About the force on a magnetic dipole

I. Campos

Departamento de Física, Facultad de Ciencias, Universidad Nacional Autónoma de México, Ciudad Universitaria, Apartado Postal 70-348, Alcaldía Coyacán, 04510, Ciudad de México.

J. A. E. Roa-Neri

Área de Física Teórica y Materia Condensada, División de Ciencias Básicas e Ingeniería, Universidad Autónoma Metropolitana Unidad Azcapotzalco, Apartado Postal 16-225, 02011, Ciudad de México.

J. L. Jiménez

Departamento de Física, División de Ciencias Básicas e Ingeniería, Universidad Autónoma Metropolitana, Iztapalapa, Av. San Rafael Atlixco # 186, Apartado Postal 21-463, Col. Vicentina, 04000, Ciudad de México.

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There is disagreement about which is the correct expression for the force on a magnetic dipole, and at least two expressions for this force have been proposed, generating a controversy between Vaidman [1985] and Franklin [2018]. Our view here exposed is that the macroscopic Maxwell equations and the constitutive relations imply, via electromagnetic momentum balance equations, several force densities which include those proposed by some authors. Therefore, the question is not which one is correct, since all are legitimate deductions of Maxwell's equations, but under what conditions they may be useful to explain some phenomena. The discussion of conceptual problems of electromagnetism is very useful to both graduate students and researchers.

Keywords: Magnetic force; magnetic dipole; Maxwell equations; momentum balance equations; hidden momentum.

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1. Introduction

Traditionally, classical electromagnetism is taught beginning with the interaction of electromagnetic fields and point charges, electric dipoles, magnetic dipoles and filamentary currents in vacuum. Some results obtained with this approach are extrapolated to the interaction of electromagnetic fields and material media. Thus, the forces electromagnetic fields exert on charges and dipoles are considered well established by the Lorentz force density. There is, however, a controversy about what is the correct expression for the force a magnetic field exerts on a magnetic dipole.

Recently, Franklin [1] revisited the controversy about what is the force on a magnetic dipole. There are two proposals,

$$\boldsymbol{F} = \boldsymbol{\nabla}\left(\boldsymbol{m}\cdot\boldsymbol{B}\right), \tag{1}$$

based on the potential energy of an isolated magnetic dipole in an external magnetic field and [2]

$$\boldsymbol{F} = \boldsymbol{\nabla}\left(m\cdot\boldsymbol{B}\right) - \frac{d}{dt}\left(\boldsymbol{m}\times\boldsymbol{E}\right). \tag{2}$$

This last expression is closely related to the force associated to hidden momentum, one of whose expressions is

$$\boldsymbol{F}_{\text{hidden mom}} = -\left(\frac{d}{dt}\boldsymbol{m}\right)\times\boldsymbol{E}. \tag{3}$$

Franklin [1] concludes that the second term in Eq. (2) is incorrect and that "There is no hidden momentum in any of magnetic dipole configuration" as Vaidman proposes [2]. Therefore, only the force Eq. (1) is correct. This problem has been treated with relativity theory with the aim of differentiating orbital angular momentum from spin angular momentum [3].

The conclusions seem imposing, since the force in Eq. (1) is based on microelectromagnetism and other magnetic forces must be derived from it, but it is not clear if there are situations in which the second term in Eq. (2) is relevant. Since an elementary dipole is independent of time, this term is zero in this case but there can be materials in which it is relevant. The term associated to hidden momentum was discussed long ago by several authors [4–7], and recently there has been renewed interest in this concept of electromagnetic momentum [8–10]. Hidden momentum is useful in dealing with the interaction of matter with static and quasi-static fields. However, we can take the macroscopic Maxwell equations with the constitutive relations as phenomenological postulates from which several force densities can be deduced through balance equations and obtain as a particular case the force on an isolated dipole with the configuration, $\boldsymbol{M}\rightarrow\boldsymbol{m}$, where $\boldsymbol{m}$ is a point magnetic dipole.

At first sight it may seem that Franklin's arguments disqualify not only the dipole configurations proposed by Vaidman [2], where hidden momentum may be relevant, but also the relevance of hidden momentum in the interaction with any dipole configuration. However, Jiménez *et al.*, have shown [11] the usefulness of hidden momentum to analyze Feynman's disk and establish that the electromagnetic field has angular momentum, even in quasi-static conditions [12]. Then, the question remains, what is the force on a magnetic dipole? On the other hand, Boyer [13] showed that the model of dipole with two magnetic monopoles imply a force

$$\boldsymbol{F} = (\boldsymbol{m}\cdot\boldsymbol{\nabla})\,\boldsymbol{B}, \tag{4}$$

while a magnetic dipole modelled with a current loop implies a force

$$\boldsymbol{F} = \boldsymbol{\nabla}\,(\boldsymbol{m}\cdot\boldsymbol{B}). \tag{5}$$

This means that the force depends on the model of dipole. We consider that the force must be independent of the model of dipole. We will see that there are forces whose usefulness depends on the particular situation studied.

2. Alternative analysis

Several force densities, electric and magnetic, have been proposed. The most usual is Lorentz's force density, sometimes considered as an axiom that must be added to Maxwell's equations, and from which other force densities are deduced. Thus, we have

$$\boldsymbol{f}_1 = (\boldsymbol{P}\cdot\boldsymbol{\nabla})\,\boldsymbol{E}, \tag{6}$$

$$\boldsymbol{f}_{\boldsymbol{J}\ \text{polarization}} = \left(\frac{\boldsymbol{\partial P}}{\boldsymbol{\partial t}}\right)\times\boldsymbol{B}, \tag{7}$$

$$\boldsymbol{f}_{\text{Helmholtz Electric}} = -\frac{1}{2}E^2\boldsymbol{\nabla}\boldsymbol{\epsilon}, \tag{8}$$

and

$$\boldsymbol{f}_{\text{Helmholtz Magnetic}} = -\frac{1}{2}H^2\boldsymbol{\nabla}\boldsymbol{\mu}. \tag{9}$$

However, these force densities can be obtained from the macroscopic Maxwell equations by transforming them and the constitutive relations into momentum balance equations whose structure is

$$\boldsymbol{\nabla}\cdot\ \overleftrightarrow{\boldsymbol{T}} = \boldsymbol{f} + \delta\boldsymbol{f}, \tag{10}$$

where $\overleftrightarrow{\boldsymbol{T}}$ is a stress tensor, $\boldsymbol{f}$ a Lorentz type force density, and $\delta\boldsymbol{f}$ a force density that depends only on the electromagnetic fields, basic and auxiliary, as can be seen below. Thus, from the most usual expression of Maxwell's equations

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{D} &= \rho,\\
\boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0,\\
\boldsymbol{\nabla}\times\boldsymbol{E} + \frac{\partial\boldsymbol{B}}{\partial t} &= 0,\\
\boldsymbol{\nabla}\times\boldsymbol{H} - \frac{\partial\boldsymbol{D}}{\partial \mathbf{t}} &= \boldsymbol{J},
\end{aligned} \tag{11}$$

a balance equation can be obtained multiplying the first and second equations by $\boldsymbol{E}$ and $\boldsymbol{B}$ and multiplying vectorially by $-\boldsymbol{D}$ and $-\boldsymbol{B}$ on the left of the third and fourth equations, and using vector and dyad identities (in particular: $\boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{v}) + \boldsymbol{u}\cdot(\boldsymbol{\nabla}\boldsymbol{v}) = \boldsymbol{\nabla}\left\{\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{v}\right\} - \frac{1}{2}[(\boldsymbol{\nabla}\boldsymbol{u})\cdot\boldsymbol{v} - (\boldsymbol{\nabla}\boldsymbol{v})\cdot\boldsymbol{u}]$, we obtain the balance equation [14]

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\left[\boldsymbol{DE}+\boldsymbol{BH}-\frac{\mathbf{1}}{\mathbf{2}}\overleftrightarrow{\boldsymbol{I}}\,(\boldsymbol{D}\cdot\boldsymbol{E}+\boldsymbol{B}\cdot\boldsymbol{H})\right]&\\
-\frac{\partial}{\partial t}(\boldsymbol{D}\times\boldsymbol{B}) &= \rho\boldsymbol{E}+\boldsymbol{J}\times\boldsymbol{B}\\
&+\frac{1}{2}\Big[\,(\boldsymbol{\nabla}\boldsymbol{E})\cdot\boldsymbol{D}-(\boldsymbol{\nabla}\boldsymbol{D})\cdot\boldsymbol{E}\\
&+(\boldsymbol{\nabla}\boldsymbol{H})\cdot\boldsymbol{B}-(\boldsymbol{\nabla}\boldsymbol{B})\cdot\boldsymbol{H}\Big],
\end{aligned} \tag{12}$$

where $\overleftrightarrow{\boldsymbol{I}}$ is the unit dyad.

A Lorentz type of force density appears,

$$f_L = \rho\boldsymbol{E}+\boldsymbol{J}\times\boldsymbol{B}, \tag{13}$$

but with total fields. The usual Lorentz force density is a limit of Eq. [13], valid only for test charges and the fields are only external fields.

Besides, we have also the force density

$$\begin{aligned}
\delta\boldsymbol{f} = \frac{1}{2}\Big[\,&(\boldsymbol{\nabla}\boldsymbol{E})\cdot\boldsymbol{D}-(\boldsymbol{\nabla}\boldsymbol{D})\cdot\boldsymbol{E}\\
&+(\boldsymbol{\nabla}\boldsymbol{H})\cdot\boldsymbol{B}-(\boldsymbol{\nabla}\boldsymbol{B})\cdot\boldsymbol{H}\Big],
\end{aligned} \tag{14}$$

which is usually omitted, since for vacuum, or homogeneous media, it is zero. It is evident that the force densities associated with forces Eqs. [6–9] do not appear. However, there are other expressions of Maxwell's equations and the constitutive relations, for example, that proposed by Panofsky and Phillips [15],

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{E} &= \frac{1}{\epsilon_0}\left(\rho-\boldsymbol{\nabla}\cdot\boldsymbol{P}\right),\\
\boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0,\\
\boldsymbol{\nabla}\times\boldsymbol{E}+\frac{\partial\boldsymbol{B}}{\partial t} &= 0,\\
\boldsymbol{\nabla}\times\boldsymbol{B}-\epsilon_0\mu_0\frac{\partial\boldsymbol{E}}{\partial t} &= \mu_0\left(\boldsymbol{J}+\frac{\boldsymbol{\partial P}}{\boldsymbol{\partial t}}+\boldsymbol{\nabla}\times\boldsymbol{M}\right).
\end{aligned} \tag{15}$$

Proceeding as before, we multiply the first and second equation by $\boldsymbol{E}$ and $\boldsymbol{B}$, and multiply vectorially by $-\boldsymbol{E}$ and

$-\boldsymbol{B}$ the third and fourth equations, and with the help of vector and dyad identities, we obtain the balance equation [16]

$$\begin{aligned}\nabla\cdot&\left[\epsilon_0\boldsymbol{E}\boldsymbol{E}+\frac{1}{\mu_0}\boldsymbol{B}\boldsymbol{B}\right.\\&\left.-\frac{1}{2}\overleftrightarrow{I}\left(\epsilon_o\boldsymbol{E}\cdot\boldsymbol{E}+\frac{1}{\mu_o}\boldsymbol{B}\cdot\boldsymbol{B}\right)\right]\\&-\epsilon_0\frac{\partial}{\partial t}(\boldsymbol{E}\times\boldsymbol{B})=\rho\boldsymbol{E}+\boldsymbol{J}\times\boldsymbol{B}+(\nabla\boldsymbol{P})\cdot\boldsymbol{E}\\&+\left(\frac{\partial}{\partial t}\boldsymbol{P}\right)\times\boldsymbol{B}+(\nabla\times\boldsymbol{M})\times\boldsymbol{B}.\end{aligned}\tag{16}$$

Now the force density Eq. (7) appears, which is in this case an implication of Maxwell's equations. This force density was used by Bohren [17] for the analysis of radiation pressure.

On the other hand, by means of the constitutive relations

$$\begin{aligned}\boldsymbol{D}&=\epsilon_0\boldsymbol{E}+\boldsymbol{P},\\\boldsymbol{B}&=\mu_0\left(\boldsymbol{H}+\boldsymbol{M}\right),\end{aligned}\tag{17}$$

we can transform Eq. (11) and (12) into Eq. (15) and (16). Furthermore, if we express the constitutive relations in the form

$$\begin{aligned}\boldsymbol{D}&=\epsilon_0\epsilon_r\boldsymbol{E}=\epsilon\boldsymbol{E},\\\boldsymbol{B}&=\mu_0\mu_r\boldsymbol{H}=\mu\boldsymbol{H},\end{aligned}\tag{18}$$

we can see that Eq. (14) leads directly to Eqs. (8) and (9). Then, it is evident that the Maxwell equations with different expressions of the constitutive relations lead to different balance equations, and therefore to different force densities. Their utility depends on the situation under analysis.

We now show that hidden momentum is part of a balance equation deduced from Maxwell's equations. First, we write these equations in the equivalent form

$$\begin{aligned}\nabla\cdot\boldsymbol{E}&=\frac{1}{\epsilon_0}\left(\rho-\nabla\cdot\boldsymbol{P}\right),\\\nabla\cdot\boldsymbol{H}&=-\nabla\cdot\boldsymbol{M},\\\nabla\times\boldsymbol{E}+\mu_0\frac{\partial\boldsymbol{H}}{\partial t}&=-\mu_0\frac{\partial\boldsymbol{M}}{\partial t},\\\nabla\times\boldsymbol{H}-\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}&=\boldsymbol{J}+\frac{\partial\boldsymbol{P}}{\partial t}.\end{aligned}\tag{19}$$

These expressions may seem strange, but they are a direct consequence of the usual form of Maxwell's equations and the constitutive relations. From these equations, following a similar procedure as before [18,19], we can obtain, again by means of vector and dyad identities, the balance equation

$$\begin{aligned}\nabla\cdot&\Big((\epsilon_0\boldsymbol{E}\boldsymbol{E}+\mu_0\boldsymbol{H}\boldsymbol{H})\\&-\frac{1}{2}\left(\epsilon_o\boldsymbol{E}\cdot\boldsymbol{E}+\mu_o\boldsymbol{H}\cdot\boldsymbol{H}\right)\overleftrightarrow{I}\Big)\\&-\epsilon_0\mu_0\frac{\partial}{\partial t}(\boldsymbol{E}\times\boldsymbol{H})=\rho\boldsymbol{E}+\mu_0\boldsymbol{J}\times\boldsymbol{H}\\&+\mu_0\left(\frac{\partial}{\partial t}\boldsymbol{P}\right)\times\boldsymbol{H}-\epsilon_0\mu_0\left(\frac{\partial}{\partial t}\boldsymbol{M}\right)\times\boldsymbol{E}\\&+\left(\boldsymbol{P}\cdot\nabla\right)\boldsymbol{E}+\mu_0\left(\boldsymbol{M}\cdot\nabla\right)\boldsymbol{H}.\end{aligned}\tag{20}$$

We can see that the force density Eq. (3) appears, as well as a term similar to Eq. (4). This is a consequence of eliminating in Maxwell's equations the magnetic field $\boldsymbol{B}$ in favor of the auxiliary field $\boldsymbol{H}$. We follow here Purcell's convention taking $\boldsymbol{B}$ as the magnetic field [20]. Then, the right member in Eq. (20) is a possible force density implied by Maxwell's equations. If there are no free charge and current densities, the balance equation reduces to

$$\begin{aligned}\boldsymbol{f_1}=\mu_0&\left(\frac{\partial}{\partial t}\boldsymbol{P}\right)\times\boldsymbol{H}-\epsilon_0\mu_0\left(\frac{\partial}{\partial t}\boldsymbol{M}\right)\times\boldsymbol{E}\\&+\left(\boldsymbol{P}\cdot\nabla\right)\boldsymbol{E}+\mu_0\left(\boldsymbol{M}\cdot\nabla\right)\boldsymbol{H}.\end{aligned}\tag{21}$$

If there are no polarizations ($\boldsymbol{P}=\boldsymbol{0}$), then the balance equation reduces to

$$\boldsymbol{f_1}=-\epsilon_0\mu_0\left(\frac{\partial}{\partial t}\boldsymbol{M}\right)\times\boldsymbol{E}+\mu_0\left(\boldsymbol{M}\cdot\nabla\right)\boldsymbol{H}.\tag{22}$$

Now, with the identity

$$\begin{aligned}\nabla\left(\boldsymbol{M}\cdot\boldsymbol{H}\right)&=\left(\boldsymbol{M}\cdot\nabla\right)\boldsymbol{H}+\left(\boldsymbol{H}\cdot\nabla\right)\boldsymbol{M}\\&+\boldsymbol{M}\times\left(\nabla\times\boldsymbol{H}\right)+\boldsymbol{H}\times\left(\nabla\times\boldsymbol{M}\right),\end{aligned}\tag{23}$$

and remembering that there are no free currents, we have

$$\boldsymbol{M}\times\left(\nabla\times\boldsymbol{H}\right)=\boldsymbol{M}\times\left(\frac{\partial}{\partial t}\boldsymbol{D}\right).\tag{24}$$

Also, we do not have polarizations, and we can write

$$\boldsymbol{M}\times\left(\frac{\partial}{\partial t}\boldsymbol{D}\right)=\epsilon_0\boldsymbol{M}\times\left(\frac{\partial}{\partial t}\boldsymbol{E}\right).\tag{25}$$

With these results, Eq. (23) becomes

$$\begin{aligned}\nabla\left(\boldsymbol{M}\cdot\boldsymbol{H}\right)&=\left(\boldsymbol{M}\cdot\nabla\right)\boldsymbol{H}+\left(\boldsymbol{H}\cdot\nabla\right)\boldsymbol{M}\\&+\epsilon_0\boldsymbol{M}\times\left(\frac{\partial}{\partial t}\boldsymbol{E}\right)+\boldsymbol{H}\times\left(\nabla\times\boldsymbol{M}\right),\end{aligned}\tag{26}$$

from which we can obtain

$$\begin{aligned}\mu_0\left(\boldsymbol{M}\cdot\nabla\right)\boldsymbol{H}=\mu_0\nabla\left(\boldsymbol{M}\cdot\boldsymbol{H}\right)-\mu_0\Big[&\left(\boldsymbol{H}\cdot\nabla\right)\boldsymbol{M}\\&+\epsilon_0\boldsymbol{M}\times\left(\frac{\partial}{\partial t}\boldsymbol{E}\right)+\boldsymbol{H}\times\left(\nabla\times\boldsymbol{M}\right)\Big].\end{aligned}\tag{27}$$

In this way, Eq. (22) is transformed into

$$\boldsymbol{f_1} = -\epsilon_0\mu_0 \left(\frac{\partial}{\partial t}\boldsymbol{M}\right) \times \boldsymbol{E} + \boldsymbol{\mu_0}\boldsymbol{\nabla}\left(\boldsymbol{M}\cdot\boldsymbol{H}\right) - \mu_0\Bigg(\left(\boldsymbol{H}\cdot\boldsymbol{\nabla}\right)\boldsymbol{M} + \epsilon_0\boldsymbol{M} \times \left(\frac{\partial}{\partial t}\boldsymbol{E}\right) + \boldsymbol{H}\times\left(\boldsymbol{\nabla}\times\boldsymbol{M}\right)\Bigg),$$

then,

$$\boldsymbol{f_1} = -\epsilon_0\mu_0\frac{\partial}{\partial t}\left(\boldsymbol{M}\times\boldsymbol{E}\right) + \boldsymbol{\mu_0}\boldsymbol{\nabla}\left(\boldsymbol{M}\cdot\boldsymbol{H}\right) - \mu_0\left(\left(\boldsymbol{H}\cdot\boldsymbol{\nabla}\right)\boldsymbol{M} + \boldsymbol{H}\times\left(\boldsymbol{\nabla}\times\boldsymbol{M}\right)\right). \tag{28}$$

Now, considering the case of an elementary dipole.

$$\boldsymbol{M} \to \boldsymbol{m}, \tag{29}$$

and assuming that $\boldsymbol{m}$ does not depend on $\boldsymbol{r}$, and that $\boldsymbol{B} = \mu_0\boldsymbol{H}$, the force density $\boldsymbol{f}_1$ results

$$\boldsymbol{f}_1 = -\epsilon_0\mu_0\frac{\partial}{\partial t}\left(\boldsymbol{m}\times\boldsymbol{E}\right) + \boldsymbol{\nabla}\left(\boldsymbol{m}\cdot\boldsymbol{B}\right), \tag{30}$$

which is the result considered by Franklin as erroneous. Here, we have shown that this expression is an implication of Maxwell's equations and the constitutive relations.

3. Conclusions

Though Franklin [1] argues that in an expression of the force on a magnetic dipole proposed by Vaidman [2], a term related to hidden momentum is erroneous, indeed what he shows is that this term is not adequate to deal with the configurations of magnetic dipoles analyzed by this author.

However, we have shown that this term is a consequence of Maxwell's equations and the constitutive relations through a momentum balance equation deduced from them. Indeed, different balance equations can be deduced from Maxwell's equations by writing them in different forms according to different ways of expressing them in terms of the constitutive relations. Therefore, different force densities derived in this way can be useful in different cases of dipole configurations. Results for point dipoles can be obtained as limits taking $\boldsymbol{P} \to \boldsymbol{p}$ and $\boldsymbol{M} \to \boldsymbol{m}$, where $\boldsymbol{p}$ is a point electric dipole and m is a point magnetic dipole.

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