# On the dynamics of a rolling spherical charged shell

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In this work, we investigate the behaviour of a charged spherical shell rolling on an inclined plane, in presence of a point charge located at the lowest part of the inclined plane. The shell generates two magnetic fields, one due to its rotation and another due to its translation, These magnetic fields affect the shell through self-inductance. On the other hand, the charge in the lowest part of the inclined plane interacts with the shell, and we find that under certain conditions the spherical shell rolls back and up the inclined plane due to the electric force. We perform a numerical analysis to study this behavior.

*Keywords:* Lagrange; electrodynamics; electric force; self-inductance; poynting vector.

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## 1. Introduction

One of the main advantages of the Lagrangian formalism over the Newtonian formalism, is the Noether's theorem [1] which, roughly speaking, states that every symmetry in the Lagrangian of a physical system with conservative forces has a corresponding conservation law. In order to apply this theorem, it is necessary that the Lagrangian is written with the same symmetries.

In this paper, we consider a charged spherical shell rolling on an inclined plane interacting with a point charge in the lowest part of the inclined plane. From the point of view of mechanics, the system has friction, inertia, angular velocity, and angular acceleration, the Newtonian standard variables (Galileo transformations). However, from the point of view of electrodynamics, the system has an electric field and a magnetic field (Lorentz transformations). If we write down the Lagrangian of this system, the Newtonian part would have quantities invariant under Galileo transformations, and the Maxwell part would have quantities invariant under Lorentz transformations. Therefore, the first question would be, is this Lagrangian correct? [2]. To avoid the above-mentioned problem, we will calculate the mechanical and electromagnetic variables at the same point and small velocities are considered. Remember that the position vector $\vec{R} = \vec{r} - \vec{r}\,'$ is where we want to measure, minus where the charge distribution is, so $\vec{r}$ will be set in the charge $q$, the black point in Fig. 1, because we want to determine the force feel the spherical shell, due to the charge $q$.

The paper is organized as follows. In Sec. 2, we present the problem and the physical implications of the system. We construct the Lagrangian and obtain the equations of motion. In Sec. 3, we solve the problem numerically and analyze the solutions for several situations. The acceleration changes of a spherical shell are analyzed in Sec. 4. Due to its electric charge and acceleration changes, the shell produces electromagnetic radiation. Finally, we present the conclusions and remarks in Sec. 5.

## 2. The problem

In this paper, we want to investigate the dynamics of a spherical shell of mass $m$ and radius $a$, made of a conducting material, carrying a uniform surface charge $\sigma$, rolling down on an inclined plane of length $L$, with a fixed angle $\alpha$ and friction, interacting with a static point charge located at the lowest part of the inclined plane, the black dot in Fig. 1.

It is known that the best way to obtain the dynamics of a system is through Lagrange's equations. However, to preserve the symmetry of the system, it is necessary to fix both reference systems, electromagnetic and mechanical, to the same point. In our case, we have decided that our reference system will be at the bottom part of the inclined plane $i.e.$, $\vec{r} = 0$, Fig. 1, and the spherical shell in the top $\vec{r}\,' = -x\cos\alpha\,\hat{x} + x\sin\alpha\,\hat{y}$. Where the $x$-axis is the horizontal, $y$-axis is the vertical and the $z$-axis, point outside the plane.

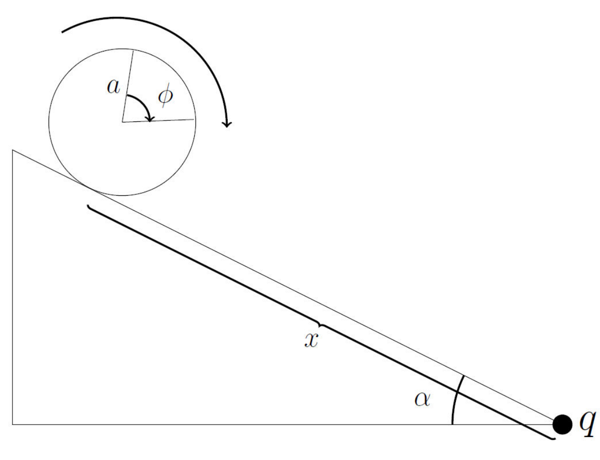


FIGURE 1. System diagram.

### 2.1. Building the Lagrangian

To solve the problem, we start by writing the generic Lagrangian, considering all the energies of the system, Eq. (1.63) of [3], $L = (1/2)mv^2 - mgy - q\,\Phi + q\,\mathbf{v}\cdot\mathbf{A}$. Since we are interested in determining the dynamics of the spherical shell, for our case, $q$ will be the charge of the spherical shell, $\Phi$ will be the potential of the point charge (the black dot in, Fig. 1), $\vec{v}$ is the velocity with which the spherical shell descends, and $\mathbf{A}$ is the magnetic vector potential generated by the charge (the black dot in, Fig. 1). Therefore, $q\,\mathbf{v}\cdot\mathbf{A} = 0$, and

$$L = T_{\text{trans}} + T_{\text{rotat}} - V_{\text{gravi}} - V_{\text{elect}}\,. \tag{1}$$

The mechanical parts are

$$T_{\text{trans}} + T_{\text{rotat}} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2,$$
$$V_{\text{gravi}} = mgx\sin\alpha\,,$$

where $x$, is the displacement along the plane (starting at the bottom of the plan), $I = (2/3)ma^2$ the moment of inertia, and $\phi$ the angle of rotation of the sphere [3, 4].

For the electric energy, Eq. (1.62) of [3], and Sec. 2.3.2 of [5], we have

$$V_{\text{elect}} = q\,\Phi(r) = \sigma(4\pi a^2)\frac{kq}{R} = \frac{q\sigma a^2}{\epsilon_0 x}\,,$$

where $R = |\vec{r} - \vec{r}\,'| = x$. Up to this point, we have obtained all the corresponding energies of the Lagrangian

### 2.2. Lagrangian and equations of motion

Finally, replacing all the energies in the Lagrangian, Eq. (1)

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{3}ma^2\dot{\phi}^2 - mgx\sin\alpha - \frac{q\sigma a^2}{\epsilon_0 x}\,.$$

The Lagrangian has two degrees of freedom and one constraint. The equations of motion are determined by the Euler-Lagrange equations. However, in this case, there is a holonomic constraint that relates the angle $\phi$ to the displacement $x$. This requires the use of the Lagrange multiplier

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \lambda\frac{\partial F}{\partial q_i},$$

where $\lambda$ is the Lagrange multiplier, and $F$ is the constraint.

To define the constraint, we analyze the angle $\phi$, and find that it is related to the displacement $x$, so that

$$x = a\phi \quad \rightarrow \quad F = x - a\phi\,.$$

From the Euler-Lagrange equations and the above constraint, the equations of motion for $q_i = x, \phi$. are

$$\begin{aligned} m\ddot{x} + mg\sin\alpha - \frac{q\sigma a^2}{\epsilon_0 x^2} &= \lambda, \\ \frac{2}{3}ma\ddot{\phi} &= -\lambda. \end{aligned} \tag{2}$$

The differential equations are second order, and one is nonlinear.

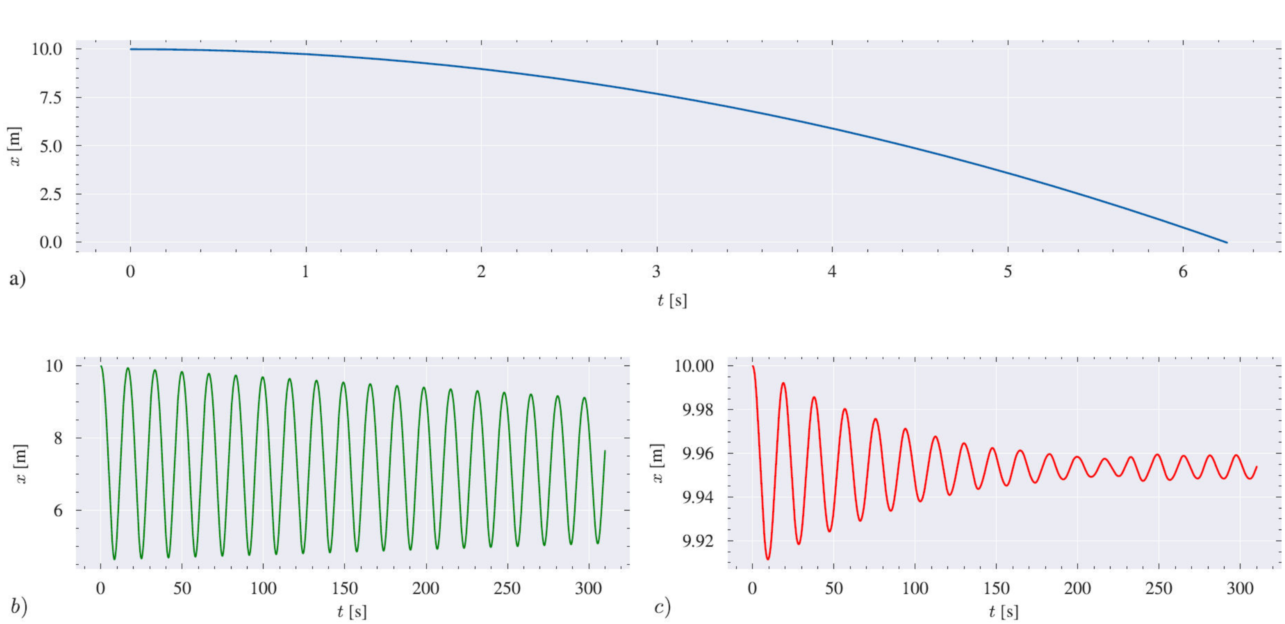


FIGURE 2. Displacement of the spherical charged shell in an electric field with: mass $m = 1$ kg, radius $a = 0.25$ m, $g = 9.81\ \mathrm{m/s^2}$, angle of the plane $\alpha = 5^o$, distance of inclined plane $L = 10$ m, charge $q = 1\,\mu$C, vacuum permittivity $\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{C^2N^{-1}m^{-2}}$, with different charge densities: a) $\sigma = 0$, b) $\sigma = 5600\,\mu\mathrm{C/m^2}$, and c) $\sigma = 12\,\mathrm{mC/m^2}$.

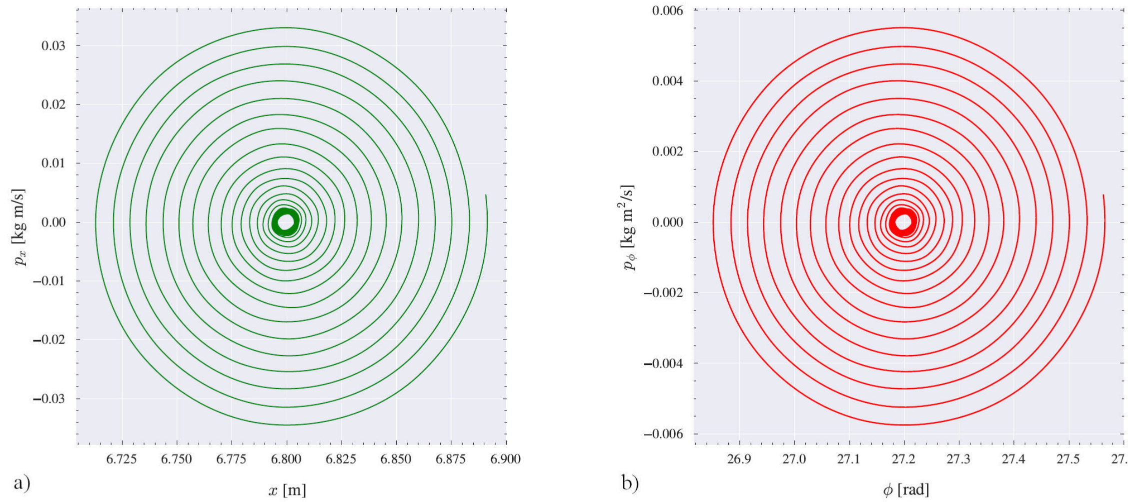


FIGURE 3. Phase space diagram of the system shown in the Fig. 2b). a) $(x, p_x)$, b) $(\phi, p_\phi)$. From the diagrams, it can be observed that the systems reaches a stability, which is not at rest.

## 3. Solving the equations of motion

To solve the equations Eq. (2), it is necessary to eliminate the $\lambda$ factor, by algebraic manipulation of the equations. It should also be noted that Eq. (2) are coupled non-linear differential equations which makes analytical solutions difficult. Instead, a numerical approach is required in this case. According to [6], in order to solve second-order differential equations using numerical methods, it is necessary to reduce their order by introducing new variables, then obtaining a system of first-order differential equations.

In this case, we introduce two new variables $v$ and $\omega$, which are the velocities, of the displacement and the rotation of the sphere, respectively

$$\dot{x} = v, \qquad \dot{\phi} = \omega, \tag{3}$$

thus Eq. (2) becomes a first-order non-linear differential equation

$$\begin{aligned} m\dot{v} + mg\sin\alpha - \frac{q\sigma a^2}{\epsilon_0 x^2} &= \lambda, \\ \frac{2}{3} ma\dot{\omega} &= -\lambda. \end{aligned} \tag{4}$$

To solve the system generated by Eq. (3) and Eq. (4), was implemented the Runge-Kutta method [7], using the SciPy library of Python. We obtained the results for distance vs. time, shown in the Fig. 2, where the behavior depends on the charge density.

In Fig. 2a) the system is purely mechanical, $i.e.$, we have turned off the electric energy; the velocity increases with distance as is expected for the motion of a particle moving on an inclined plane. In Fig. 2b) and Fig. 2c) the charge density has been increased, and is enough to affect the dynamics of the system, since the system now oscillates, $i.e.$, the sphere rises and falls on the inclined plane with an oscillatory behavior, this is caused by the fact that when the spherical shell moves downward, it is closer to the other charge $q$, and due to the electric force $\vec{F} = q\,\vec{E}$, Fig. 4, the spherical shell is repelled, causing the sphere to move upward and, consequently, the sphere will present a loss of velocity, due to the friction, gravity, and to the decay of the Coulomb force, causing it to rotate downward again, repeating this behavior periodically. The system continues oscillating over time, $i.e.$, the system never reaches the rest. This can be verified in the phase space diagrams, Fig. 3.

On the other hand, it is known that a moving charge produces a magnetic vector potential $\vec{A}$, in the same direction of movement of the charge. Therefore, when analyzing the dynamics of the spherical shell there are two magnetic vector potential, one due to rotation of the sphere and the other due to translation, analyzing these two magnetic field we have that. When the sphere rotates around its own axis, it generates a magnetic field that leaves the sheet plane and points in the $\hat{z}$ direction. Also when the sphere is displaced, the magnetic field (caused by this displacement) leaves the plane of the sheet and also points in the $\hat{z}$ direction. These two magnetic fields interact with the sphere through self-inductance, resulting in the appearance of a magnetic force $\vec{F} = q\,\vec{v} \times \vec{B}$ on the sphere, where $q$, the charge of the sphere, $\vec{v}$, the velocity when moving, and $\vec{B}$, the superposition of the two mag-

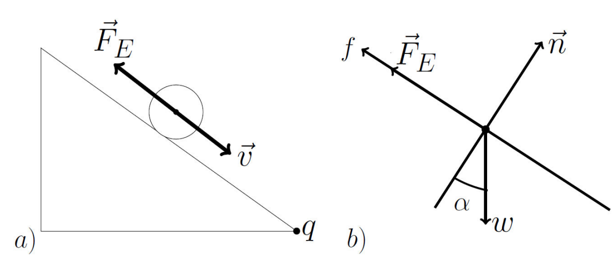


FIGURE 4. a) Electric force of the charge $q$, on the sphere in its downward motion. b) The external forces exerted on a spherical shell, lying on an inclined plane. It is important to note that the friction will change direction depending on whether the sphere is going up or down.

netic fields mentioned above. The more charge the sphere has, the greater the magnetic field will be, and therefore the greater the magnetic force, and according to the right-hand rule, the sphere should slow down in its downward path due to self-inductance. We analyze such situation numerically, but the magnetic force is too small that we can only see this behavior when we consider very large charges.

## 4. Energy loss

Since the inclined plane has friction, the system is losing energy. From the electromagnetic point of view, we can quantify the energy flux density loss through the Poynting vector, since the spherical shell starts to move from rest, accelerates and slows down, there is an exchange of kinetic energy to electromagnetic energy, which is reflected as electromagnetic radiation emitted by the shell. Using the generalization analogous of electric dipole radiation Eq. (11.59) of [5], we can quantify the radiation. Bear in mind that there are two magnetic fields. The Poynting vector

$$\vec{S} = \frac{\mu_0}{16c^3}(a^2\ddot{I})^2\frac{\sin^2\theta}{r^2}\,\hat{r} = \frac{\mu_0(\sigma\pi a^2\sin\theta)^2}{2c^3r^2}\,\dddot{\phi}^2\,\hat{r},$$

where $\vec{I} = l\vec{K} = l(\sigma\vec{v})$. As long as there is a change in acceleration, we will have electromagnetic radiation.

## 5. Conclusion and remarks

With this didactic problem, we wanted to show how to deal with a mechanical-electromagnetic problem, with a holonomic constraint, and to show the behavior of the fields individually and the hierarchy that each one of them has.

As we have shown, the sphere initially starts rolling down the plane until it reaches a rest point, and immediately goes back upward, starting an oscillating movement (Fig. 2).

This behavior is maintained without the shell coming to rest, which can be shown in the graphic of the phase space.

This behavior is because of the electric force is much stronger than the gravitational and magnetic force, and this dominates at short distances. However, since the charge $q$, is small, the electric force does not completely dominate over the gravitational force.

In the other hand, there are different numerical methods to solve the system. According to [8], the Runge-Kutta is the best method to solve second order differential equations.

Finally, we want to emphasize that there is self-inductance, but it is too small to modify the aforementioned physical behavior, due to the low speed and low charge of the spherical shell.

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