -\frac{\partial \Pi}{\partial x}, \qquad E_z = -\frac{1}{c}\frac{\partial \Pi}{\partial t}. \tag{3}$$

The function Π is a scalar potential whose existence follows from the homogeneous Maxwell equations and the invariance of the electromagnetic field under translations in a spatial direction (taken here as the z-axis). Note that there are no restrictions about the sources of the field. By contrast with the standard potentials, $\mathbf{A}$ and φ, the scalar potential Π has no gauge freedom. (The only freedom allowed by Eqs. (1) and (2) is the addition of a trivial constant to Π.) Furthermore, we shall show in Sec. 7 that Π is invariant under the proper Lorentz transformations preserving the direction of the z-axis.

Even though for some purposes Eqs. (3) is all we require [see Eq. (11)], we shall find some implications of the assumed symmetry on the remaining components of the electromagnetic field. Writing the z-component of Faraday's law $\nabla \times \mathbf{E} = -(1/c)\, \partial\mathbf{B}/\partial t$ in the form

$$\frac{1}{c}\frac{\partial B_z}{\partial t} + \frac{\partial E_y}{\partial x} + \frac{\partial(-E_x)}{\partial y} = 0, \tag{4}$$

which is similar to the form of the divergence of a vector field in Cartesian coordinates, we conclude that there exist locally functions, f, g, h, of (x, y, t) only such that

$$B_z = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}, \quad E_y = \frac{\partial h}{\partial y} - \frac{1}{c}\frac{\partial g}{\partial t},$$
$$E_x = \frac{\partial h}{\partial x} - \frac{1}{c}\frac{\partial f}{\partial t}. \tag{5}$$

By contrast with Π, the functions f, g, h are not uniquely defined.

2.2. Electromagnetic fields invariant under rotations

Now we shall assume that the electromagnetic field is invariant under rotations about the z-axis, which means that the components of the fields with respect to the orthonormal basis induced by the circular cylindrical coordinates, (ρ, ϕ, z), do not depend on ϕ. Taking into account that in these coordinates

$$0 = \nabla \cdot \mathbf{B} = \frac{1}{\rho}\frac{\partial(\rho B_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z},$$

the condition $\partial B_\phi/\partial\phi = 0$ implies that

$$\frac{\partial(\rho B_z)}{\partial z} = -\frac{\partial(\rho B_\rho)}{\partial \rho}, \tag{6}$$

and from the ρ- and z-components of Faraday's law we have

$$-\frac{\partial E_\phi}{\partial z} = -\frac{1}{c}\frac{\partial B_\rho}{\partial t}, \qquad \frac{1}{\rho}\frac{\partial(\rho E_\phi)}{\partial \rho} = -\frac{1}{c}\frac{\partial B_z}{\partial t}. \tag{7}$$

Equations (6) and (7) imply the local existence of a function $\Lambda(\rho, z, t)$, defined up to an additive constant, such that

$$\rho B_z = \frac{\partial \Lambda}{\partial \rho}, \qquad \rho B_\rho = -\frac{\partial \Lambda}{\partial z}, \qquad \rho E_\phi = -\frac{1}{c}\frac{\partial \Lambda}{\partial t}. \tag{8}$$

Thus, three of the components of the electromagnetic field are given in terms of the gauge-independent function Λ whose existence is a consequence of the homogeneous Maxwell equations and the rotational invariance of the fields.

On the other hand, the ϕ-component of Faraday's law can be written in the form

$$\frac{1}{c}\frac{\partial B_\phi}{\partial t} + \frac{\partial E_\rho}{\partial z} + \frac{\partial(-E_z)}{\partial \rho} = 0, \tag{9}$$

which has the form of the divergence of a vector field in Cartesian coordinates. Hence, this equation is locally equivalent to the existence of three functions, f, g, h, of (ρ, z, t) only such that

$$B_\phi = \frac{\partial f}{\partial z} - \frac{\partial g}{\partial \rho}, \qquad E_\rho = \frac{\partial h}{\partial \rho} - \frac{1}{c}\frac{\partial f}{\partial t},$$
$$-E_z = \frac{1}{c}\frac{\partial g}{\partial t} - \frac{\partial h}{\partial z}. \tag{10}$$

By contrast with Λ, the functions f, g, h are not defined in a unique way.

3. Equations of motion of a test particle. Elementary approach

Now we consider the motion of a test charged particle, of mass m and charge q, in the framework of Newtonian mechanics, if the electromagnetic field is invariant under translations along the z-axis. Making use of Newton's second law and the Lorentz force, we have

$$m\ddot{z} = \frac{q}{c}(cE_z + \dot{x}B_y - \dot{y}B_x), \tag{11}$$

which, in view of Eqs. (3), amounts to

$$m\ddot{z} = \frac{q}{c}\left(-\frac{\partial \Pi}{\partial t} - \dot{x}\frac{\partial \Pi}{\partial x} - \dot{y}\frac{\partial \Pi}{\partial y}\right) = -\frac{q}{c}\frac{\mathrm{d}\Pi}{\mathrm{d}t},$$

and to the existence of the constant of motion

$$m\dot{z} + \frac{q}{c}\Pi = \text{const.} \tag{12}$$

Similarly, noting that the ϕ-component of the acceleration is given by

$$\begin{aligned} \hat{\phi}\cdot\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(\hat{\phi}\cdot\mathbf{v}) - \mathbf{v}\cdot\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}t} = \frac{\mathrm{d}(\rho\dot{\phi})}{\mathrm{d}t} + \mathbf{v}\cdot\dot{\phi}\hat{\rho} \\ &= \frac{\mathrm{d}(\rho\dot{\phi})}{\mathrm{d}t} + \dot{\rho}\dot{\phi} = \frac{1}{\rho}\frac{\mathrm{d}(\rho^2\dot{\phi})}{\mathrm{d}t}, \end{aligned} \tag{13}$$

we have

$$\frac{\mathrm{d}(m\rho^2\dot{\phi})}{\mathrm{d}t} = \frac{q}{c}\rho\,\hat{\phi}\cdot(c\mathbf{E}+\mathbf{v}\times\mathbf{B}) = \frac{q}{c}\rho\,(cE_\phi + \dot{z}B_\rho - \dot{\rho}B_z)$$

and if the electromagnetic field is invariant under rotations about the z-axis, making use of Eqs. (8),

$$\frac{\mathrm{d}(m\rho^2\dot{\phi})}{\mathrm{d}t} = \frac{q}{c}\left(-\frac{\partial\Lambda}{\partial t} - \dot{z}\frac{\partial\Lambda}{\partial z} - \dot{\rho}\frac{\partial\Lambda}{\partial\rho}\right) = -\frac{q}{c}\frac{\mathrm{d}\Lambda}{\mathrm{d}t}.$$

Hence,

$$m\rho^2\dot{\phi} + \frac{q}{c}\Lambda = \text{const.} \tag{14}$$

4. Electromagnetic potentials for the electromagnetic fields invariant under translations or rotations

In this section we shall show that in the cases considered above we can give expressions for the usual potentials of the electromagnetic field. The main result of this section is that these potentials can be chosen in such a way that they share the same symmetry as the corresponding electromagnetic field.

Recalling the standard expressions for the electromagnetic fields in terms of the potentials $\mathbf{A}$ and φ,

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla\times\mathbf{A}, \tag{15}$$

and comparing with Eqs. (3) and (5) we see that if the electromagnetic field is invariant under translations along the z-axis the potentials $\mathbf{A}$ and φ can be chosen according to

$$A_x = f, \qquad A_y = g, \qquad A_z = \Pi, \qquad \varphi = -h. \tag{16}$$

Since the functions Π, f, g and h do not depend on z, $\mathbf{A}$ and φ are invariant under translations along the z-axis.

In the case where the electromagnetic fields are invariant under the rotations about the z-axis, taking into account that $\mathbf{B} = \nabla\times\mathbf{A}$ amounts to

$$\begin{aligned} B_\rho &= \frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}, \qquad B_\phi = \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}, \\ B_z &= \frac{1}{\rho}\left[\frac{\partial(\rho A_\phi)}{\partial\rho} - \frac{\partial A_\rho}{\partial\phi}\right], \end{aligned} \tag{17}$$

comparison with Eqs. (8) and (10) shows that we can choose the potentials in the form

$$A_\rho = f, \qquad \rho A_\phi = \Lambda, \qquad A_z = g, \qquad \varphi = -h. \tag{18}$$

This shows that if the electromagnetic fields are invariant under rotations about an axis, the usual electromagnetic potentials can be chosen in such a way that they are also invariant under these rotations.

5. Equations of motion of a test particle. Lagrangian approach

In the framework of Newtonian mechanics, the standard Lagrangian for a charged particle, with mass m and electric charge q, in an electromagnetic field defined by the potentials $\mathbf{A}$ and φ is given by

$$L = \frac{1}{2}m\mathbf{v}^2 + \frac{q}{c}\mathbf{A}\cdot\mathbf{v} - q\varphi. \tag{19}$$

Hence, if the electromagnetic field is invariant under translations along the z-axis, the potentials can be chosen in the form (16) and the Lagrangian (19) expressed in Cartesian coordinates becomes

$$\begin{aligned} L &= \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{c}(A_x\dot{x} + A_y\dot{y} + A_z\dot{z}) - q\varphi, \\ &= \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{c}(f\dot{x} + g\dot{y} + \Pi\dot{z}) + qh. \end{aligned} \tag{20}$$

Since the functions f, g, h and Π are functions of x, y, t only, the coordinate z is ignorable and its conjugate momentum

$$\frac{\partial L}{\partial\dot{z}} = m\dot{z} + \frac{q}{c}\Pi. \tag{21}$$

is conserved. This constant of motion coincides with that given in Eq. (12).

Similarly, if the electromagnetic field is invariant under rotations about the z-axis, the potentials can be chosen in the form (18) and the Lagrangian (19) expressed in terms of the cylindrical coordinates is

$$\begin{aligned} L &= \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) + \frac{q}{c}(A_\rho\dot{\rho} + \rho A_\phi\dot{\phi} + A_z\dot{z}) - q\varphi, \\ &= \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) + \frac{q}{c}(f\dot{\rho} + \Lambda\dot{\phi} + g\dot{z}) + qh. \end{aligned} \tag{22}$$

Now ϕ is an ignorable coordinate and its conjugate momentum

$$\frac{\partial L}{\partial\dot{\phi}} = m\rho^2\dot{\phi} + \frac{q}{c}\Lambda \tag{23}$$

is a constant of motion, which coincides with that given by Eq. (14).

The corresponding results in the framework of relativistic mechanics are very similar to those given above. Making use of the Lagrangian [3]

$$L = -mc^2\sqrt{1 - \frac{\mathbf{v}^2}{c^2}} + \frac{q}{c}\mathbf{A}\cdot\mathbf{v} - q\varphi$$

one readily finds that the constants of motion (21) and (23) maintain their form with m replaced by the "relativistic mass" $m/\sqrt{1-\frac{\mathbf{v}^2}{c^2}}$.

6. An example. Charged particle in the field of a magnetic monopole

It must be emphasized that the constants of motion (12) and (14) are, necessarily, gauge-independent and do not depend on the coordinates being employed. In this sense, it is convenient to notice, for instance, that Eqs. (8) are equivalent to

$$\nabla\Lambda = (\hat{z}\times\mathbf{r})\times\mathbf{B}, \qquad \frac{1}{c}\frac{\partial\Lambda}{\partial t} = -(\hat{z}\times\mathbf{r})\cdot\mathbf{E}, \qquad (24)$$

where $\mathbf{r}$ is the position vector of an arbitrary point.

The electromagnetic field produced by a magnetic monopole, with magnetic charge $q_{\rm m}$, placed at the origin, would be $\mathbf{E}=\mathbf{0}$ and $\mathbf{B}=q_{\rm m}\mathbf{r}/r^3$, which is invariant under rotations about any axis passing through the origin. According to Eqs. (24), the function Λ associated with the invariance under the rotations about the z-axis can be taken as $-q_{\rm m}z/r$ and therefore the constant of motion (14) is $m(x\dot{y}-y\dot{x})-qq_{\rm m}z/(cr)$. There are two similar expressions for the constants of motion associated with the invariance of the field under rotations about the x- and y-axes. Together, these constants form the conserved vector

$$\mathbf{r}\times m\dot{\mathbf{r}} - \frac{qq_{\rm m}}{c}\frac{\mathbf{r}}{r}.$$

Whereas the invariance under the rotations about, *e.g.*, the z-axis can be explicitly exhibited in the electromagnetic potentials, the simultaneous invariance under rotations about different axes *cannot* be exhibited in the potentials or the Lagrangian.

In the case of an infinitely long solenoid, as that considered in the study of the Aharonov–Bohm effect, the magnetic field in the exterior of the solenoid should be equal to zero, but there must be a nonzero vector potential. According to Eqs. (8), (10) and (18), the functions f,g,h can be taken equal to zero and, taking into account the Stokes theorem, the function Λ must be a constant equal to $\Phi/2\pi$, where Φ is the magnetic flux through a cross-section of the solenoid. In the framework of classical mechanics the presence of the constant Λ in the constant of motion (14) does not make an essential difference. However, as we know, in the quantum mechanical version of the problem, a nonzero flux Φ produces observable effects.

7. Relation with the Lorentz transformations in 2+1 dimensions

As we have seen, when an electromagnetic field is invariant under translations along the z-axis, the Cartesian components of the field are functions of (ct,x,y) only, and the potentials required to express them can be chosen as functions of (ct,x,y) only [see Eqs. (3) and (5) or (16)], so that one is led to consider the $2+1$ space-time with coordinates $x^0=ct$, $x^1=x$ and $x^2=y$. Furthermore, the components of the electromagnetic field are naturally grouped into two sets of three components each [see Eqs. (3) and (5)]. In fact, Eqs. (3) can be expressed in the simple form

$$\phi_\alpha = -\partial_\alpha\Pi, \qquad \text{with}$$
$$(\phi^0,\phi^1,\phi^2)\equiv(-E_z,B_y,-B_x). \qquad (25)$$

The lower case Greek indices, $\alpha,\beta,\ldots$, take the values $0,1$ and 2 and these indices are raised or lowered with the aid of the 3×3 matrices $(\eta_{\alpha\beta})=\mathrm{diag}\,(-1,1,1)=(\eta^{\alpha\beta})$ (hence, *e.g.*, $\phi_0=-\phi^0=E_z$ but $\phi_1=\phi^1=B_y$).

Making use of the well-known formulas for the transformation of the components of the electromagnetic field in the case of a boost in the x-direction (see, *e.g.*, sec. 11.10 of Ref. [3]), we find

$$\begin{pmatrix} -E'_z \\ B'_y \\ -B'_x \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -E_z \\ B_y \\ -B_x \end{pmatrix}, \qquad (26)$$

where $\beta=v/c$, v is the velocity of the primed inertial frame with respect to the unprimed one, and $\gamma=(1-\beta^2)^{-1/2}$. This means that ϕ^α transforms as a $2+1$-vector under the proper Lorentz transformations that leave invariant the z-axis. Clearly, under rotations in the xy-plane the components of ϕ^α transform appropriately.

Equation (4) can be written in the form $\partial_\alpha\psi^\alpha=0$, with $(\psi^0,\psi^1,\psi^2)\equiv(B_z,E_y,-E_x)$, and we can verify that ψ^α also transforms as a $2+1$-vector under the proper Lorentz transformations that leave invariant the z-axis, in fact,

$$\begin{pmatrix} B'_z \\ E'_y \\ -E'_x \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} B_z \\ E_y \\ -E_x \end{pmatrix}. \qquad (27)$$

The vector ψ^α can also be expressed in a covariant manner. Equations (5) are equivalent to $\psi^\alpha=\varepsilon^{\alpha\beta\gamma}\partial_\beta s_\gamma$, with $\varepsilon^{012}=1$ and $(s_0,s_1,s_2)=(h,f,g)$.

By combining the two $2+1$-vectors ϕ^α and ψ^α we can form three invariants: $\phi^\alpha\psi_\alpha=\mathbf{E}\cdot\mathbf{B}$ (which is one of the two well-known Lorentz invariants of the electromagnetic field), $\phi^\alpha\phi_\alpha=-E_z{}^2+B_x{}^2+B_y{}^2$, and $\psi^\alpha\psi_\alpha=-B_z{}^2+E_x{}^2+E_y{}^2$. The last two quantities are separately invariant under the proper Lorentz transformations that leave invariant the z-axis and their difference, $\psi^\alpha\psi_\alpha-\phi^\alpha\phi_\alpha$, corresponds to the second Lorentz invariant of the electromagnetic field.

The results of this section may seem curious but are part of a more general behavior. For instance, in the case of a time-independent electromagnetic field, one half of the components of the field (those corresponding to the electric field) can be expressed in terms of a time-independent scalar potential (defined up to an additive constant), which is invariant under the Lorentz transformations that leave invariant the time axis (that is, the ordinary rotations). The other half of the components of the field (those corresponding to the magnetic field) can be expressed in terms of a time-independent vector that is not uniquely defined. With the electric and magnetic field vectors we can form three invariants under rotations, $\mathbf{E}\cdot\mathbf{B}$, $\mathbf{E}\cdot\mathbf{E}$ and $\mathbf{B}\cdot\mathbf{B}$, and the difference between the last two is invariant under all the Lorentz transformations.

8. Final remarks

As we have shown, the possibility of having electromagnetic potentials or Lagrangians invariant under some transformations is not a trivial matter. The concept of symmetry is very often used loosely, without the required precision.

The terms $q\Pi/c$ or $q\Lambda/c$, appearing in Eq. (12) and (14), respectively, need not coincide with the z-component of the linear or the angular momentum of the electromagnetic field as usually defined [4]. As argued in Ref. [4], the reason is that in the derivation of the expression for the density of linear momentum or angular momentum of the electromagnetic field one takes into account the interaction of the field with its sources and the test charge considered here may not be the only source of electric field present.

Acknowledgement

The author wishes to thank the referee for helpful comments.

1. G. F. Torres del Castillo and A. Moreno Ruiz, Symmetries of the equations of motion that are not shared by the Lagrangian, arXiv:1705.08446 (2017).
2. G. F. Torres del Castillo, *An Introduction to Hamiltonian Mechanics*, (Springer, Cham, 2018). Sec. 2.5. https://doi.org/10.1007/978-3-319-95225-3.
3. J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1998).
4. G. F. Torres del Castillo, The linear and the angular momentum stored in a distribution of charges in a magnetic field, *Rev. Mex. Fís.* **69** (2023) 050701. https://doi.org/10.31349/RevMexFis.69.050701.