Linear and angular momentum stored in a distribution of charges in a magnetic field. The other side of the story

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The linear or the angular momentum stored in an arbitrary electric charge distribution in the presence of a magnetic field is defined by calculating the linear or the angular momentum transferred to the electric charge distribution by a time-dependent magnetic field, which is initially zero and after some time reaches the desired final value. The component of the transferred linear momentum along some axis depends only on the final magnetic field if and only if the magnetic field is invariant under translations along this axis. Similarly, the component of the transferred angular momentum along some axis depends only on the final magnetic field if and only if the magnetic field is invariant under rotations about this axis.

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1. Introduction

In a recent paper [1] the linear and the angular momentum stored in a distribution of charges in a magnetic field was calculated by considering the linear momentum and the angular momentum transferred to a point charge displaced from infinity to a final position in a given static magnetic field. It was shown that the component of the linear momentum along some axis transferred to a point charge is independent of the path if and only if the magnetic field is invariant under translations along that axis, and that the component of the angular momentum along some axis transferred to a point charge is independent of the path if and only if the magnetic field is invariant under rotations about that axis.

In this paper the same calculation is addressed in a different way. We consider an arbitrary *static* electric charge distribution and a time-dependent magnetic field which is initially zero and after some time acquires some final predetermined value. According to Faraday's law, the variable magnetic field will induce an electric field which will exert some force and some torque on the stationary electric charge (by contrast, the magnetic field does not produce forces or torques on the electric charges, since they are static). In order to maintain the charges in their positions, some linear or angular momentum must be transferred to the charge distribution. This linear or angular momentum transferred to the charges is defined as the linear or angular momentum stored in the system formed by the charges and the magnetic field. This procedure is similar to that employed to define the energy stored in the magnetic field (see, *e.g.*, Ref. [2]).

It is a remarkable fact that the two procedures yield the same expressions for the linear and the angular momentum of the magnetic field, taking into account that in Ref. [1] the calculation is based on the Lorentz force on an electric charge moving in a static magnetic field, while the calculation presented in this paper is based on the application of Faraday's law of induction to find the force on a static charge in a time-dependent magnetic field.

In Sec. 2 we summarize the findings of Ref. [1] in order to compare them with the results derived here. In Sec. 3 we calculate the linear or the angular momentum that has to be transferred to a static electric charge distribution when the magnetic field attains some predetermined value beginning from zero. We show that the imposition of the adequate symmetry of the final magnetic field is crucial in order for the transferred momentum to depend on the final configuration only. Throughout this paper it is assumed that the reader is acquainted with the basic notions of electrodynamics (as presented, *e.g.*, in Refs. [2, 3]) and vector calculus.

2. Summary of previous results

As shown in Ref. [1], if one considers a static magnetic field, $\mathbf{B}$ (produced by electric currents or by permanent magnets), in order to displace a point charge, q, along a path C, one has to transfer to the point charge a linear momentum given by (in cgs units)

$$\Delta \mathbf{p}_{\rm t} = \frac{q}{c}\int_C \mathbf{B}\times {\rm d}\mathbf{r}, \tag{1}$$

where c is the speed of light in vacuum. The z-component of this linear momentum, for instance, is path-independent if and only if the magnetic field is invariant under the translations along the z-axis (that is, $\partial\mathbf{B}/\partial z = 0$). Hence, if this symmetry condition is satisfied, one can define the z-component of the linear momentum of the system formed by the magnetic field and the point charge at some final position by Eq. (1), making use of *any* path C starting at infinity and ending at the final position of the charge. With the aid of Gauss's law, the z-component of the stored linear momentum is expressed in the form

$$p_z = \frac{1}{4\pi c}\int (\mathbf{E}\times\mathbf{B})_z \mathrm{d}v, \tag{2}$$

where $\mathbf{E}$ is the electric field produced by the point charge when it reaches its final position.

In a similar way, the angular momentum transferred to the point charge is given by [1]

$$\Delta\mathbf{L}_\mathrm{t} = \frac{q}{c}\int_C \mathbf{r}\times(\mathbf{B}\times\mathrm{d}\mathbf{r}). \tag{3}$$

The z-component of this angular momentum, for instance, is path-independent if and only if the magnetic field is invariant under the rotations about the z-axis. Then, if this symmetry condition is satisfied, one can define the z-component of the angular momentum of the system formed by the magnetic field and the point charge at some final position by Eq. (3), making use of any path C starting at infinity and ending at the final position of the charge. Again, with the aid of Gauss's law, the z-component of the stored angular momentum is given by

$$L_z = \frac{1}{4\pi c}\int [\mathbf{r}\times(\mathbf{E}\times\mathbf{B})]_z \mathrm{d}v, \tag{4}$$

where $\mathbf{E}$ is the electric field produced by the point charge when it arrives at its final position.

3. Alternative approach

One can expect that the expressions (2) and (4) be obtainable by considering a different process: Instead of displacing the electric charges from infinity to their final positions in a given static magnetic field, we can imagine that the charges are maintained fixed all the time (with respect to some *inertial frame*) and now we build up the magnetic field starting from zero. In this way, we will have a magnetic field changing with the time, which, according to Faraday's law, must induce some electric field (different from, and unrelated to, that produced by the static charges). This induced electric field gives rise to a force and a torque on the electric charges and, in order to keep the charges motionless, we would have to apply an equilibrating force and torque on the charges, which amounts to a transfer of linear and angular momentum. This linear or angular momentum can be defined as the linear or angular momentum stored in the system.

These ideas seem pretty simple but there are two difficulties: Faraday's law allows us to calculate the rotational of the induced electric field, not the induced field itself, which is what we need to calculate forces and torques. Furthermore, the reasoning given in the previous paragraph is valid regardless of the symmetry of the magnetic field, which is essential in the findings of Ref. [1].

3.1. Stored linear momentum

We shall consider a fixed charge distribution characterized by a charge density ρ_c and a time-dependent magnetic field, $\mathbf{B}$, which induces an electric field $\mathbf{E}$ (different from the electrostatic field, $\mathbf{E}'$, produced by the charge distribution). According to the elementary formulas, the z-component of the force produced on the charge distribution by the induced electric field is given by

$$F_z = \int \rho_\mathrm{c} E_z \,\mathrm{d}v. \tag{5}$$

According to Gauss's law, $\nabla\cdot\mathbf{E}' = 4\pi\rho_\mathrm{c}$, and making use of Gauss's theorem we can write Eq. (5) in the form

$$F_z = \frac{1}{4\pi}\int (\nabla\cdot\mathbf{E}')E_z \,\mathrm{d}v = -\frac{1}{4\pi}\int \mathbf{E}'\cdot\nabla E_z \,\mathrm{d}v.$$

Thus, making use of Newton's second law, if p_z is the z-component of the linear momentum that we have to transfer to the charges in order to keep them motionless,

$$\begin{aligned}\frac{\mathrm{d}p_z}{\mathrm{d}t} &= \frac{1}{4\pi}\int\left(E'_x\frac{\partial E_z}{\partial x} + E'_y\frac{\partial E_z}{\partial y} + E'_z\frac{\partial E_z}{\partial z}\right)\mathrm{d}v\\ &= \frac{1}{4\pi}\int\left[E'_x\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) + E'_y\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\right.\\ &\left. + E'_x\frac{\partial E_x}{\partial z} + E'_y\frac{\partial E_y}{\partial z} + E'_z\frac{\partial E_z}{\partial z}\right]\mathrm{d}v,\end{aligned}$$

and, employing Faraday's law, $\nabla\times\mathbf{E} = -(1/c)\,\partial\mathbf{B}/\partial t$, we have

$$\begin{aligned}\frac{\mathrm{d}p_z}{\mathrm{d}t} &= \frac{1}{4\pi c}\int\left(E'_x\frac{\partial B_y}{\partial t} - E'_y\frac{\partial B_x}{\partial t}\right)\mathrm{d}v\\ &+ \frac{1}{4\pi}\int\left(E'_x\frac{\partial E_x}{\partial z} + E'_y\frac{\partial E_y}{\partial z} + E'_z\frac{\partial E_z}{\partial z}\right)\mathrm{d}v\end{aligned}$$

or, equivalently,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[p_z - \frac{1}{4\pi c}\int(\mathbf{E}'\times\mathbf{B})_z\mathrm{d}v\right] &= \frac{1}{4\pi}\\ \times\int\left(E'_x\frac{\partial E_x}{\partial z} + E'_y\frac{\partial E_y}{\partial z} + E'_z\frac{\partial E_z}{\partial z}\right)\mathrm{d}v. \end{aligned}\tag{6}$$

Thus if $\mathbf{B}$ is invariant under translations along the z-axis (that is, all the Cartesian components of $\mathbf{B}$ can be functions of (x, y, t) only) and, therefore, the Cartesian components of the induced electric field are also independent of z, then the right-hand side of Eq. (6) vanishes and, at any instant,

$$p_z = \frac{1}{4\pi c}\int(\mathbf{E}'\times\mathbf{B})_z\mathrm{d}v, \tag{7}$$

which agrees with Eq. (2).

If the components of the induced electric field depend on z, then the value of the integral on the right-hand side of Eq. (6) may be different from zero (keep in mind that the static field $\mathbf{E}'$ is arbitrary) and the transferred linear momentum will depend not only on the final value of the magnetic field, but also on the detailed manner in which the magnetic field achieves its final value. Hence, in order to have a well-defined value for the stored linear momentum at any moment or field configuration, the magnetic field must be invariant under the translations along the z-axis.

3.2. Stored angular momentum

Proceeding in a similar manner, we shall calculate the z-component of the angular momentum that has to be transferred to a fixed charge distribution when a magnetic field is turned on. For this calculation it is convenient to make use of the circular cylindrical coordinates, (ρ, ϕ, z), and the orthonormal basis $\{\hat{\rho}, \hat{\phi}, \hat{z}\}$ associated with these coordinates. According to the elementary definitions, the z-component of the torque on the charge distribution produced by the induced electric field $\mathbf{E}$ is

$$\tau_z = \int \rho_{\rm c} (\mathbf{r} \times \mathbf{E})_z \, {\rm d}v = \int \rho_{\rm c} \rho E_\phi \, {\rm d}v.$$

(Note the presence of the charge density, $\rho_{\rm c}$, and the coordinate ρ in the last equation.)

Again, with the aid of Gauss's law and Gauss's theorem, we have

$$\tau_z = \frac{1}{4\pi} \int (\nabla \cdot \mathbf{E}') \, \rho E_\phi \, {\rm d}v = -\frac{1}{4\pi} \int \mathbf{E}' \cdot \nabla (\rho E_\phi) \, {\rm d}v. \quad (8)$$

If L_z is the z-component of the angular momentum that we have to transfer to the charges in order to keep them motionless,

$$\begin{aligned} \frac{{\rm d}L_z}{{\rm d}t} &= \frac{1}{4\pi} \int \left(E'_\rho \frac{\partial(\rho E_\phi)}{\partial \rho} + E'_\phi \frac{1}{\rho} \frac{\partial(\rho E_\phi)}{\partial \phi} + E'_z \frac{\partial(\rho E_\phi)}{\partial z} \right) {\rm d}v \\ &= \frac{1}{4\pi} \int \Bigg[E'_\rho \left(\frac{\partial(\rho E_\phi)}{\partial \rho} - \frac{\partial E_\rho}{\partial \phi} \right) \\ &\quad + E'_z \left(\frac{\partial(\rho E_\phi)}{\partial z} - \frac{\partial E_z}{\partial \phi} \right) \\ &\quad + E'_\rho \frac{\partial E_\rho}{\partial \phi} + E'_\phi \frac{\partial E_\phi}{\partial \phi} + E'_z \frac{\partial E_z}{\partial \phi} \Bigg] {\rm d}v \end{aligned}$$

which, making use of Faraday's law, amounts to

$$\begin{aligned} &\frac{\rm d}{{\rm d}t} \left\{ L_z - \frac{1}{4\pi c} \int [\mathbf{r} \times (\mathbf{E}' \times \mathbf{B})]_z {\rm d}v \right\} \\ &\quad = \frac{1}{4\pi} \int \left(E'_\rho \frac{\partial E_\rho}{\partial \phi} + E'_\phi \frac{\partial E_\phi}{\partial \phi} + E'_z \frac{\partial E_z}{\partial \phi} \right) {\rm d}v \end{aligned} \quad (9)$$

[cf. Eq. (6)]. This equation shows that if the time-dependent magnetic field and the induced electric field are invariant under rotations about the z-axis, then, at any instant, the angular momentum stored in the system is

$$L_z = \frac{1}{4\pi c} \int [\mathbf{r} \times (\mathbf{E}' \times \mathbf{B})]_z {\rm d}v, \quad (10)$$

which coincides with Eq. (4).

Equation (9) shows that in order for the stored angular momentum to have a value depending only on the final value of the magnetic field, and not on the detailed form in which the field changes with the time, it is necessary that the induced electric field be invariant under rotations about the z-axis.

4. Discussion

It should be remarked that the expressions found in the preceding section coincide with those obtained in Ref. [1], which is highly satisfactory. If we have a configuration of electric and magnetic fields with the appropriate symmetry, the transferred linear or angular momentum is the same whether we consider a fixed magnetic field and calculate the magnetic force on charges brought from infinity or we consider a fixed charge distribution in a magnetic field that is initially zero and evolves into its predetermined non-zero value.

In the approach followed in Ref. [1] the symmetry guarantees that the total transferred linear or angular momentum is path-independent, while in the approach followed here, the symmetry guarantees that the total transferred linear or angular momentum does not depend on the precise manner in which the magnetic field goes from zero to its final value.

The attribution of a linear or angular momentum to a charge distribution in a magnetic field solves the so-called Feynman's paradox (see, *e.g.*, Ref. [4]).

As pointed out in Ref. [1], the Maxwell equations lead to the definition of a density of linear momentum for the electromagnetic field, which is given by $\mathbf{E} \times \mathbf{B}/4\pi c$, without having to assume that the electromagnetic field possesses some symmetry. Despite the similarity with Eq. (2), in the standard derivation of the density mentioned above one considers the interaction of the electric and magnetic fields with the charges and currents producing them, whereas in the calculations presented here we have considered the force on a fixed electric charge distribution, neglecting the effect of the electric field produced by this charge on the sources of the magnetic field. In fact, in the calculations given in Sec. 3 it was not necessary to specify if the magnetic field is produced by currents or by permanent magnets.

It should be emphasized that the usual (gauge-dependent) electromagnetic potentials were not necessary in the calculations presented above (nor in the calculations presented in Ref. [1]).

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