# Gravity train of variable mass

Y. Kajiyama[i]

*Faculty of Education, Gifu Shotoku Gakuen University,*
*1-1 Takakuwa-nishi, Yanaizu, Gifu, 501-6194, Japan,*
*e-mail: kajiyama@gifu.shotoku.ac.jp*

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The gravity train is a hypothetical vehicle that travels in a tunnel across Earth, and its motion due to only gravity is a simple harmonic oscillator. It can travel at a constant velocity by appropriately exhausting fuel. In this case, the train obeys the equation of motion for a variable mass system. We discuss the mass and energy reduction of the gravity train during travel and find the optimal conditions for economical travel. This study is suitable for university students in physics classes.

*Keywords:* Gravity train; gravity tunnel; variable mass system.

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## 1. Introduction

If there is a tunnel across the Earth of constant density and a train moves through the tunnel due to gravity, it moves as a harmonic oscillator because the gravitational force is proportional to the distance from the center of Earth. This train (the gravity train) travels between any two points on the surface of Earth in about 42 minutes [1]. Many authors have studied this classical problem so far. The brachistochrone for the gravity train is a hypocycloid [2], and a non-uniform assumption of Earth reduces the travel time [3]. The rotation of Earth [4], the relativistic effects [5], and the drag and contact friction effects [6] were also discussed. Reference [7] is a worthy note on the history of the gravity tunnel.

Let us consider the motion of the gravity train with a constant velocity. Appropriately exhausting fuel can control to keep the velocity of the gravity train. However, it causes the mass reduction of the gravity train. If an object of mass $m$ moves like a rocket with exhausting fuel of the relative velocity $\vec{u}$, the following equation of motion for a variable mass system can describe its motion:

$$m\frac{d\vec{v}}{dt} = \vec{F} + \vec{u}\frac{dm}{dt}, \tag{1}$$

where $\vec{F}$ is the external force exerted on the system. This problem has been discussed in the context of physics education by many authors [8–11].

In this manuscript, we consider the motion of the gravity train with a constant velocity. The force $\vec{F}$ in Eq. (1) is the gravitational force exerted on the train, which obeys Hooke's law. The condition of $d\vec{v}/dt = 0$ is satisfied by appropriately exhausting fuel forward or backward during its motion. Thus, university students can readily solve Eq. (1) and find the mass (fuel) reduction rate of the travel. They can find the most economical way by solving the extreme value problem. This study will be suitable for university students in physics courses.

## 2. Theoretical studies

Consider a train traveling from point A to point B through the center of Earth, point C, due to gravity as shown in Fig. 1. We assume that Earth is a sphere of radius $R = 6.37 \times 10^6$ m with uniform density. The gravitational force acting on the train of mass $m$ is $F = -m\omega^2 x$, where $\omega = \sqrt{g/R}$ and $g = 9.81\ \mathrm{m/s^2}$ is the gravitational acceleration. The train moves at a constant velocity $v_0$ by exhausting fuel like rocket propulsion. The train reduces mass $dm_e = -dm$ by exhausting forward (in the positive direction of $x$) or backward (in the negative direction of $x$) at a constant relative speed $u$ to maintain its velocity. From Eq. (1), the equation of motion for the train is

$$0 = -m\omega^2 x + \epsilon u\frac{dm}{dt}, \tag{2}$$

where $\epsilon = \pm 1$ is a sign parameter to specify the fuel exhaust direction, and $dm/dt < 0$. From Eq. (2), one finds

$$\begin{aligned} x < 0 \ &\rightarrow\ \epsilon = +1 \ \text{(forward jet)}, \\ x > 0 \ &\rightarrow\ \epsilon = -1 \ \text{(backward jet)}. \end{aligned} \tag{3}$$

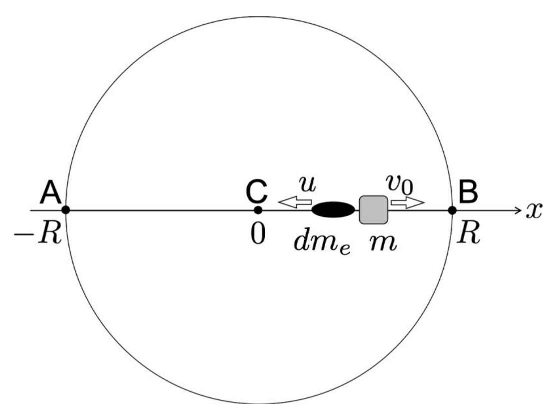


FIGURE 1. Schematic diagram of the gravity train at a constant velocity $v_0$ with the exhaust velocity $u$ relative to the train.

The train starts from point A at $x = -R$ and $t = 0$, and reaches point B at $x = R$ and $t = T = 2R/v_0$. The position of the train, $x(t)$, is thus given by

$$x(t) = v_0 t - R. \tag{4}$$

Hereafter, we parametrize the velocities $v_0$ and $u$ as

$$v_0 = \alpha\sqrt{gR}, \quad u = \beta\sqrt{gR}, \tag{5}$$

and use the dimensionless position parameter

$$\tilde{x} = \frac{x}{R} = \alpha\omega t - 1, \quad -1 \leq \tilde{x} \leq 1. \tag{6}$$

If one wants to travel in $T = \pi\sqrt{R/g} = 42$ minutes like Cooper's estimation [1], $\alpha = 2/\pi$, and thus

$$v_0 = \frac{2}{\pi}\sqrt{gR} = 5.0 \times 10^3 \text{ m/s}, \tag{7}$$

is required. As for $\beta$, it would be helpful to see the specific values of each rocket, such as $\beta < 0.316$ for a solid-fuel rocket and $\beta = 1.90 - 26.56$ for an electrostatic ion thruster [12]. As examples, we will perform numerical calculations for $\beta = 0.316(u = 2.5 \text{ km/s})$ and $12.6(u = 100 \text{ km/s})$.

By using $dm/dt = \alpha\omega dm/d\tilde{x}$, we rewrite Eq. (2) as

$$\frac{dm}{d\tilde{x}} = \frac{\epsilon m}{\alpha\beta}\tilde{x}, \tag{8}$$

and its general solution of Eq. (8) is

$$m(\tilde{x}) = Ce^{\frac{\epsilon}{2\alpha\beta}\tilde{x}^2}, \tag{9}$$

where $C$ is an integration constant. From the boundary condition at point A, $m_\mathrm{A} = m(-1)$, we find $C = m_\mathrm{A}e^{-(1/2\alpha\beta)}$. Thus, the position dependence of the mass is

$$m(\tilde{x}) = m_\mathrm{A} \exp\left[\frac{1}{2\alpha\beta}\left(\epsilon\tilde{x}^2 - 1\right)\right]. \tag{10}$$

The final mass at point B, $m_\mathrm{B} = m(1)$, is

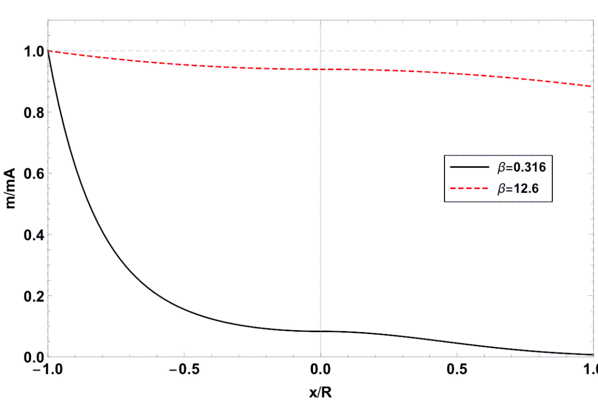


FIGURE 2. The position dependence of the mass ratio $m/m_\mathrm{A}$ for $\beta = 0.316$ (black solid curve) and 12.6 (red dashed curve).

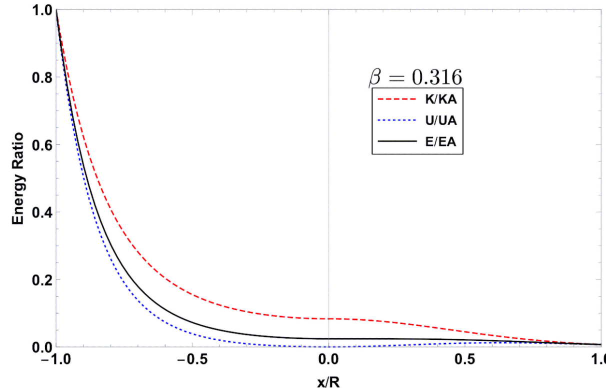


FIGURE 3. The position dependence of the energies relative to their initial values for $\beta = 0.316$.

$$m_\mathrm{B} = m_\mathrm{A}e^{-\frac{1}{\alpha\beta}}, \tag{11}$$

and the train reduces its mass by $\Delta m = m_B - m_A$ as the fuel jets.

Figure 2 shows the position dependence of $m(\tilde{x})/m_\mathrm{A}$ from Eqs. (10) for $\beta = 0.316$ (black solid curve) and 12.6 (red dashed curve) with Eq. (7) for $v_0$. The mass reduction during the travel is less for large values of $\beta$ because the momentum of the exhausted fuel relative to the train is proportional to $\beta m_e$: large values of $\beta$ require less amount of $m_e$.

Figures 3 and 4 show the position dependence of the energies relative to their initial values at point A for $\beta = 0.316$ and 12.6, respectively. The kinetic energy $K = (1/2)mv_0^2$ behaves in the same way as $m$ shown in Fig. 2. The potential energy $U = (1/2)m\omega^2x^2$ is different from a quadratic function of $x$ because of the position dependence of $m$. In fact, one can see the behavior from its derivative

$$\frac{d}{d\tilde{x}}\left(\frac{U}{U_\mathrm{A}}\right) = \frac{\tilde{x}}{\alpha\beta}\left(2\alpha\beta + \epsilon\tilde{x}^2\right)\frac{m(\tilde{x})}{m_\mathrm{A}}, \tag{12}$$

where $U_\mathrm{A} = (1/2)m_\mathrm{A}gR$. For $\tilde{x} < 0$, the sign of Eq. (12) is always negative, and it is a decreasing function in this region. For $\tilde{x} > 0$, the sign of Eq. (12) changes at the point

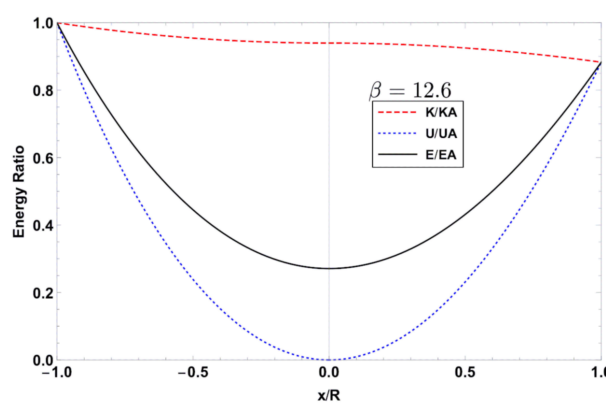


FIGURE 4. The position dependence of the energies relative to their initial values for $\beta = 12.6$.

$$\tilde{x} = \sqrt{2\alpha\beta} = \begin{cases} 0.634 \ (\beta = 0.316), \\ 4.00 \ (\beta = 12.6). \end{cases} \tag{13}$$

The blue dotted curve representing $U/U_{\rm A}$ has the maximum at $\tilde{x} = 0.634$ in Fig. 3, while it is an increasing function in Fig. 4 because the maximum given in Eq. (13) is outside the range of $\tilde{x}$.

The black solid curves represent the ratio of the total mechanical energy $E/E_{\rm A}$, where $E = K + U$, in Figs. 3 and 4. Its derivative is

$$\frac{d}{d\tilde{x}}\left(\frac{E}{E_{\rm A}}\right) = \frac{\tilde{x}}{\alpha\beta(\alpha^2+1)} \times \left[2\alpha\beta + \epsilon(\alpha^2 + \tilde{x}^2)\right] \frac{m(\tilde{x})}{m_{\rm A}}, \tag{14}$$

which is negative for $\tilde{x} < 0$ and $E/E_{\rm A}$ is a decreasing function in this region. For $\tilde{x} > 0$, $E/E_{\rm A}$ has the maximum at the point $\tilde{x} = \sqrt{\alpha(2\beta - \alpha)}$. If the maximum exists in the region $0 < \tilde{x} < 1$, the parameter $\beta$ must satisfy $1/\pi < \beta < 1/\pi + \pi/4$ for $\alpha = 2/\pi$. However, it is outside the region in our present case because $0.316 < 1/\pi$ and $12.6 > 1/\pi + \pi/4$. Thus, for $\tilde{x} > 0$, $E/E_{\rm A}$ is a decreasing function in Fig. 3 and an increasing function in Fig. 4.

The next problem is to find the most economical way to travel by solving the extreme value problem of $E$ with respect to $v_0$. The energy loss, $\Delta E$, during the travel is

$$\Delta E = E_{\rm B} - E_{\rm A} = -\left(1 - \frac{m_{\rm B}}{m_{\rm A}}\right) E_{\rm A} = -U_{\rm A}\left(1 - e^{-\frac{1}{\alpha\beta}}\right)\left(\alpha^2+1\right). \tag{15}$$

Figure 5 shows the $\alpha$ dependence of the dimensionless value $|\Delta E|/U_{\rm A}$. The red dashed curve has the minimum value near $\alpha = 1$, while the black solid curve is a monotonically increasing function of $\alpha$. Thus, one can find the most economical traveling velocity for large values of $\beta$.

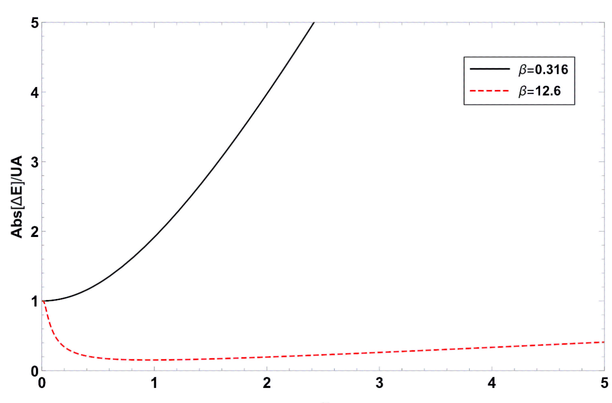


FIGURE 5. The dimensionless value of the absolute value of $\Delta E$, $|\Delta E|/U_{\rm A}$. The red dashed curve ($\beta = 12.6$) has the minimum, unlike the black solid curve ($\beta = 0.316$).

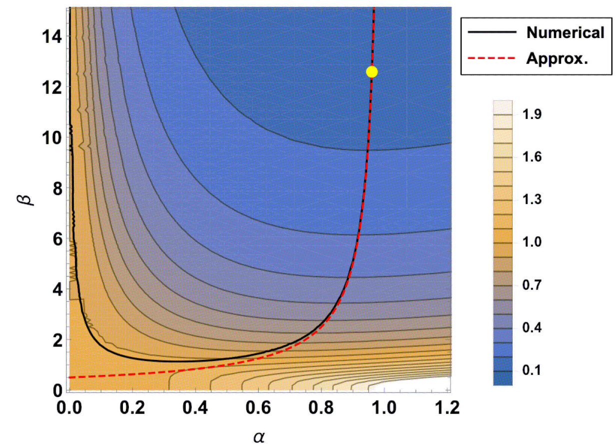


FIGURE 6. Contour plot for $d\Delta E/d\alpha = 0$ (black solid) and its approximation curve Eq. (20) (red dashed). Color plot of $|\Delta E|/U_{\rm A}$ is also depicted. The yellow point at $(\alpha, \beta) = (0.96, 12.6)$ corresponds to the minimum of the red dashed curve in Fig. 5.

To do this, we solve the extreme value problem of $\Delta E$. From the differentiation of $\Delta E$ by $\alpha$,

$$\frac{d\Delta E}{d\alpha} = \frac{m_{\rm B} g R}{2\beta}\left[1 + \frac{1}{\alpha^2} + 2\alpha\beta\left(1 - e^{\frac{1}{\alpha\beta}}\right)\right], \tag{16}$$

the extreme condition $d\Delta E/d\alpha = 0$ gives

$$1 + \frac{1}{\alpha^2} + 2\alpha\beta\left(1 - e^{\frac{1}{\alpha\beta}}\right) = 0. \tag{17}$$

Figure 6 shows the color plot for $|\Delta E|/U_{\rm A}$ in the $\alpha - \beta$ plane. The dark blue regions have small values of $|\Delta E|/U_{\rm A}$ and more economical ways of travel. For large values of $\beta$, one can reduce the energy loss during the travel. The black solid curve represents the extreme condition Eq. (17), and there is the lower bound $\beta \gtrsim 1$ for the existence of the minimum. To solve Eq. (17) approximately, we assume that $\alpha \simeq 1$ and $|1/\beta| \ll 1$, corresponding to the red dashed curve in Fig. 5. In this parameter region, the expansion

$$1 - e^{\frac{1}{\alpha\beta}} \simeq -\frac{1}{\alpha\beta}\left(1 + \frac{1}{2\alpha\beta}\right), \tag{18}$$

simplifies the extreme condition Eq. (17) as

$$\alpha^2 + \frac{1}{\beta}\alpha - 1 = 0. \tag{19}$$

Its appropriate solution is

$$\alpha = \sqrt{1 + \frac{1}{4\beta^2}} - \frac{1}{2\beta} \simeq 1 - \frac{1}{2\beta}, \tag{20}$$

which is depicted by the red dashed curve in Fig. 6. For $\beta = 12.6$, Eq. (20) gives $\alpha = 0.9603$, which is consistent with the numerical solution of Eq. (17) by `Mathematica`, $\alpha = 0.9600$. The yellow point at $(\alpha, \beta) = (0.96, 12.6)$, corresponding to the minimum of the red dashed curve in Fig. 5, is on the contour plots of both the numerical and approximate solutions. One can easily find some properties of the

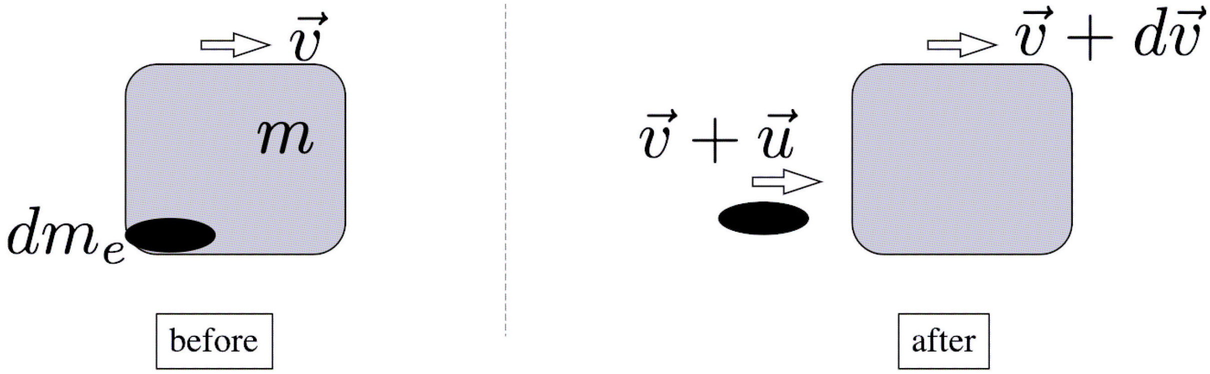


FIGURE 7. Schematic diagram for a variable mass system.

extreme from Eq. (20) instead of Eq. (17). There is no extreme value of $\alpha$ for $\beta = 0.316$, because it gives a negative value of $\alpha$. On the contrary, $\beta \to \infty$ when $\alpha \to 1$, and there are no positive solutions of $\beta$ for $\alpha > 1$.

## 3. Conclusions

We have studied the motion of a gravity train moving at a constant velocity. The train exhausts fuel forward and backward, reducing its mass to maintain its velocity. A variable mass system describes the motion of the train. Students can readily solve the equation of motion of the train and solve the minimum energy condition in an approximate form. One finds that the optimal velocity of the train is $v_0 \simeq \sqrt{gR}$ and that a large exhaust velocity $u$ is preferred.

## Appendix A

We give a brief derivation of the equation of motion Eq. (1) for a variable mass system [8–11].

Consider a train of total mass $m + dm_e$ moving at a velocity $\vec{v}$ as shown in Fig. 7 (before). The train changes its velocity to $\vec{v} + d\vec{v}$ by exhausting the fuel of mass $dm_e$ with a velocity $\vec{v} + \vec{u}$ measured from the rest frame as in Fig. 7 (after). Notice that $\vec{u}$ is the relative velocity of the exhausted fuel as defined in the main text. The change of the total momentum $d\vec{p}$ is

$$\begin{aligned} d\vec{p} &= m\left(\vec{v} + d\vec{v}\right) + dm_e\left(\vec{v} + \vec{u}\right) - \left(m + dm_e\right)\vec{v} \\ &= m d\vec{v} - \vec{u} dm, \end{aligned} \tag{A.1}$$

where the relation $dm_e = -dm$ has been used in the last expression. The equation of motion of the system is $d\vec{p}/dt = \vec{F}$, where $\vec{F}$ is a force exerting on the whole system. Thus, Eq. (A.1) becomes Eq. (1),

$$m\frac{d\vec{v}}{dt} = \vec{F} + \vec{u}\frac{dm}{dt}. \tag{A.2}$$

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$i$. Is an associate professor teaching physics in the Faculty of Education, Gifu Shotoku Gakuen University, Japan. He has a Ph.D. in elementary particle physics from Kanazawa University. His research interests include theoretical physics, high energy physics, cosmology, and physics education.

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