An experiment for the study of projectile motion

M. S. Kovačević[a], L. Kuzmanović[a], S. Kovačević[b], and M. M. Milošević[a]

[a]*Faculty of Science, University of Kragujevac, Kragujevac, Serbia.*
[b]*The First Grammar School of Kragujevac, Kragujevac, Serbia.*

Received 30 August 2023; accepted 26 February 2024

The classic demonstration experiment of the motion of a point mass thrown at an angle to the horizon is studied. Several measurements of the range of a projectile launched sphere are carried out and compared with the results that were obtained by an analytical approach. The sphere's motion can be treated as two independent movements: a linear uniform movement in the horizontal direction and a uniformly accelerated motion in the vertical direction. The value of the launching velocity obtained from kinematics is compared with those predicted by the law of mechanical energy conservation. The conclusion is that the model of a frictionless sliding sphere is far from explaining the experimental result. The model could be improved proposing that the sphere rolls without sliding (pure rolling) on the platform including the rotational kinetic energy. Finally, the fact that the sphere does not settle completely on the launch rail was considered using an effective radius of rotation. Observed from the three proposed models, the last one is the closest to the obtained experimental value. These activities can also improve students' understanding of the concept of projectile motion.

Keywords: Projectile motion; energy conservation; rolling; mechanical energy conservation.

DOI: https://doi.org/10.31349/RevMexFisE.21.020217

1. Introduction

The problem of the motion of a projectile is an important topic and of great interest to teachers and students. Understanding projectile motion is necessary for students in middle schools, high schools, and universities. It is, therefore, significant for students to have some knowledge of the mechanical characteristics of projectile motion. There are a lot of publications that include numerical simulations with analytical solutions of this problem. The study of parabolic motion, in the absence of any drag force, is a common example in introductory physics courses. The theory of parabolic motion allows you to analytically determine the trajectory and all-important characteristics of the movement of the projectile. The solution of the projectile motion under the uniform gravitational field and without any resistant force, which is a "parabola", is well known [1,2]. However, including the air resistance forces into the study of the motion complicates the problem and makes it difficult to obtain analytical solutions. Many such solutions of a projectile motion with quadratic resistance have been obtained [3-6]. For the construction of the analytical solutions various methods are used - both the traditional approaches [7-11], and the modern methods [12]. So, the solution of the projectile motion by means of simple approximate analytical formulas under air resistance has great methodological and educational importance.

Here is a new simple and convincing demonstration based on the energy concept that we use to introduce the topic of projectile motion. It consists of the projectile launching-platform and a mechanism that permits varying angles of launch angle. The experiment can be effectively integrated into classroom demonstrations or become part of laboratory activities. It can be used to show interesting connections between topics such as rolling motion, conservation of mechanical energy, and projectile motion. This experiment's design could become a solid addition to students' lab experiences.

2. The apparatus

An inexpensive apparatus for demonstrating the laws of projectile motion is shown in Fig. 1. It consists of a launching platform and a mechanism that permits varying the angle of elevation (the angle α degrees from the horizontal $0 < \alpha < 90°$C). A smooth spherical ball with radius R is positioned at height H on the launching platform. Note that three positions were drawn: the first one at the start position at the height H of the launching platform; the second one at the bottom of the curved path with height h; and the third one when the sphere reaches the workbench. This experiment permits a quantitative study of the motion of the projectile; it is interesting to compare the initial launching velocity value obtained by kinematics with those predicted by the mechanical energy.

As you can see in Fig. 1, the apparatus consists of two wooden boards, horizontal and vertical, that form a right angle. This is a stand on which a plastic launching platform is mounted. In our experiment, as a plastic launching guide, we used the lid part of the channel that is used to fix cables on the walls of the rooms (cross-section is shown in Fig. 2). This is inexpensive and it can be found for 1 euro per meter. One end of the plastic guide is attached to the vertical part of the tripod, while the other end is bent at a certain angle (the angle can be changed using the angle positioner). At this end of the rail plastic guide, there is a platform in the horizontal plane where the distance D of the ball can be easily measured. As a projectile, we used a wooden ball that is common in the

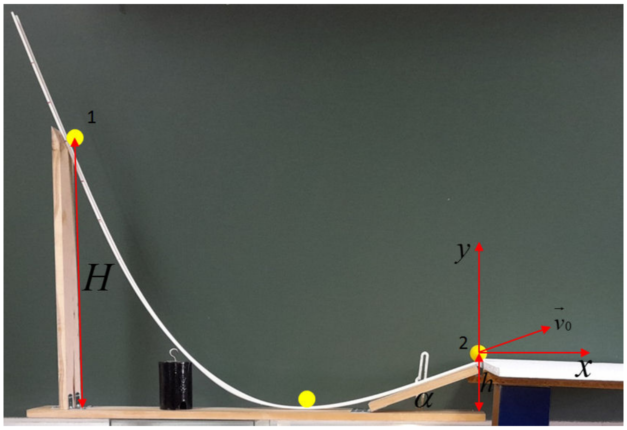

FIGURE 1. Apparatus dedicated to the study of projectile motion.

children's toy sets. We chose the ball so that its dimensions correspond to the width of the plastic rail that we used as a jumping platform.

First, it is required that the launching platform be horizontally leveled. The experimental procedure must ensure that the experiment is always conducted under the same conditions, that is to say, the same starting height H for the sphere (this point is labeled with 1 in the Fig. 1).

3. The governing equation

In projectile motion, a particle is launched into the air with a speed v_0 and at an angle α (as measured from a horizontal x-axis). The sphere's motion can be treated as two independent movements, a linear uniform movement in the horizontal direction and a uniformly accelerated motion in the vertical direction. During flight, its horizontal acceleration is zero and its vertical acceleration is $-g$ (downward on a vertical y-axis). If air resistance is neglected, the equations of motion for the particle (while in flight) can be written as

$$x = v_{0x}t = (v_0 \cos\alpha)t, \tag{1a}$$

$$y = v_{0y}t - \frac{gt^2}{2} = (v_0 \sin\alpha)t - \frac{gt^2}{2}, \tag{1b}$$

$$v_x = v_{0x} = v_0 \cos\alpha, \tag{1c}$$

$$v_y = v_{0y} - gt = v_0 \sin\alpha - gt, \tag{1d}$$

$$v_y^2 = (v_0 \sin\alpha)^2 - 2gy. \tag{1e}$$

The trajectory (path) of a particle in projectile motion is parabolic and is given by

$$y = x \tan\alpha - \frac{gx^2}{2(v_0 \cos\alpha)^2}. \tag{2}$$

The particle's horizontal range D, which is the horizontal distance from the launch point to the point at which the particle returns to the launch height, is

$$D = \frac{v_0^2}{g} \sin 2\alpha. \tag{3}$$

4. Projectile motion and conservation of energy: Results and discussion

We will let the sphere slide down on a frictionless incline (launching platform) to measure the range of the projectile D (Fig. 2). In practical realization a spherical object rolls along a track made of a plastic U-shaped strip. The width of the groove in the platform is L. The sphere used in the experiment was a wooden ball of diameter $2R$. At the beginning, the starting position of the ball on the platform is at height H. Since the sphere's diameter is larger than the groove's width $(2R > L)$ (see Table I), the sphere has contact with the groove only at two points as shown in Fig. 2.

From the horizontal range formula, which derives from the kinematic equations, we can express the initial launching velocity as

$$v_0 = \sqrt{\frac{gD}{\sin 2\alpha}}. \tag{4}$$

For an elevation angle $\alpha = 19.5°$C, we obtained the initial velocity of $v_0 = 2.48$ m/s. It is interesting to compare this value of v_0 obtained from kinematic equations with the velocity of the sphere's center of mass v_{CM} predicted using mechanical energy conservation. Three models were considered: the frictionless sliding sphere model, the model that includes the rotation of the sphere, and finally the model with the sphere of radius R that rolls down the groove of width L and has an effective radius, K, which should be used to relate both the angular velocity and the CM velocity.

TABLE I. Numerical values obtained by measurements and calculations.

H (m)	h (m)	D (m)	R (m)	L (m)	Sphere velocity (the initial velocity v_{CM})	
0.7	0.13	0.394	0.0239	0.0188	v_0 (m/s)	$\frac{v_{CM}-v_0}{v_0}$ (%)
The result obtained from the kinematic equations					2.48	
Law of conservation of mechanical energy: *Model 1*					3.34	34.8
Law of conservation of mechanical energy: *Model 2*					2.83	14
Law of conservation of mechanical energy: *Model 3*					2.66	7.2

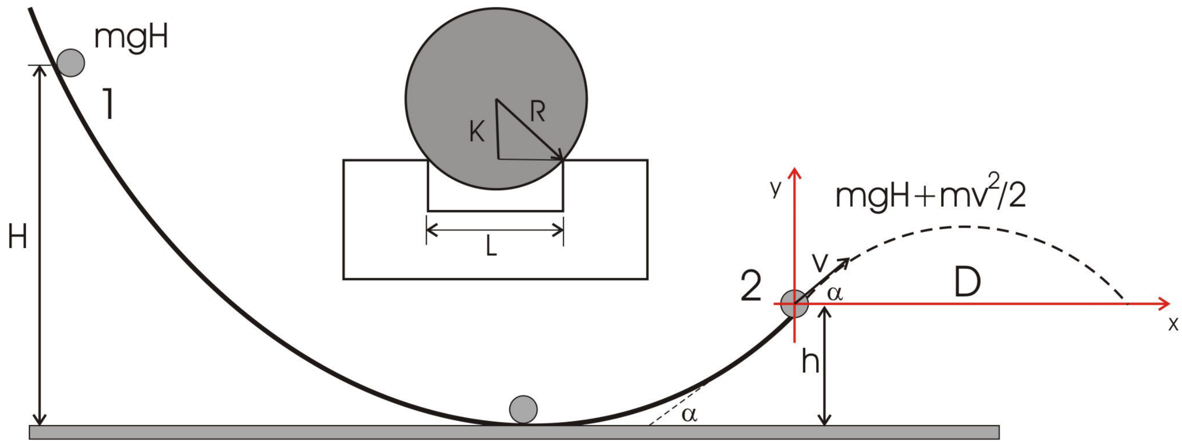

FIGURE 2. Rolling ball for projectile motion. Measurement of range D of projectile and effective radius of rotation K ($K < R$).

Model 1: The spherical ball of mass m and radius R starts moving from a height H (point 1 in Fig. 2) and has the initial potential energy mgH. Leaving the groove (point 2 in Fig. 2), the sphere's potential energy is mgh, and its kinetic energy is $mv_0^2/2$ (it's assumed that $v_0 = v_{CM}$). In this model, the movement of the sphere ball corresponds to the frictionless sliding of the sphere. When the sphere rolls down an incline on a track, the final mechanical energy should be equal to the initial one, considering the sphere as a particle that slides without friction over the platform. The law of conservation of mechanical energy has the following form:

$$mgH = mgh + \frac{mv_{CM}^2}{2}. \tag{5}$$

From this equation the velocity v_{CM} is given by

$$v_{CM} = \sqrt{2g(H-h)}. \tag{6}$$

Using the height of the platform $H = 0.7$ m and the height $h = 0.13$ m, and the standard value of the local acceleration of gravity, we obtain the estimate of the CM velocity as $v_{CM} = 3.34$ m/s. Comparing this result to the value obtained from Eq. (4), we find a high relative error: $(v_{CM} - v_0)/v_0 = 34.8\%$. We can conclude that the model of a sphere sliding without friction down the steep plane's groove is far from explaining the experimental result and is not adequate for determining the initial launch velocity of the projectile motion. We can improve this model by proposing that the sphere rolls without sliding (pure rolling) on the platform.

Model 2: This model considers the rotation of the sphere as it rolls down the steep plane's groove. To fulfill this condition, we need to include the rotational kinetic energy on the left side of the Eq. (5) (the rotation energy of sphere is $E_{rot} = I\omega^2$, where $I = 2mR^2/5$ is the moment of inertia in relation to the axis of rotation passing through the center of the sphere, and ω is the sphere's the angular velocity). With the rotation considered, the law of conservation of mechanical energy has the form:

$$mgH = mgh + \frac{mv_{CM}^2}{2} + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\omega^2. \tag{7}$$

Remembering that in the case of pure rolling $v_{CM} = \omega R$, the solution of Eq. (7) can be written as:

$$v_{CM} = \sqrt{\frac{10}{7}g(H-h)}. \tag{8}$$

Using the values of g, H and h cited before, we obtain $v_{CM} = 2.83$ m/s. Applying the analogous procedure, we find that this still corresponds to a considerable relative error of $(v_{CM} - v_0)/v_0 = 14\%$. It is obvious that this model gives a much smaller relative error compared to the result in Model 1. Further improvement of the model is the fact that the sphere does not settle completely on the launch rail, as can be seen in detail of the scheme presented in Fig. 2.

Model 3: When the sphere of radius R rolls down the groove of width L, it has contact with the edges of the groove at only two points (Fig. 2). Figure 2 shows that there is an effective radius, K, which should be used to relate both the angular velocity and the CM velocity. From Fig. 2 we find that the effective radius of rotation of the sphere is $K = (R^2 - L^2/4)^{1/2}$, where L is the width of the launch rail. Even when considering the pure rolling, the relation for angular velocity $\omega = v_{CM}/R$ needs to be replaced by $\omega = v_{CM}/K$. In this case, the law of conservation of mechanical energy becomes:

$$mgH = mgh + \frac{mv_{CM}^2}{2} + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\frac{v_{CM}^2}{K^2}. \tag{9}$$

After some algebraic manipulations we can write CM velocity v_{CM} (initial launching velocity) as

$$v_{CM} = \sqrt{\frac{2}{1+\dfrac{2}{5[1-(L/2R)^2]}}g(H-h)}. \tag{10}$$

Using the values of R and L that given in Table I, we obtain $v_{CM} = 2.66$ m/s, which corresponds to a reasonably low relative error when compared with the experimental data in Eq. (4): $(v_{CM} - v_0)/v_0 = 7.2\%$. The estimated relative error is the smallest, and this model seems to get closer to the obtained experimental value.

Frequently, this experiment is presented in experimental physics handouts assuming that the sphere always executes a pure rolling all the way until being launched, allowing the use of the mechanical energy conservation. This assumption is the reason for the systematic difference between the predicted energy balance and the one obtained by experimental measurements. From the experimental point of view, it is necessary to have a data acquisition system in real time to provide the sphere's position at every instant. The experiment can be improved by introducing the sensors and video analysis but that wouldn't be an inexpensive tool. This could reveal interesting characteristics of the movement that are frequently overloaded with the traditional approach. From the mathematical point of view, the curved platform implies a continuous variation of the inclination angle of the launching rail, causing the variation of both the normal and the friction force along the way. Then, the equations of motion are nonlinear, and its analytical solutions are not available, which prevents the dynamic modelling.

5. Conclusion

This paper describes an inexpensive experiment that provides a demonstration of a projectile motion. The experiment is used to determine the initial (launch) velocity of the ball launched from the inclined launching platform. This demonstration consists of a spherical ball rolling down the launching ramp, gaining the initial velocity. This paper describes the three different models of projectile motion for the physics classroom. These models are based on the mechanical energy conservation. The motion on an incline provides an effective means for the discussion of the conservation of mechanical energy and the transformation of potential into kinetic energy. The energy concept is fundamentally important for describing and analyzing projectile motion. Within the subject that is regarded in this work, it should be pointed out that, strictly speaking, the conservation of mechanical energy is valid only when no energy is transferred across the boundary of the system being considered. Here, the system being considered is a small sphere moving down an inclined plastic launching platform. The spherical ball has contact with the platform at only two points. In this scenario, very little energy is transferred to the track due to low friction. Through this experiment, the activities of students were emphasized to enhance students' understanding of the scientific concept of mechanical energy conservation. First was determined the initial velocity v_0 of the projectile motion by kinematic equations. The value of v_0 was compared with the values of the v_{CM} obtained by the energy concept in the proposed models. From the three proposed models, when considering the pure rolling of a sphere on the rail of the platform using the effective radius K, this model seems to get the closest to the obtained experimental value. This simple apparatus will provide a verification of the projectile theory that is acceptable to most students. When implementing this laboratory investigation, we propose to assign each student group a particular angle for their experiment. After performing the measurement of the projectile range, the students are challenged to use conservation of energy and kinematic equations to compare the calculations of the initial velocity. Then the students carry out the comparison to see how close the initial velocity v_0 obtained by kinematic equations is to the v_{CM} velocity predicted by mechanical energy conversation. This experiment can provide the basis for more advanced experiments such as the study of two-dimensional projectile motion in which the resistance acting on an object moving in air is proportional to the square of the velocity of the object (quadratic resistance law).

Acknowledgments

This work was supported by the Serbian Ministry of Science, Technological Development, and Innovations (Agreement No. 451-03-65/2024-03/200122).

1. C. Kittel, W. D. Knight, and M. A. Ruderman, Mechanics, Berkeley Physics Course Vol. 1, 2nd ed. (Mc Graw Hill Book Company, New York, 1973).
2. D. Halliday and R. Resnick, Fundamentals of Physics (Willey, New York, 1986).
3. P. S. Chudinov, Analytical investigation of point mass motion in midair, *Eur. J. Phys.* **25** (2004) 73, https://dx.doi.org/10.1088/0143-0807/25/1/010
4. P. S. Chudinov, The motion of a point mass in a medium with a square law of drag, *J. Appl. Math. Mech.* **65** (2001) 421, https://doi.org/10.1016/S0021-8928(01)00047-8
5. P. S. Chudinov, The motion of a heavy particle in a medium with quadratic drag force, *Int. J. Nonlinear Sci. Numer. Simul.* **3** (2002) 121, https://doi.org/10.1515/IJNSNS.2002.3.2.121
6. G. W. Parker, Projectile motion with air resistance quadratic in the speed, *Am. J. Phys.* **45** (1977) 606, https://doi.org/10.1119/1.10812
7. J. Benacka, Solution to projectile motion with quadratic drag and graphing the trajectory in spreadsheets, *Int. J. Math. Educ. Sci. Technol.* **41** (2010) 373, https://doi.org/10.1080/00207390903189229
8. A. Vial, Horizontal distance travelled by a mobile experiencing a quadratic drag force: normalized distance and parameteriza-

tion, *Eur. J. Phys.* **28** (2007) 657, https://dx.doi.org/10.1088/0143-0807/28/4/005

9. J. W. Gibson, Convenient equations for projectile motion, *Am. J. Phys.* **29** (1961) 623, https://doi.org/10.1119/1.1937861

10. H. Erlichson, Maximum projectile range with drag and lift, with particular application to golf, *Am. J. Phys*. **51** (1983) 357, https://doi.org/10.1119/1.13248

11. A. Tan, C. H. Frick, and O. Castillo, The fly ball trajectory: an older approach revisited, *Am. J. Phys*. **55** (1987) 37, https://doi.org/10.1119/1.14968

12. K. Yabushita, M. Yamashita, and K. Tsuboi, An analytic solution of projectile motion with the quadratic resistance law using the homotopy analysis method, *J. Phys. A: Math. Theor*. **40** (2007) 8403, https://dx.doi.org/10.1088/1751-8113/40/29/015