Perturbations of planetary orbit parameters due to decreasing in stellar mass and the expansion of the universe from a classical approach

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In this analysis, we found the disturbances caused by the decrease in stellar mass and the expansion of the universe, to three fundamental parameters that represented the stability of a planetary orbit: the period, the semi-major axis and the eccentricity. First, by assuming much greater the mass of the star than the planetary, the star-planet interaction is reduced to a single-body problem with origin of reference system lying in the greater mass; and, through the mathematical formalism of the central forces, the variations of the three orbital parameters to be considered were obtained. As a result, the variations corresponding to the period and semi-major axis have been characterized in their mathematical structure by the terms that describe each phenomenon; namely, $\xi \sim 2.16 \times 10^{-21}$ for the case of Sun, and $\ddot{\alpha}/\alpha \sim 3 \times 10^{-36}$ concerning the decrease in stellar mass and the expansion of the universe, respectively. In the case of eccentricity, it is shown that this parameter is an invariant quantity under the disturbances produced by these two cosmological phenomena.

Keywords: Universe expansion; stellar mass; orbital period; semi-major axis; eccentricity; central forces.

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1. Introduction

The dynamics of Solar System is a fundamental topic of physics, which is teaching from a bachelor to graduate level university. From the theory it begins from two body problem interacting via a central force, extended to the three body system, crossing the study of perturbations of the inner orbit planets associated to its orbital elements variations as a time function. Last topic makes part of an area of knowledge much bigger, belonging to the study of stability of the Solar System, where orbital motion of one inner planet has disturbances not only due to effect of interaction of remaining planets but also to the effects associated with the inner dynamic of the star, until cosmological effects. Perturbations of the orbits in the inner planets and the stability in the Solar System can be introduced in the classroom as it was approached from the historical and theoretical point of view, approaches that contribute to the construction of learning environments around mechanics. It can be presented from Johannes Kepler, who with the help of the observations made by Tycho Brahe, published his three laws and pointed out the discrepancy between the observations made by Regiomontanus about the movements of the planets Jupiter and Saturn later. In 1687, Isaac Newton in his text on optics expressed his doubts about the stability of the Solar System when he saw compromised disturbances due to other planets and comets [1]; furthermore, he raised the idea that these disturbances could accumulate and destroy it. For this problem, he raised the possibility of divine intervention that could correct the planetary orbits. In 1776, Halley's tables were reproduced, where the irregularities in the behavior of the orbits of Jupiter and Saturn were evidenced. Due to these irregularities, the French Academy of Sciences offered a prize for the one who could solve the aforementioned problem. This was of vital importance, since the stability of the Solar System was directly involved even more when the law of gravitation was already formulated along with Kepler's laws. Leonhard Euler set about working on the problem and was awarded prizes in 1748 and 1752 for laying the foundations of perturbation methods, and for showing that secular variations in the mean motion of Jupiter and Saturn were induced in Newton's laws. However, based on observations, it was known that Euler's results were wrong [2]. In 1773 Pierre Simon Laplace described the secular invariance of the semi-major axis of the planets; however, he was concerned that his results had been shown by Laplace to be secular invariant of the semi-major axis by considering only the first few terms in the expansion of average perturbations; but his results were in discrepancy with the observations. Nevertheless a comparison between Halley's data and his results allowed him to conclude that the variations in the movements of Jupiter and Saturn were due to their mutual action, building a better model in which he considered the Jupiter-Saturn system, which coincided with observations without recourse to an empirical secular term. After the work of Lagrange and Laplace, the stability of the Solar System seems to be explained. The semi-principal axes of the orbits had no long-term variations, and their eccentricities and inclinations showed only small variations that do not allow the orbits to cross and the planets to collide. Urbain Jean Joseph Le Verrier in 1840 was based on the work done by Lagrange and Laplace, and considered the effects of the higher order terms in the perturbation series, with the idea that the third order terms could be larger than the third order terms, terms of second order, which in his opinion compromised the convergence of the solutions.

Henri Poincaré in 1892-1899 demonstrated that it is impossible to integrate the equations of motion of three bodies that interact with each other through the action of gravity, and the impossibility of finding an analytical solution in an infinite time interval, which makes the system not be integrable and that the behavior of the system cannot be predicted, creating a type of instability [3]. For Le Verrier the problem lies in considering the higher order terms in the expansion due to perturbations, while for Poincaré the problem lies in the convergence of the series. Kolmogorov in 1954 analyzed the problem of the convergence of the series of perturbations of celestial mechanics and showed that for perturbed Hamiltonian systems non-degenerate, close to the non-regular solutions described by Poincaré, there are still regular quasi-periodic trajectories that span a torus in phase space [4]. Arnold in 1963 showed that for a sufficiently small disturbance, the set of invariant tori foliated by quasi-periodic trajectories is of strictly positive measure; as it tends to unity when the disturbance tends to zero [5]. Moser in 1962 established the same type of results for less strong conditions that do not require the analyticity of the Hamiltonian. Since the quasi-periodic tori are isolated, an infinitely small variation of the initial conditions will convert a neatly stable quasi-periodic solution into a solution chaotic and unstable. Said in other words, weakly perturbed Hamiltonian systems tend to maintain their stability. As higher is the disturbance parameter, is lower the probability that the system being stable.

At a predictive level, Jackes Laskar using computational methods in the 1980s, 1990s and in 2009, he showed that resonances destroy predictability because they amplify the gravitational effects by periodically joining the bodies [2, 6]. Laskar recreated the orbital movements with small variations of the initial conditions that give rise to 2500 dynamics possible for the next billions of years; in the cases, Venus, Mercury, The Earth or Mars collided with each other or with the Sun. Computational methods can provide very good approximations of the solutions of the planets over thousands of years, but they would not be able to give answers to the questions about the stability of the Solar System.

Another topic associated to the Solar System stability concerns not only inner planets but also astrophysics phenomena mainly to the dynamics of the Star, *i.e.* the Sun and cosmological effects, making this topic more extended. The classical Kepler problem could be introduced in a didactic way by "local" effects associated to the disturbance of the orbits of the inner planets by interaction between them, but also the decreasing in the mass of the Sun, and "global" effects related to the expansion of the universe. For example, in this sense the generalizes of the Newtonian condition for hydrostatic equilibrium is modeled with Tolman-Oppenheimer-Volkoff (TOV) equation, which presents "local" and "global" solutions as analytical strategy [7–9] to describe a star as perfect fluid.

Along with the interaction between the planets when they move around the star by gravitational action, the expansion of the universe as a "global" phenomenon and the decreasing in stellar mass as a "local", are equally important factors when the stability of the planetary orbits is the subject to consider [2, 10, 11]. In general problems that concern cosmological phenomena are mostly explained in the field of general relativity, but to understand these interesting topics from another formalism contemplated in any course of physics graduation, we present this analysis using a classical approach; more properly, the central forces. This formalism offers the advantage that from a disturbance in the conditions of stability and circularity, a number of physical consequences follow that do not necessarily involve a complex calculation; so, it turns out to be an excellent alternative method to obtain concrete and approximate results.

In Sec. 2, reduction to one-body problem is made with their respective invariant quantities that are deduced from Noether's theorem; furthermore, the conditions for circularity are defined, resulting in the Kepler problem and Hooke's law like two special cases. In Sec. 3, the mathematical representation of orbital parameters, that have the main study in this work, is shown. Perturbations to these orbital parameters caused by the decrease in stellar mass and the expansion of the universe have been performed in Secs. 4, 5 respectively. The conclusions and final remarks are done in Sec. 6.

2. The two body problem

The two-body problem is represented by two masses (M and m) interacting through some potential that depends on the mutual distance $V(\mathbf{r})$. This system can be reduced to an one-body problem if the center of mass, whose associated vector position $\mathbf{r}$ (Fig. 1), represents an inertial system reference. Furthermore, if $M \gg m$, the center of mass is approximately located within the largest mass, allowing to chose a reference system whose origin lies in this mass. Dynamics of two body can be described from the Lagrange formalism: It contains the kinetic energy and $V(\mathbf{r})$ the potential energy. Then, the Lagrangian relative to the body m is

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 - V(\mathbf{r}). \tag{1}$$

To determine the conserved quantities, Noether's theorem allows us to obtain a more complete meaning. For a rotation of

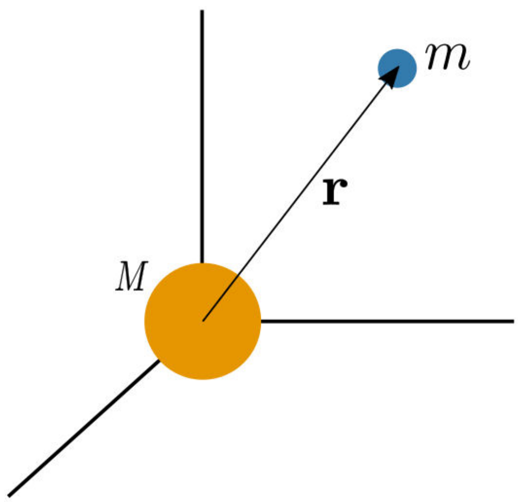

FIGURE 1. Two body problem.

type $\mathbf{r}' = \zeta \mathbf{r}$ (with ζ constant), it is obtained that $L'(\dot{\mathbf{r}}', \mathbf{r}') = L(\dot{\mathbf{r}}, \mathbf{r})$. It allow us to deduce the conservation of angular momentum $\mathbf{L}$,[i] bringing with it the restriction of the movement of mass m to a perpendicular plane. Given this symmetry, and since the motion of the planet will be restricted to an elliptical orbit according to Kepler's first law, it allow us to write L in polar coordinates r, θ:

$$L = \frac{m}{2}[\dot{r}^2 + r^2\dot{\theta}^2] - V(\mathbf{r}). \tag{2}$$

Again, the first two terms make up the kinetic energy (in polar coordinates), and the second term is the potential energy. The Euler-Lagrange‘s equations for r and θ are [12]

$$m[\ddot{r} - r\dot{\theta}^2] = F(r), \tag{3}$$

$$m[r\ddot{\theta} + 2\dot{r}\dot{\theta}] = 0, \tag{4}$$

where $F(r) = -\partial V/\partial r$ is the central force, and the last equation is related with the conserved angular momentum $l = mr^2\dot{\theta}$. On the other hand, by Noether's theorem, the explicit independence of time in the Lagrangian Eq. (1) allow us to get the total energy of system E as a conserved quantity. Taking these two considerations into account, the energy equation in polar coordinates is

$$E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V(r). \tag{5}$$

It is possible to relate energy equation with the angular momentum l defined the effective potential $V'(r) = l^2/2mr^2 + V(r)$, and the total energy of the system is

$$E = \frac{1}{2}m\dot{r}^2 + V'(r). \tag{6}$$

In the case of $V(r) = -kr^{-1}$, and for an effective force $F' = 0$ (corresponding to V'), it is guaranteed a circular orbit for a single distance r_0. That explicitly means

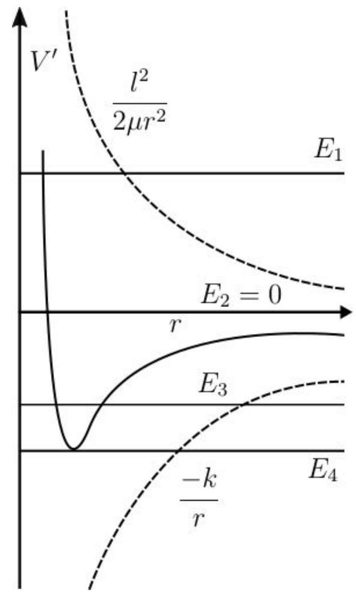

FIGURE 2. Effective potential V' as a function of r.

$$E = -\frac{k}{r_0} + \frac{l^2}{2mr_0^2}, \tag{7}$$

$$0 = -\frac{k}{r_0^2} + \frac{l^2}{mr_0^3}. \tag{8}$$

This situation is represented in the Fig. 2, where the minimum of the effective potential curve coincides with the energy E_4, and the movement is only possible for a radius r_0; this leads to $\dot{r} = 0$, and therefore, the orbit is circular.

Bertrand's theorem generalizes this result, establishing that an energy slightly deviated from one that produces a circular orbit, would still produce a closed orbit. This disturbance is translated into a harmonic movement around $u = 1/r$ into Eq. (3), that is

$$u''(\theta) + u(\theta) = -\frac{m}{l^2 u(\theta)}F(1/r), \tag{9}$$

$$u''(\theta) + u(\theta) = J(u), \tag{10}$$

where for small deviation in the energy around r_0, $J(u)$ can be development in Taylor series of the force with respect to the radius r_0, $J(u) = u_0 + (u - u_0)J'(u_0)$, and its solution is described by the equation [12]:

$$u = u_0 + A\cos\beta\theta, \tag{11}$$

where A is the amplitude which depends on the energy deviation from the value corresponding to the circular orbit, and

$$\beta^2 = 3 + \frac{r_0}{F(r_0)}\frac{dF(r_0)}{dr}. \tag{12}$$

If the deviations into energy are not according to a circular motion, Taylor expansion must have higher terms into Eq. (9) up to third order, that is

$$u''(\theta) + u(\theta) = u_0 + xJ'(u_0) + \frac{x^2}{2}J''(u_0) + \frac{x^3}{6}J'''(u_0), \qquad (13)$$

with $x = (u - u_0)$. Bertrand proved that adding harmonic solutions to the Eq. (11), the only forces that give rise to closed orbits are the law of the inverse squared of the distance $\mathbf{F} = -kr^{-2}\hat{\mathbf{r}}$ (Kepler problem) and Hooke's law $\mathbf{F} = -kr\hat{\mathbf{r}}$.

3. The Kepler problem

From Eq. (6) and using the definition of the angular momentum $l = mr^2\dot{\theta}$, the differential equations

$$\frac{d\theta}{dr} = \frac{l}{mr^2\sqrt{\frac{2(E-V')}{m}}}, \qquad (14)$$

$$\frac{dt}{dr} = \frac{1}{\sqrt{\frac{2}{m}(E-V')}}, \qquad (15)$$

are obtained. Using Euler's substitutions, the solution for the first equation is

$$\frac{1}{r} = \frac{mk}{l^2}\left[1 + \sqrt{1 + \frac{2El^2}{mk^2}}\cos\theta\right]. \qquad (16)$$

Connected with the previous eq., there is the Laplace-Runge-Lenzt vector [12]:

$$\mathbf{A} = \mathbf{p}\times\mathbf{L} - mk\frac{\hat{\mathbf{r}}}{r}, \qquad (17)$$

whose magnitude $A^2 = m^2k^2 + 2mEl^2$ is another constant in the Kepler's problem, and also can be related with the radial orbit Eq. (16)

$$\frac{1}{r} = \frac{mk}{l^2}\left[1 + \frac{A}{mk}\cos\theta\right]. \qquad (18)$$

First orbital element emerges from previous equation: the eccentricity, which by comparing with the equation of a conic in polar coordinates is

$$e = \sqrt{1 + \frac{2El^2}{mk^2}}, \qquad (19)$$

$$= \frac{A}{mk}, \qquad (20)$$

where for different energy values E (and hence e) orbits are classified as parabola $E = 0$, hyperbolic $E > 0$ and ellipse $E < 0$ [12]. For elliptical orbit $0 < e < 1$ and $e = 0$ for a circle. Both cases as a closed orbit generates the semi-major axis a, the second orbital element, which is a function of the total energy E

$$a = -\frac{k}{2E}, \qquad (21)$$

$$a = \frac{l^2}{mk(1-e^2)}, \qquad (22)$$

and Eq. (16) can be written as

$$r = \frac{a(1-e^2)}{1 + e\cos\theta}. \qquad (23)$$

The period arises from the second differential equation, Eq. (15), which the integration by parts leads to

$$T = \pi k\sqrt{\frac{m}{-2E^3}}. \qquad (24)$$

It is easy to verify the Kepler's third law using Eq. (21) and (24)

$$\frac{T^2}{a^3} = \frac{4\pi^2 m}{k}, \qquad (25)$$

which establishes semi-major axis is connected with the orbit period.

4. Decreasing in stellar mass

The properties that describe each star vary with the object in question. Some appropriate parameters are the radius, mass and the superficial temperature. However, the time in which the star's mass is converted into energy and then irradiated, is the most fundamental parameter since it is common to all stars. Thus, if it is supposed that the mass is converted at a constant rate, it will be convenient to obtain a simpler expression. The ratio of stellar mass with respect to time is calculated from certain physical principles. Firstly, the Stefan-Boltzmann law states that for a black body the luminosity is

$$L_\odot = \sigma A T^4, \qquad (26)$$

where $\sigma = 5.67 \times 10^{-8}$ J/m^{2}sK^{-4}, A is the emission area and T the temperature. In the stars the fusion nuclear reactions create helium, carbon, magnesium, oxygen, sulfur, neon, nickel, cobalt, and 4 different isotopes of iron. The teacher should assist or give hints as necessary. The students should end up with the following fusion relationships:

$$(4\text{H}) \rightarrow \text{He} + 2e^+ + 2\nu + \text{energy}, \qquad (27)$$

$$(4\text{He}) \rightarrow 12\text{C} + \text{energy}. \qquad (28)$$

If the luminosity is related with energy E by $L_\odot \equiv E/t$, it is obtained that the mass is converted into energy by the equation $L_\odot = m_\odot c^2/t$ (using Einstein's relation $E = mc^2$). For a constant rate, ones has:

$$\dot{m}_\odot = \frac{4\pi r_\odot^2\sigma T^4}{c^2} = (7.91 \times 10^{-24})r_\odot^2 T^4, \qquad (29)$$

where $r_\odot$ and T are the radius and temperature of the star, respectively. Thus, the stellar mass decreases with respect to an initial mass M_0 obeying the equation

$$M_\odot(t) = M_0 - \dot{m}_\odot t. \qquad (30)$$

Making a suitable changing in the mass decreases Eq. (29) as $\dot{m}_\odot = \xi M_0$ with $\xi = 7.91 \times 10^{-24} r_\odot^2 T^4 M_0^{-1}$, the stellar mass Eq. (30) is given by

$$M_\odot(t) = M_0(1 - \xi t), \tag{31}$$

$$= M_\odot e^{-\xi t}, \tag{32}$$

where the last one equation is obtained by using a first order approximation $e^{-x} \approx 1 - x$ for $\dot{m}_\odot \ll M_0$. This also implies a variation in k of type $k(t) = k_0 e^{-\xi t}$. To study the effects of a variable mass on the semi-major axis (and under a central force), Jeans relation [13]:

$$\frac{d}{dt}\left(\frac{M_\odot}{2a}\right) = \frac{1}{a}\frac{dM_\odot}{dt}, \tag{33}$$

leads to the equality

$$\frac{1}{a}\frac{da}{dt} - \xi = 0, \tag{34}$$

whose solution is:

$$a(t) = a_0 e^{\xi t}. \tag{35}$$

The integration constant a_0 is interpreted as the initial semi-major axis. With the variation of a and $M_\odot$, it is effectively verified that the product $M_\odot a$ is constant. For energy, Eq. (21) allow us to obtain

$$E(t) = E_0 e^{-2\xi t}, \tag{36}$$

where $E_0 = -k_0/2a_0$. In the case of eccentricity, Eq. (20) establishes that this quantity remains constant over time; *i.e.*, the shape of the planetary orbit is unchanged. Using Eq. (20) and expressing k with Eq. (21),

$$e = \sqrt{1 + \frac{2(E_0 e^{-2t})l^2}{m(k_0^2 e^{-2t})}}, \tag{37}$$

$$e = \sqrt{1 + \frac{2E_0 l^2}{m k_0^2}}. \tag{38}$$

These results are also used to find the variation of the orbital period by Eq. (24)

$$T(t) = T_0 e^{2\xi t}, \tag{39}$$

with

$$T_0 = \pi k_0 \sqrt{\frac{m}{-2E_0^3}}. \tag{40}$$

With this, the fulfillment of Kepler's third law is also confirmed, where

$$\frac{T^2}{a^3} = \frac{4\pi^2 m}{k} \rightarrow \frac{T_0^2}{a_0^3} = \frac{4\pi^2 m}{k_0}. \tag{41}$$

The equation obtained in Ref. [13] is based on the fact that while the stellar mass decreases, the energy of the object in orbit increases along with its orbital parameters, and the product $M_\odot a$ remains constant. For this, the conservation of angular momentum l must be assumed at all times.

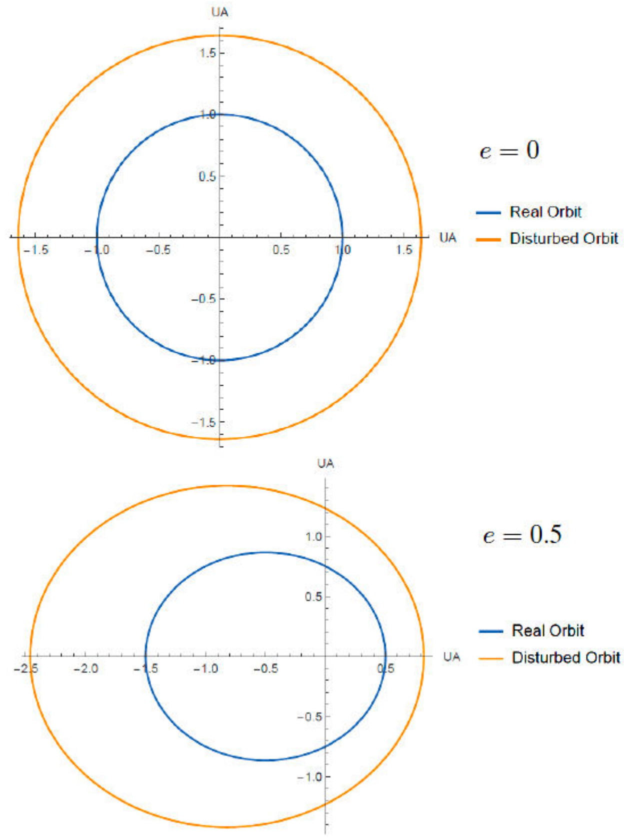

FIGURE 3. Orbits for planets at $a_0 = 1.0$ UA with eccentricities values of $e = 0$ and $e = 0.5$. In this case, the star has 1 solar mass, 500 solar radio, and 5 solar temperature. The time spent for disturbed orbits was $t = 50.000$ years.

As eccentricity does not change, we can rewrite the orbit equation including Eq. (35) as

$$r = \frac{a_0 e^{\xi t}(1 - e^2)}{1 + e\cos\theta}. \tag{42}$$

In Fig. 3, orbits with eccentricities values of $e = 0$ and $e = 0.5$ are obtained, reproducing these results for planets with $a_0 = 1$ UA value. In this case, the star has 1 solar mass, 500 solar radio, and 5 solar temperature. The time spent for disturbed orbits was $t = 50.000$y.

5. Expansion of the universe

The expansion of the universe is defined from Hubble's law [14]:

$$\dot{r} = H(t)r, \tag{43}$$

being H the Hubble parameter and r coincide with the distance measured from the center of mass (greater mass) in a problem of two bodies. Regarding H, it is useful to use the scale factor

$$\alpha(t) = \frac{r(t)}{r_0}, \tag{44}$$

being r_0 the distance measured at an initial time t_0. This expression allow us to re-express the Hubble parameter as

$$H(t) = \frac{\dot{\alpha}(t)}{\alpha(t)}. \tag{45}$$

On the other hand, the discovery of the accelerated expansion of the universe [15], suggests an acceleration in Hubble's law. This result can be expressed by combining Eqs. (43) and (44) as:

$$\ddot{r} = \dot{H}r + H\dot{r}, \tag{46}$$

$$\ddot{r} = \frac{\ddot{\alpha}}{\alpha}r. \tag{47}$$

Current values [10] indicate that $\ddot{\alpha}/\alpha = 3 \times 10^{-36}$ s^{-2}. For the two body problem given by Se. 2, a "force" expression can be associated to expansion effect using Eq. (46), that is

$$F_e = m\ddot{r} = m\frac{\ddot{\alpha}}{\alpha}r, \tag{48}$$

that can be included into radial Eq. (3)

$$m\ddot{r} = -\frac{k}{r^2} + \frac{l^2}{mr^3} + m\frac{\ddot{\alpha}}{\alpha}r. \tag{49}$$

It can be established a circular orbit condition for the force and energy Eqs. (8), given as:

$$E = E_0 - \frac{\ddot{\alpha}}{\alpha}\frac{m}{2}r_0^2, \tag{50}$$

$$0 = F_0' + \frac{\ddot{\alpha}}{\alpha}mr_0, \tag{51}$$

where

$$E_0 = \frac{-k}{r_0} + \frac{l_0^2}{2mr_0^2}, \quad \text{and} \quad F_0' = \frac{-k}{r_o^2} + \frac{l_0^2}{mr_0^3}. \tag{52}$$

Like energy, Eq. (50) can be understood as the square of the disturbed angular momentum

$$l^2 = l_0^2 - \frac{\ddot{\alpha}}{\alpha}m^2r_0^4, \tag{53}$$

where $l_0^2 = mkr_0$. To find the variation in the period, it is necessary to remember that $T = 2\pi/\dot{\theta}$, and l is related to the angular frequency by $l = mr_0^2\dot{\theta}$. Therefore, $\dot{\theta}$ is expressed as a function $\ddot{\alpha}/\alpha$ by the equation

$$\dot{\theta} = \sqrt{\frac{k}{mr_0^3} - \frac{\ddot{\alpha}}{\alpha}} = \sqrt{\frac{k}{mr_0^3}}\sqrt{1 - \frac{\ddot{\alpha}}{\alpha}\frac{mr_0^3}{k}}. \tag{54}$$

Applying the approximation $(1 + \epsilon x)^n \approx 1 + n\epsilon x$ for small values of ϵ, a new period expression is obtained, given by

$$T = T_0\left(1 + \frac{\ddot{\alpha}}{\alpha}\frac{T_0^2}{8\pi^2}\right), \tag{55}$$

where $T_0 = 2\pi r_0^{3/2}\sqrt{m/k}$. For the semi-major axis, that coincides with constant radius r_0, a result is obtained by a similar procedure. In this case, Eq. (21) leads to the equation $T^2/a^3 = 4\pi^2 m/k$ along with the approximation already used $(1+\epsilon x)^n \approx 1+n\epsilon x$, makes the semi-major axis changes as

$$a = -\frac{k}{2\left(E_0 - \frac{\ddot{\alpha}}{\alpha}\frac{m}{2}r_0^2\right)}, \tag{56}$$

$$a = -\frac{k}{2E_0}\left(1 + \frac{\ddot{\alpha}}{\alpha}\frac{mr_0^2}{2E_0}\right), \tag{57}$$

$$a = a_0\left(1 - \frac{\ddot{\alpha}}{\alpha}\frac{T_0^2}{4\pi^2}\right), \tag{58}$$

where $a_0 = -k/2mE_0$ and $a_0^3 = kT_0^2/4m\pi^2$. Equation (58) shows that for a circular orbit radius is reduced, a bizarre result. Like the decrease in stellar mass in a central force problem, the fact that the expansion of the universe does not have a preferential direction would suppose an unalteration in the eccentricity of the orbits. To prove this, we use the disturbed energy and square of the angular momentum, in order to substitute in the definition of eccentricity Eq. (20). The modified product using the Eq. (20) and (53) is

$$El^2 = \left(E_0 - \frac{\ddot{\alpha}}{\alpha}\frac{m}{2}r_0^2\right)\left(l_0^2 - \frac{\ddot{\alpha}}{\alpha}m^2r_0^4\right), \tag{59}$$

which is solved under certain considerations: the multiplication between the last two terms of each expression is negligible insofar as the factor $(\ddot{\alpha}/\alpha)^2$ is too small in comparison to the other terms; secondly, the substitution of E_0 and l_0^2 for their corresponding values, nullify any total contribution related to $\ddot{\alpha}/\alpha$. In this way, it is verified that the product $El^2 = E_0l_0^2$ (undisturbed value), leaving the eccentricity of a planetary orbit unchanged.

On the other hand, Bertrand's theorem can be applied by including the effect of expansion to study the behavior of closed orbit when energy is slightly disturbed. This can be performed using Eq. (12) but including the associated force given by the Eq. (48), that is:

$$\beta^2 = 3 - \frac{r}{-\frac{k}{r^2} + \frac{\ddot{\alpha}}{\alpha}mr}\left[\frac{d}{dr}\left(-\frac{k}{r^2} + \frac{\ddot{\alpha}}{\alpha}mr\right)\right]_{r=r_0}, \tag{60}$$

$$\beta^2 = 4 - 3\left(1 - \frac{\ddot{\alpha}}{\alpha}\frac{mr_0^3}{k}\right)^{-1}. \tag{61}$$

Using again the period $T_0^2 = 4\pi^2r_0^3m/k$,

$$\beta = \sqrt{1 - \frac{\ddot{\alpha}}{\alpha}\frac{3T_0^2}{4\pi^2}}, \tag{62}$$

$$\beta = 1 - \frac{3}{8}\frac{\ddot{\alpha}}{\alpha}\frac{T_0^2}{\pi^2}, \tag{63}$$

where previous Eq. (63) is obtained using $(1+x)^n \approx 1+nx$. The orbit equation can be established by using Eq (11), written as

$$\frac{1}{r} = \frac{1}{r_0[1+e\cos\beta\theta]}, \tag{64}$$

where $A = s_0 e$ and e is the eccentricity which is unchanged. Thus, the orbit equation keeps an ellipse shape but is not totally a closed. It can be understood since β has to be a rational number, and as it is given from the Eq. (63), this is not the case. However, when $\ddot{\alpha}/\alpha = 0$ is obtained, $\beta = 1$ *i.e.* a circular orbit.

Interestingly, we can get the disturbed energy $(\ddot{\alpha}/\alpha)(m/2)r_0^2$ (from which most of the results were generated) from the Eq. (50) by means of a simple dimensional analysis [16]: let f an arbitrary function such that $f = f(V, m, r, H)$ where V is an energy potential. There is a dimensionless quantity Π that satisfies $\Pi = H^a m^b r^c V$. Taking the units of $[V] = \left[ML^2/T^2\right]$, and through a system of equations, it reads that: $a = -2$, $b = -1$, $c = -2$. This results in equivalence:

$$\Pi = \frac{V}{H^2 m r^2}, \tag{65}$$

or more properly

$$V = \Pi H^2 m r^2. \tag{66}$$

As we can see, $\Pi = -1/2$.

6. Conclusions and perspectives

We study the local and global effects for the orbital parameters for a two body system with a central force coming from mass stellar variation and the cosmological expansion. The most interesting result was that eccentricity is an invariant quantity under the phenomena considered here. This could have been foreseen since the phenomena that gave rise to the extra terms were of the center type; that is, they presented a symmetry in angular directions, and the only dependency was the radial variable. If that were not the case, anisotropies into expansion effect would have into account, but this does not make part of this study. In the semi-major axis and period, the decrease of stellar mass causes a disturbance that vary with time; that result makes sense since the rate (considered constant here) at which the mass is lost varies over time. In the case of the expansion of the universe both quantities are directly proportional to the term $\ddot{\alpha}/\alpha$, and as in the case of the stellar mass, their disturbance depends on time. This can be seen if the value of H changes for different times in which it is measured. Also, this work could be used as a heuristic example in a classical subject. The implementation of the new factors due to the expansion of the universe, under the analysis of Bertrand's theorem and a new effective potential, can yield new results that in principle should be more rigorous from a mathematical point of view. In this case, the problem lies in that the term $(\ddot{\alpha}/\alpha)(m/2)r^2$ has no algebraic relationship with the centrifugal potential $(1/2)(l^2/mr^2) - (k/r)$; that is, it is not possible to obtain a new force of the form $f(r) = -(k/r^{3-\beta^2})$ (a necessary condition to give rise to a closed orbit and stable). On the other hand, the results in stellar mass were obtained thanks to the mass of the star is much greater than the planetary one. However, if $\dot{m}_\odot \approx M_0$ in magnitude, and there is a planetary system whose stellar mass is comparable to the mass of the orbital planet, such approximations would not be the most indicated. For these cases, the perturbations to the orbital factors must be studied from another analysis.

For the study to the three body problem and extended to inner planets into Solar System, the effects due to decreasing in the stellar mass and the expansion of the universe as perturbations into the elemental orbits can be neglected at first instance, but as numerical studies presented by Laskar [2, 6] these perturbations could appear for huge scales of time and being taken into account.

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i. This conservation will help to re-express the kinetic energy of the system given by the Eq. (5).

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