Non-conventional coherent states

A. Plastino[a,b] and G. L. Ferri[c]

[a]*Departamento de Física, Universidad Nacional de La Plata, Argentina,*
[b]*Argentina's IFLP-CCT-Conicet-Consejo Nacional de Investigaciones Científicas y Tecnológicas.*
[c]*Universidad Nacional de La Pampa, Sta. Rosa, Argentina.*

Received 11 October 2023; accepted 22 March 2024

We analyze properties of excited states of a special Hamiltonian H, endowed with a displaced quadratic potential V such that its ground state is a HO-Glauber coherent state of amplitude α. For large enough real α, we show that the first excited state of this Hamiltonian is also a coherent state of minimum uncertainty.

Keywords: Harmonic oscillator; Glauber states; exact analytic forms; excited states.

DOI: https://doi.org/10.31349/RevMexFis.22.010214

1. Introduction

Coherent states in quantum mechanics exhibit properties that resemble classical fields, making them a special and unique class of quantum states with classical-like behavior. The radiation field in a coherent state is a prominent example where these classical-like properties are particularly evident. In the context of quantum optics, a coherent state is a particular type of quantum state that can describe the electromagnetic radiation field of a light mode. The coherent state is often denoted as α, where α is a complex number representing the coherent amplitude. Some properties of the radiation field in a coherent state that resemble classical fields are:

- Classical-Like Mean Field: In a coherent state, the expectation value (mean) of the field's electric field amplitude is proportional to the coherent amplitude α. This property resembles the classical concept of a well-defined mean field amplitude.

- Minimum Uncertainty: Coherent states are eigenstates of the annihilation operator (lowering operator), and they exhibit the minimum possible uncertainty in the field's electric field amplitude and phase. This minimum uncertainty is reminiscent of classical fields, where both amplitude and phase can be precisely determined.

- Gaussian Probability Distribution: The probability distribution of measuring a particular field amplitude in a coherent state follows a Gaussian distribution. This distribution is characteristic of classical fields with well-defined coherent amplitudes.

- Stable Over Time: Coherent states are temporally stable and do not exhibit significant quantum fluctuations in the field's amplitude over time. This stability is akin to the classical behavior of classical waves.

- Classical Intensity Interference: The interference patterns observed in coherent light beams are similar to those produced by classical waves, as coherent states maintain their coherence properties even after superposition with other coherent states.

- Classical-Like Optical Interference: In classical optics, coherent light sources are known to produce optical interference patterns with bright and dark fringes. The interference of coherent states exhibits similar bright and dark fringes, akin to classical optical interference patterns.

These classical-like properties of the radiation field in a coherent state arise from the specific nature of the coherent state itself. Coherent states are superpositions of Fock states (number states) with a specific coherent amplitude, and this superposition results in the quantum state exhibiting properties that closely resemble classical fields. The classical-like behavior of coherent states has practical implications and applications in quantum optics and quantum information processing, where coherent states are used in quantum communication, quantum cryptography, and quantum key distribution due to their robustness against certain types of noise and their resemblance to classical fields.

As we see, coherent states are highly significant in several areas of physics, particularly in quantum mechanics, quantum optics, quantum information, quantum computing, and quantum metrology. Their mathematical properties and close connection to classical states make them valuable tools for both theoretical and experimental investigations in these fields.

1.1. Non conventionaal coherent states

Introducing **new** types of coherent states, a done for example in the excellent Ref. [1], can deepen our understanding of quantum systems. By examining different mathematical representations and properties, we can gain insights into the fundamental nature of quantum mechanics and its connections to classical physics. It allows us to explore the boundaries and possibilities of coherent states beyond the standard formulations. We mention below some possibilities:

1) Enhanced Description of Physical Systems: Different physical systems may have unique characteristics that are not fully captured by standard coherent states. By discovering new types of coherent states tailored to specific systems, we can provide more accurate and comprehensive descriptions of these systems. This can lead to better models and predictions for a wide range of phenomena, including molecular vibrations, quantum gases, and complex quantum networks.

2) Applications in Quantum Technologies: Coherent states are essential resources in various quantum technologies, including quantum communication, quantum computing, and quantum metrology. Finding new types of coherent states can potentially enhance the capabilities and performance of these technologies. For example, new coherent states with improved properties, such as higher squeezing or reduced noise, could enable more efficient quantum information processing or more precise quantum measurements.

3) Quantum Control and Manipulation: Coherent states are often used as starting points for quantum control and manipulation techniques. Discovering new types of coherent states with specific properties could provide alternative avenues for manipulating quantum systems. This could lead to advancements in quantum control methods, such as state preparation, state engineering, and quantum feedback control, enabling better manipulation and utilization of quantum resources.

4) Fundamental Physics: Coherent states have connections to various fundamental principles and phenomena in physics, such as symmetry, quantization, and entanglement. Exploring new types of coherent states may uncover previously unknown connections and deepen our understanding of these fundamental aspects. This can contribute to the development of new theoretical frameworks and concepts in physics.

In summary, the discovery of new types of coherent states has the potential to enrich our understanding of quantum systems, improve our ability to describe and manipulate physical systems, enhance quantum technologies, and shed light on fundamental principles of physics. It is an exciting avenue of research with promising prospects for advancing both theoretical and applied physics [1].

1.2. Our goal

We will here analyze some hopefully interesting properties of *excited states* of a special Hamiltonian H, endowed with a displaced quadratic potential V such that its ground state is an HO-Glauber coherent state /CS). The amplitude of this CS is called α. Our main result is that, for large enough real α, the pertinent first excited state of this special Hamiltonian is also a coherent state of minimum uncertainty.

1.3. Basic considerations

For starters, let us recapitulate notions regarding Glauber coherent states of the harmonic oscillator (HO) of amplitude $|\alpha\rangle$ [2–4]. A Glauber state (CS) α is a very special sort of quantum state, the one of minimum uncertainty, so that it most resembles a classical state. Note that α is a complex variable.

These Glauber states are employed in manifold ways, for instance with regard to the quantum harmonic oscillator, the electromagnetic field, etc. They portray a maximal kind of coherence and a classical type of behavior. Our states $|\alpha\rangle$ are normalized, that is, $\langle\alpha|\alpha\rangle = 1$. They provide a resolution of the identity operator

$$\int \frac{\mathrm{d}^2\alpha}{\pi}\, |\alpha\rangle\langle\alpha| = 1, \tag{1}$$

a completeness relation for the coherent states [4]. The Glauber states $|\alpha\rangle$ for the harmonic oscillator are, by definition, eigenstates of the annihilation operator $\hat{a}$: The eigenvalues are complex. In other words,

$$\alpha = \frac{q+ip}{\sqrt{2}}, \tag{2}$$

which satisfy $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ [4].

It is a text book recipe that the $n-$th HO eigenfunction reads

$$\phi_n(x) = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{4}} \mathcal{H}_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right). \tag{3}$$

$\mathcal{H}_n$ is Hermite's $n-$th order generalized function

$$\mathcal{H}_n(x) = \left(\pi^{\frac{1}{2}}2^n n!\right)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_n(x), \tag{4}$$

and H_n is the associated Hermite polynomial. In the x-representation one writes for the coherent state

$$\psi_\alpha(x) = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\phi_n(x), \tag{5}$$

and also

$$\psi_\alpha(x) = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{4}} e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\mathcal{H}_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right). \tag{6}$$

For convenience, we set below $\sqrt{m\omega/\hbar} = 1$. We use the fact that the quantum harmonic oscillator possesses natural scales for length and energy, which can be used to simplify the notation. These can be found by nondimensionalization. The Hamiltonian simplifies to

$$H = (1/2)\left(-\frac{d^2}{dx^2} + x^2\right). \tag{7}$$

Thus, for the HO one thus has

$$\phi_n(x) = \mathcal{H}_n\left(x\right), \tag{8}$$

while for its coherent states (CS) one writes

$$\psi_\alpha(x) = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \mathcal{H}_n\left(x\right). \tag{9}$$

2. Compact analytic form for a Glauber coherent state (CS)

The CS can be presented in a compact analytic form in quantum mechanics' x representation, as reported in [6]. To do so, one starts with the annihilation operator for the one-dimensional harmonic oscillator

$$\hat{a} = \frac{\hat{x} + i\hat{p}}{\sqrt{2}}, \tag{10}$$

that in the x-representation is expressed as

$$\hat{a}(x) = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right). \tag{11}$$

Precisely, Glauber stated that coherent states are the eigenfunctions of $\hat{a}(x)$ (α is a complex quantity)

$$\hat{a}(x)\psi_\alpha(x) = \frac{1}{\sqrt{2}}\left(x\psi_\alpha(x) + \frac{d\psi_\alpha(x)}{dx}\right) = \alpha\psi_\alpha(x). \tag{12}$$

Thus,

$$\frac{d\psi_\alpha(x)}{dx} = (\sqrt{2}\alpha - x)\psi_\alpha(x), \tag{13}$$

a quite simple enough equation of solution

$$\psi_\alpha(x) = Ce^{-\frac{x^2}{2}} e^{\sqrt{2}\alpha x}. \tag{14}$$

C is determined using the normalization process requirement

$$\int_{-\infty}^{\infty} |\psi_\alpha(x)|^2 dx = |C|^2 \int_{-\infty}^{\infty} e^{-x^2} e^{\sqrt{2}(\alpha+\alpha^*)x} dx = 1. \tag{15}$$

From the precedent equation, we gather that

$$\int_{-\infty}^{\infty} |\psi_\alpha(x)|^2 dx = |C|^2 e^{\frac{(\alpha+\alpha^*)^2}{2}} \times \int_{-\infty}^{\infty} e^{-\left(x - \frac{\alpha+\alpha^*}{\sqrt{2}}\right)^2} dx = 1. \tag{16}$$

Using the Table [7] we get

$$\int_{-\infty}^{\infty} e^{-\left(x - \frac{\alpha+\alpha^*}{\sqrt{2}}\right)^2} dx = \sqrt{\pi}. \tag{17}$$

Thus,

$$C = \pi^{-\frac{1}{4}} e^{-\frac{(\alpha+\alpha^*)^2}{4}}. \tag{18}$$

Remind that, in obvious notation, $\alpha = \alpha_R + i\alpha_I$ and $\alpha + \alpha^* = 2\alpha_R$. Also, $\exp\alpha x = \exp(\alpha_R x)\exp(i\alpha_I x)$. The second factor is just a phase factor. Accordingly, $\psi_{(\alpha)}(x)$ attains the compact form

$$\psi_\alpha(x) = \pi^{-\frac{1}{4}} e^{-(\alpha_R)^2} e^{(-\frac{x^2}{2})} e^{(\sqrt{2}\alpha_R x)}, \tag{19}$$

where, let us repeat, we omitted a phase factor $\exp i\sqrt{2}\alpha_I x$.

Properties of the above cast ψ function are detailedly discussed in Ref. [8], where it is specified just how ψ is identical with the Glauber Fock expansion usually used to define coherent states. A peculiar detail fact can here be mentioned. The form of ψ is identical to that of a wave packet of Harmonic Oscillator eigenfunctions studied in Ref. [9]. See also Ref. [10].

2.1. ψ-partner potential

Appropriate small changes lead to a Gaussian $\psi(y)$ with $(y^2/2) = (x/\sqrt{2} - \alpha_R)^2$, $y = x - \sqrt{2}\alpha_R$ and $\psi_\alpha(y) = (1/\pi^4)e^{-y^2/2}$.

It is clear that, from Schrödinger equation, for ψ we have a partner potential $V(x)$ and an energy eigenvalue E given by

$$2(V(y) - E) = \frac{\psi''(y)}{\psi(y)}, \tag{20}$$

so that

$$(1/2)\frac{\psi''(y)}{\psi(y)} = y^2/2 - 1/2 = V - E, \tag{21}$$

implying

$$V(y) = y^2/2; \quad y = x - \sqrt{2}\alpha_R, \tag{22}$$

and

$$E = 1/2. \tag{23}$$

Accordingly, Glauber coherent states can be regarded as eigenfunctions of an α_R-displaced quadratic potential V.

3. Excited states of the displaced potential V

In an excellent wprk, Agarwal and Tara, in 1991 introduced a new class of states defined as 'm' times application of creation operator to Coherent States known as Excited Coherent States (ECS) or Photon Added Coherent States (PACS) [1].

We look here seek instead for purely quantum excited states of the Hamiltonian whose potential is V of (22). These states are to be compared to those of [1, 11]. We set $y = x - \sqrt{2}\alpha_R$ and then look for the overlap $O_{m,n}$ between the m-th eigenstate of the displaced potential and the n-th one of the ordinary Harmonic Oscillator. H_n are Hermite polynomials. We are led to

$$O_{m,n} = \frac{1}{2^m\sqrt{n!}\sqrt{\pi}} \int_{-\infty}^{\infty} dx \, \exp(-x^2) \times H_m(x-\sqrt{2}\alpha_R)\, H_n(x+\sqrt{2}\alpha_R). \quad (24)$$

Recourse to the Table in [7] yields for the overlap

$$O_{m,n} = \frac{1}{2^m\sqrt{n!}} \left[\sqrt{m!}(\sqrt{2}\alpha_R)^{n-m} L_m^{n-m}(4\alpha_R)^2)\right]. \quad (25)$$

L_m^n are associated Laguerre polynomials and one requires $m \leq n$. Remind that

$$L_m^n(x) = \sum_{i=0}^{m} (-1)^i (m+n)! x^i / [(m-i)!(n+i)!i!]. \quad (26)$$

Now, the ψ_m associated to of the displaced quartic potential are expressed, in terms of the HO eigenstates ϕ_n as

$$\psi_m = \sum_n O_{m,n}\phi_n = \sum_n c_n\phi_n. \quad (27)$$

The m-th excited state can be regarded as a Glauber state [1, 11]. Note however that Eq. (24) can be used only if $m \leq n$. If this is not so, one must interchange sub-indexes in that equation. In particular, for $m = 1$ we have

$$\xi = \psi_1 = \left[\sum_{n\geq 1} \frac{1}{\sqrt{n!}} (\sqrt{2}\alpha_R)^{n-1} L_1(4\alpha_R^2)^{n-1}\phi_n\right] + (1/2)\phi_0. \quad (28)$$

$|O_{m,n}|^2$ is of course the probability $|c_n|^2$ that ψ_m coincides with the ordinary OH state ϕ_n. Also, $L_1^{n-1}(x) = n - x$. Thus,

$$\xi = \psi_1 = \left[\sum_{n\geq 1} \frac{(\sqrt{2}\alpha_R)^{n-1}}{\sqrt{n!}} (n - 4\alpha_R^2)^{n-1}\phi_n\right] + (1/2)\phi_0. \quad (29)$$

4. ξ-Uncertainties

We assume now that α is real. We pass here to our main concern: the quantum uncertainties associated to our excites state ξ. Remember that our coherent state and also ground state of the displaced Hamiltonian is

$$\psi_\alpha(x) = \pi^{-1/4} e^{-\alpha^2} e^{-\frac{x^2}{2}} e^{\sqrt{2}\alpha x}, \quad (30)$$

while the pertinent first excited state reads

$$\xi_\alpha(x) = a^\dagger \Psi_\alpha(x) = \frac{1}{\sqrt{2}} \left(x - \frac{d}{dx}\right) \psi_\alpha(x), \quad (31)$$

i.e.,

$$\xi_\alpha(x) = \frac{\pi^{-1/4} e^{-\alpha^2}}{\sqrt{2}} (2x - \sqrt{2}\alpha) e^{-\frac{x^2}{2}} e^{\sqrt{2}\alpha x}, \quad (32)$$

whose normalization requires

$$\int_{-\infty}^{+\infty} |\xi_\alpha(x)|^2 dx = 1 + \alpha^2, \quad (33)$$

and then

$$\xi_\alpha(x) = \frac{\pi^{-1/4} e^{-\alpha^2}}{\sqrt{2}\sqrt{1+\alpha^2}} (2x - \sqrt{2}\alpha) e^{-\frac{x^2}{2}} e^{\sqrt{2}\alpha x}, \quad (34)$$

Below, we repeatedly use Ref. [7].

4.1. Mean value of x

We have

$$\langle x \rangle = \int_{-\infty}^{+\infty} x |\xi_\alpha(x)| dx, \quad (35)$$

and

$$\langle x \rangle = \sqrt{2}\alpha \frac{2+\alpha^2}{1+\alpha^2}. \quad (36)$$

4.2. Mean value of x^2

We start with

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 |\xi_\alpha(x)| dx, \quad (37)$$

and then

$$\langle x^2 \rangle = \frac{3 + 13\alpha^2 + 4\alpha^4}{2(1+\alpha^2)}. \quad (38)$$

4.3. x-Uncertainty

It is

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}, \quad (39)$$

i.e.,

$$\Delta x = \sqrt{\frac{3+\alpha^4}{2(1+\alpha^2)^2}}. \quad (40)$$

Note that there cannot be squeezing in Δx.

4.4. Mean value of p

Focus attention on

$$\langle p\rangle = \int_{-\infty}^{+\infty} \xi_\alpha^*(x)\hat{p}\xi_\alpha(x)dx, \tag{41}$$

$$\hat{p} \equiv -i\hbar\frac{d}{dx}, \tag{42}$$

and then

$$\langle p\rangle = -i\hbar\int_{-\infty}^{+\infty} \xi_\alpha^*(x)\frac{d}{dx}\xi_\alpha(x)dx, \tag{43}$$

i.e.,

$$\langle p\rangle = 0. \tag{44}$$

4.5. Mean value of p^2

Begin with

$$\langle p^2\rangle = -\hbar^2\int_{-\infty}^{+\infty} \xi_\alpha^*(x)\frac{d^2}{dx^2}\xi_\alpha(x)dx, \tag{45}$$

leading to

$$\langle p^2\rangle = \hbar^2\frac{3+\alpha^2}{2(1+\alpha^2)}. \tag{46}$$

4.6. p-Uncertainty

One has

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}, \tag{47}$$

and

$$\Delta p = \hbar\sqrt{\frac{3+\alpha^2}{2(1+\alpha^2)}}. \tag{48}$$

Note that there cannot be squeezing in Δp.

4.7. ξ-Uncertainty

It becomes

$$\Delta x\Delta p = \frac{1}{2(1+\alpha^2)}\sqrt{\frac{(3+\alpha^2)(3+\alpha^4)}{1+\alpha^2}}, \tag{49}$$

our main result.

We plot this uncertainty in Fig. 1. It is seen that for large enough α our ξ is a coherent state, our main result.

Figure 1 plots, for the excited states ξ, the associated Heisenberg uncertainty versus value of real amplitude α. If this amplitude is large enough we reach minimum uncertainty, which makes ξ a coherent state.

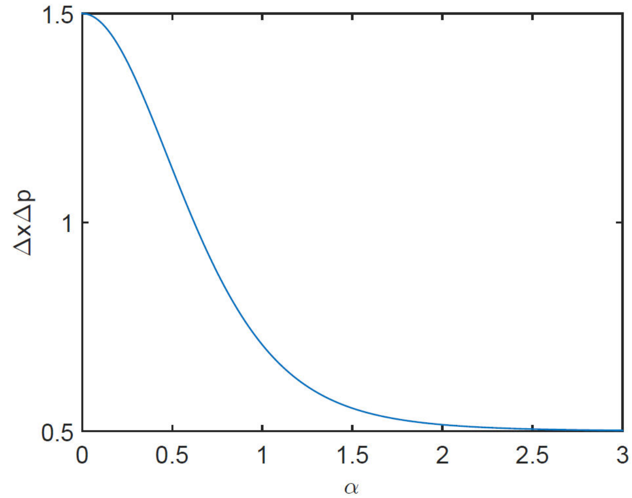

FIGURE 1. Excited states ξ-associated Heisenberg uncertainty versus value of real amplitude α. If this amplitude is large enough we reach minimum uncertainty, which makes ξ a coherent state.

5. Uncertainty ands second excited state η

Let us construct the second excited state from the first one ξ:

$$\xi_\alpha(x) = \frac{\pi^{-1/4}e^{-\alpha^2}}{\sqrt{2}\sqrt{1+\alpha^2}}(2x-\sqrt{2}\alpha)e^{-\frac{x^2}{2}}e^{\sqrt{2}\alpha x}, \tag{50}$$

that is

$$\eta_\alpha^*(x) = a^\dagger\xi = \frac{1}{\sqrt{(2)}}\left(x - \frac{d}{dx}\right)\xi_\alpha(x), \tag{51}$$

i.e.,

$$\eta_\alpha^*(x) = \frac{\pi^{-1/4}e^{-\alpha^2}}{2\sqrt{1+\alpha^2}} \times(-2+2\alpha^2-4\sqrt{2}\alpha x+4x^2)e^{-\frac{x^2}{2}}e^{\sqrt{2}\alpha x}. \tag{52}$$

Normalization of η entails:

$$\int_{-\infty}^{+\infty}|\eta_\alpha^*(x)|dx = \frac{2}{1+\alpha^2}(2+4\alpha^2+\alpha^4), \tag{53}$$

and

$$\eta_\alpha(x) = \frac{\pi^{-1/4}e^{-\alpha^2}}{2\sqrt{2+4\alpha^2+\alpha^4}} \times(-2+2\alpha^2-4\sqrt{2}\alpha x+4x^2)e^{-\frac{x^2}{2}}e^{\sqrt{2}\alpha x}. \tag{54}$$

A rather laborious and tedious manipulation gives the Heisenberg uncertainty versus α that we plot in Fig. 2. We see that for (real) α large enough we attain minimum uncertainty and thus coherent states.

Figure 2 plots, for the excited states η, the associated Heisenberg uncertainty versus value of real amplitude α. If this amplitude is large enough we reach minimum uncertainty, which makes η a coherent state.

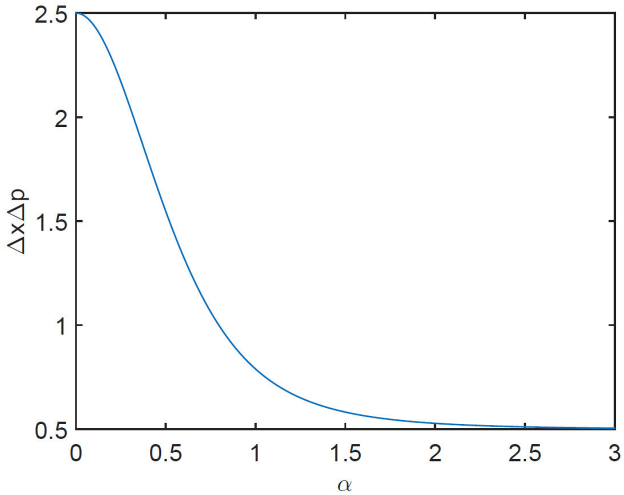

FIGURE 2. Second excited state η-associated Heisenberg uncertainty versus value of real amplitude α. If this amplitude is large enough we reach minimum uncertainty, which makes η a coherent state.

6. Conclusions

- The ordinary Glauber coherent states of amplitude α are just ground states eigenfunctions of an α_R displaced harmonic oscillator of potential $V(x)$.
- The Hamiltonian whose displaced potential is $V(x)$ has, of course, excited states
- These excited states are non-conventional coherent states of amplitude α.
- We have here established their form in terms of the eigenstates of the ordinary HO.
- The n-th excited state can be regarded as a Glauber state to which one adds n photons.

Note that Excited Coherent states of [11] instead exhibit mixtures of both coherent states (which are quantum mechanical analogs of classical oscillator) and Fock states (strictly quantum with no classical analog). Thus, they represent Quantum fluctuations in simple quantum coherent states [11].

Our excited coherent states instead are simply quantum excited states of an α-displaced potential V. For large enough real α we reach minimum uncertainty, which, as we have showed here, makes the first and second excited state a new kind of coherent states. We speculate that the same happens with higher excited states of our peculiar Hamiltonian.

Availability statement

All necessary data are contained within the text.

Authors' declaration

Conflicts of interest

The authors have no conflicts to disclose.

1. G. S. Agarwal and K. Tara, Nonclassical properties of states generated by the excitations on a coherent state, *Phys. Rev. A* **43** (1991) 492, `https://doi.org/10.1103/PhysRevA.43.492`
2. R. J. Glauber, *Phys. Rev. Lett*. **10** (1963) 84, `https://doi.org/10.1103/PhysRevLett.10.84`.
3. R. J. Glauber, The Quantum Theory of Optical Coherence, *Phys. Rev.* **130** (1963) 2529, `https://doi.org/10.1103/PhysRev.130.2529`
4. R. J. Glauber, *Phys. Rev*. **131** (1963) 493.
5. R. J. Glauber, Coherent and Incoherent States of the Radiation Field, *Phys. Rev.* **131** (1963) 2766, `https://doi.org/10.1103/PhysRev.131.2766`.
6. G. L. Ferri *et al.*, New mathematics for the nonadditive Tsallis' scenario, *Int. J. Mod. Phys. B* **31** (2017) 1750151, 10.1142/S021797921750151X
7. I. Gradshteyn and I. Ryzhik, Table of Integrals, Series, and Products (Academic Press, 1965)
8. A. Plastino and M. Rocca, Teaching strategy for introducing beginners to Coherent States, *Rev. Mex. Fis. E* **65** (2019) 191, `https://doi.org/10.31349/RevMexFisE.65.191`.
9. E. SchrÂ¨odinger, Der stetige Übergang von der Mikro- zur Makromechanik, *Naturwissenschaften* **14** (1926) 664, `https://doi.org/10.1007/BF01507634`.
10. V. V. Dodonov, V. I. Mankov, *Theory of Nonclassical States of Light* (Taylor and Francis, London, 2003); V V Dodonovy, Y. A. Korennoyy, V. I. Mankov, Y. A. Moukhin, *Quantum Semiclass. Opt.* **8** (1996) 413.
11. AA. Dehghani, B. Mojaveri, and A. A. Alenabi, Excitation and depression of coherent state of the simple harmonic oscillator, *J. Math. Phys*. **60** (2019) 083501, https://doi.org/10.1063/1.5055915