Calculation of the Wigner angle by means of vectors

G.F. Torres del Castillo

Instituto de Ciencias, Benemérita Universidad Autónoma de Puebla, 72570 Puebla, Pue., México.

Received 18 January 2024; accepted 6 March 2024

It is shown that each Lorentz transformation leaving invariant one spatial axis can be represented by a single complex vector. This fact is employed in the calculation of the Wigner angle (which arises in the composition of two boosts in arbitrary directions) and of the aberration of light.

Keywords: Lorentz transformations; Wigner angle; light aberration.

DOI: https://doi.org/10.31349/RevMexFisE.22.020201

1. Introduction

Among the counterintuitive results of the special theory of relativity is the fact that if two inertial reference frames, S and S′, are moving in different directions with respect to a third inertial frame, S_0, with their Cartesian axes parallel to those of S_0 (see Fig. 1), then the axes of S′ are rotated with respect to those of S through an angle known as the Wigner angle (see, *e.g.*, Ref. [1] and the references cited therein). This curious behavior is a consequence of the algebraic properties of the Lorentz transformations, which give the relation between the space-time coordinates determined by different inertial reference frames.

The elementary formalism employed in the study of the Lorentz transformations is that of the 4×4 real matrices (see, *e.g.*, Refs. [2,3]) but, if one considers Lorentz transformations involving only two spatial directions (as in the determination of the Wigner angle), it suffices to employ 3×3 real matrices (see, *e.g.*, Ref. [4]). In these two cases it is possible to represent the 4×4 or 3×3 real matrices corresponding to the Lorentz transformations by 2×2 matrices whose entries are complex, real, or double numbers (see, *e.g.*, Refs. [4–8]).

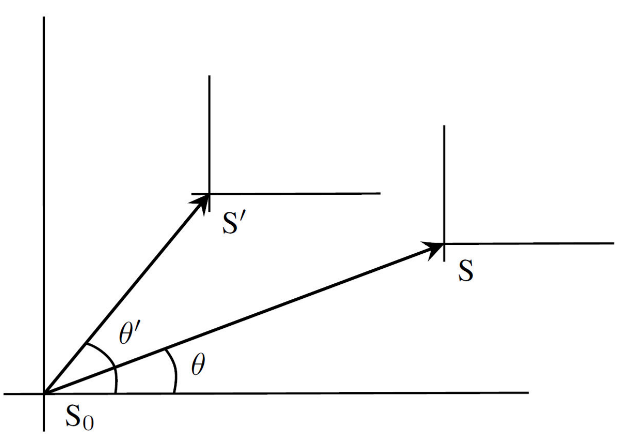

FIGURE 1. The inertial reference frames S and S′ move with respect to the inertial reference frame S_0 with their spatial axes parallel to those of S_0 in directions making angles θ and θ' with respect to the x-axis of S_0.

The Wigner angle, in particular, can be calculated making use of any of these sets of matrices.

As shown in Ref. [4], each Lorentz transformations involving only two spatial directions, can be represented by a single two-component spinor. Given two such spinors we can calculate two different scalar products which must correspond to invariants related to the two Lorentz transformations represented by those spinors. The modulus of one of these invariants gives the relative velocity between the frames represented by the spinors, and its argument gives the angle between their spatial axes.

Taking into account that the spinor formalism is not widely employed, the aim of this paper is to present an elementary derivation of the results of Ref. [4], making use of the vector formalism. In Sec. 2 we show that a single complex vector represents a Lorentz transformation in the restricted case where only two spatial directions are considered and in Sec. 3 we show that the products between these vectors and their conjugates reproduce the invariants constructed by means of two-component spinors in Ref. [4]. In Sec. 4 we obtain the standard expressions for the relativistic Doppler effect and the aberration of light by combining the usual wave four-vector corresponding to a plane wave with the complex vector representing an inertial frame.

It is assumed that the reader is acquainted with the basic notions of special relativity and the standard four-vector formalism.

2. Vector description of the Lorentz transformations

In the standard approach, the Lorentz transformations are viewed as coordinate transformations. The basic example of a Lorentz transformation considered in the elementary textbooks on special relativity is that corresponding to two inertial frames whose Cartesian axes coincide at $t = 0$ and the primed axes move with respect to the unprimed ones with velocity v along the x-axis (the so-called standard configuration, see, *e.g.*, Ref. [3]). Then, the Cartesian coordinates of any event with respect to these frames are related by

$$\begin{aligned} ct' &= \gamma\big(ct - \frac{v}{c}x\big), \\ x' &= \gamma\big(x - \frac{v}{c}ct\big), \\ y' &= y, \end{aligned} \tag{1}$$

together with $z' = z$, where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

In what follows we shall restrict ourselves to transformations such that $z' = z$, so that we can omit the coordinates z and z'.

Defining the *rapidity*, w, by

$$\tanh w = \frac{v}{c}, \tag{2}$$

Eqs. (1) are equivalent to

$$\begin{aligned} ct' &= (\cosh w)ct - (\sinh w)x, \\ x' &= (\cosh w)x - (\sinh w)ct, \\ y' &= y. \end{aligned} \tag{3}$$

By analogy with the position vector of a point of the three-dimensional Euclidean space, $\mathbf{r} = x\,\mathbf{e}_1 + y\,\mathbf{e}_2 + z\,\mathbf{e}_3$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are unit vectors along a set of Cartesian axes, we shall consider the position vector of an event, $ct\,\mathbf{e}_0 + x\,\mathbf{e}_1 + y\,\mathbf{e}_2$ making use of the space-time coordinates of the event with respect to an inertial frame. According to Eqs. (3), this position vector is also given by

$$\begin{aligned} &ct'\,\mathbf{e}_{0'} + x'\,\mathbf{e}_{1'} + y'\,\mathbf{e}_{2'} \\ &\quad = [(\cosh w)ct - (\sinh w)x]\,\mathbf{e}_{0'} \\ &\quad + [(\cosh w)x - (\sinh w)ct]\,\mathbf{e}_{1'} + y\,\mathbf{e}_{2'} \\ &\quad = ct\,[(\cosh w)\,\mathbf{e}_{0'} - (\sinh w)\,\mathbf{e}_{1'}] \\ &\quad + x\,[-(\sinh w)\,\mathbf{e}_{0'} + (\cosh w)\,\mathbf{e}_{1'}] + y\,\mathbf{e}_{2'} \end{aligned}$$

and, therefore (since ct, x and y are arbitrary),

$$\begin{aligned} \mathbf{e}_0 &= (\cosh w)\,\mathbf{e}_{0'} - (\sinh w)\,\mathbf{e}_{1'}, \\ \mathbf{e}_1 &= -(\sinh w)\,\mathbf{e}_{0'} + (\cosh w)\,\mathbf{e}_{1'}, \\ \mathbf{e}_2 &= \mathbf{e}_{2'} \end{aligned} \tag{4}$$

or, equivalently,

$$\begin{aligned} \mathbf{e}_{0'} &= (\cosh w)\,\mathbf{e}_0 + (\sinh w)\,\mathbf{e}_1, \\ \mathbf{e}_{1'} &= (\sinh w)\,\mathbf{e}_0 + (\cosh w)\,\mathbf{e}_1, \\ \mathbf{e}_{2'} &= \mathbf{e}_2. \end{aligned} \tag{5}$$

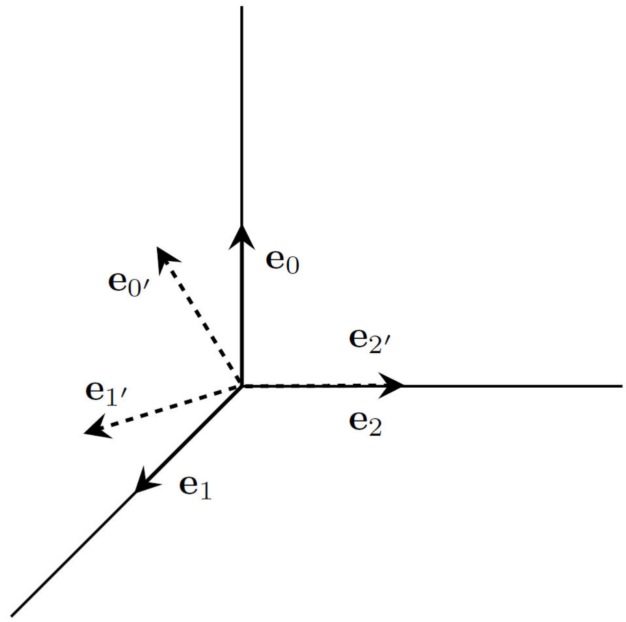

FIGURE 2. The Lorentz transformation (3) corresponds to the change of basis given by Eqs. (5). The vectors $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2$ are shown with solid lines and the vectors $\mathbf{e}_{0'}, \mathbf{e}_{1'}, \mathbf{e}_{2'}$ are shown with dashed lines.

Thus, the Lorentz transformation (3) can also be viewed as the change of basis (5) (see Fig. 2).

Similarly, an ordinary rotation in the xy-plane through an angle ϕ:

$$\begin{aligned} ct' &= ct, \\ x' &= x\cos\phi + y\sin\phi, \\ y' &= -x\sin\phi + y\cos\phi, \end{aligned}$$

corresponds to the change of basis

$$\begin{aligned} \mathbf{e}_{0'} &= \mathbf{e}_0, \\ \mathbf{e}_{1'} &= (\cos\phi)\,\mathbf{e}_1 + (\sin\phi)\,\mathbf{e}_2, \\ \mathbf{e}_{2'} &= -(\sin\phi)\,\mathbf{e}_1 + (\cos\phi)\,\mathbf{e}_2. \end{aligned} \tag{6}$$

The fact that Eqs. (5) and (6) indeed correspond to Lorentz transformations can be verified directly making use of the metric tensor, g, defined by

$$g(\mathbf{e}_0, \mathbf{e}_0) = -1, \quad g(\mathbf{e}_1, \mathbf{e}_1) = 1, \quad g(\mathbf{e}_2, \mathbf{e}_2) = 1,$$

and $g(\mathbf{e}_i, \mathbf{e}_j) = 0$ for $i \neq j$. That is,

$$g(\mathbf{e}_i, \mathbf{e}_j) = g_{ij} \tag{7}$$

for $i, j = 0, 1, 2$, with

$$(g_{ij}) \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

The change of basis $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2\} \mapsto \{\mathbf{e}_{0'}, \mathbf{e}_{1'}, \mathbf{e}_{2'}\}$ corresponds to a Lorentz transformation if and only if

$$g(\mathbf{e}_{i'}, \mathbf{e}_{j'}) = g_{ij}. \tag{9}$$

For instance, considering the transformation given by Eqs. (5), due to the bilinearity of g, we have

$$\begin{aligned} &g(\mathbf{e}_{0'}, \mathbf{e}_{0'}) \\ &= g\big((\cosh w)\mathbf{e}_0 + (\sinh w)\mathbf{e}_1, (\cosh w)\mathbf{e}_0 + (\sinh w)\mathbf{e}_1\big) \\ &= -\cosh^2 w + \sinh^2 w = -1, \end{aligned}$$

and

$$\begin{aligned} &g(\mathbf{e}_{0'}, \mathbf{e}_{1'}) \\ &= g\big((\cosh w)\mathbf{e}_0 + (\sinh w)\mathbf{e}_1, (\sinh w)\mathbf{e}_0 + (\cosh w)\mathbf{e}_1\big) \\ &= -\cosh w \sinh w + \sinh w \cosh w = 0. \end{aligned}$$

Any Lorentz transformation of the restricted class where $z' = z$ can be specified giving three vectors, $\{\mathbf{e}_{0'}, \mathbf{e}_{1'}, \mathbf{e}_{2'}\}$, as linear combinations of $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2\}$, satisfying the conditions (9). However, there is some redundancy in this; it is enough to specify two of the vectors $\mathbf{e}_{i'}$, the third one is, up to sign, the cross product of the other two. (Since we are dealing with a vector space with the metric tensor (8), the components of the cross product of the vectors a^i and b^j are given by $c^i = g^{ij}\varepsilon_{jkl}a^k b^l$, where, as usual, (g^{ij}) is the inverse of the matrix (g_{ij}), ε_{ijk} is the Levi-Civita symbol, with $\varepsilon_{012} = 1$, and there is sum over repeated indices.)

In order to establish a simple relation with the results of Ref. [4], we shall make use of the (spacelike) vectors $\mathbf{e}_{1'}$ and $\mathbf{e}_{2'}$ and we shall combine them in the complex vector

$$\mathbf{M}' \equiv \mathbf{e}_{1'} - \mathrm{i}\,\mathbf{e}_{2'} \tag{10}$$

(the minus sign accompanying the imaginary unit is not essential, it is included in order to get agreement with the expressions employed in Ref. [4]). According to Eqs. (9), we have

$$g(\mathbf{M}', \mathbf{M}') = 0 \quad \text{and} \quad g(\overline{\mathbf{M}}', \mathbf{M}') = 2, \tag{11}$$

with the bar denoting complex conjugation.

For instance, for the Lorentz transformation (5), $\mathbf{M}' = (\sinh w)\,\mathbf{e}_0 + (\cosh w)\,\mathbf{e}_1 - \mathrm{i}\,\mathbf{e}_2$, and for the Lorentz transformation (6), $\mathbf{M}' = \mathrm{e}^{\mathrm{i}\phi}(\mathbf{e}_1 - \mathrm{i}\,\mathbf{e}_2)$.

In the case where the primed reference frame moves with respect to the unprimed one with rapidity w in the direction making an angle θ with respect to the x-axis, and with its spatial axes rotated through an angle ϕ with respect to those of the unprimed frame,

$$\begin{aligned} \mathbf{M}' = \mathrm{e}^{\mathrm{i}(\phi-\theta)}\big[&\sinh w\,\mathbf{e}_0 + (\cosh w \cos\theta + \mathrm{i}\sin\theta)\mathbf{e}_1 \\ &+ (\cosh w \sin\theta - \mathrm{i}\cos\theta)\mathbf{e}_2\big] \end{aligned} \tag{12}$$

then, the only real, future-pointing vector, $\mathbf{e}_{0'}$, satisfying the conditions

$$g(\mathbf{M}', \mathbf{e}_{0'}) = 0, \qquad g(\mathbf{e}_{0'}, \mathbf{e}_{0'}) = -1,$$

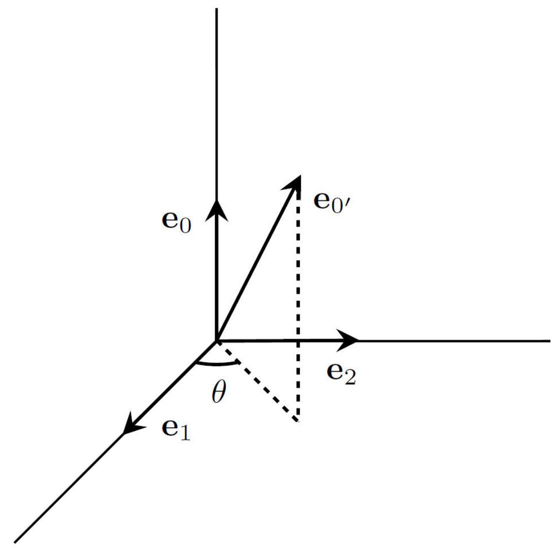

FIGURE 3. The basis vectors $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2$ are shown together with the vector $\mathbf{e}_{0'}$ given by Eq. (13).

is

$$\mathbf{e}_{0'} = \cosh w\,\mathbf{e}_0 + \sinh w \cos\theta\,\mathbf{e}_1 + \sinh w \sin\theta\,\mathbf{e}_2 \tag{13}$$

(see Fig. 3).

The unprimed frame is represented by the vector

$$\mathbf{M} = \mathbf{e}_1 - \mathrm{i}\,\mathbf{e}_2. \tag{14}$$

The product $g(\mathbf{M}, \mathbf{M}')$ must be invariant under Lorentz transformations and, therefore, it must represent some geometric property of the pair of frames represented by $\mathbf{M}$ and $\mathbf{M}'$. Making use of (12) and (14) one finds that

$$\begin{aligned} g(\mathbf{M}, \mathbf{M}') &= \mathrm{e}^{\mathrm{i}(\phi-\theta)}\big[(\cosh w \cos\theta + \mathrm{i}\sin\theta) \\ &\quad - \mathrm{i}(\cosh w \sin\theta - \mathrm{i}\cos\theta)\big] \\ &= \mathrm{e}^{\mathrm{i}(\phi-2\theta)}(\cosh w - 1). \end{aligned} \tag{15}$$

(It may be noticed that $\mathrm{e}^{\mathrm{i}(\phi-2\theta)}(\cosh w - 1) = 2(\mathrm{e}^{\mathrm{i}\phi/2-\theta}\sinh w/2)^2$. The expression $\mathrm{e}^{\mathrm{i}\phi/2-\theta}\sinh w/2$ is one of the invariants obtained through the spinor formalism in Ref. [4].)

A second invariant associated with the pair of frames repesented by $\mathbf{M}$ and $\mathbf{M}'$ is given by $g(\overline{\mathbf{M}}, \mathbf{M}')$ and using again Eqs. (12) and (14) one finds that

$$\begin{aligned} g(\overline{\mathbf{M}}, \mathbf{M}') &= \mathrm{e}^{\mathrm{i}(\phi-\theta)}\big[(\cosh w \cos\theta + \mathrm{i}\sin\theta) \\ &\quad + \mathrm{i}(\cosh w \sin\theta - \mathrm{i}\cos\theta)\big] \\ &= \mathrm{e}^{\mathrm{i}\phi}(\cosh w + 1). \end{aligned} \tag{16}$$

(In this case, one can verify that $\mathrm{e}^{\mathrm{i}\phi}(\cosh w + 1) = 2(\mathrm{e}^{\mathrm{i}\phi/2}\cosh w/2)^2$. $\mathrm{e}^{\mathrm{i}\phi/2}\cosh w/2$ is another invariant found in Ref. [4] making use of the spinor formalism.) Thus, the argument of the complex number $g(\overline{\mathbf{M}}, \mathbf{M}')$ is the angle formed by the spatial axes of S$'$ with respect to those of S. (Note the order of the vectors. The argument of $g(\overline{\mathbf{M}}', \mathbf{M})$ is the angle formed by the spatial axes of S with respect to those of S$'$.)

3. The Wigner angle

Now we shall make use of the results obtained above in the calculation of the Wigner angle. The relevant facts are that each frame is completely represented by a single complex vector (10) and that the products (calculated with the aid of the metric tensor g) between these vectors are invariant under Lorentz transformations. According to the discussion presented at the Introduction, we shall consider three inertial reference frames, S_0, S and S′ sharing their z-axes. S and S′ move with respect to S_0 with their spatial axes parallel to those of S_0 in possibly different directions, forming angles θ and θ' with respect to the x-axis of S_0 (see Fig. 1).

Since S and S′ have their spatial axes parallel to those of S_0, they are represented by the complex vectors [see Eq. (12)]

$$\begin{aligned}\mathbf{M} = \mathrm{e}^{-\mathrm{i}\theta}\big[&\sinh w\,\mathbf{e}_0 + (\cosh w\cos\theta + \mathrm{i}\sin\theta)\mathbf{e}_1\\ &+ (\cosh w\sin\theta - \mathrm{i}\cos\theta)\mathbf{e}_2\big]\end{aligned} \tag{17}$$

and

$$\begin{aligned}\mathbf{M}' = \mathrm{e}^{-\mathrm{i}\theta'}\big[&\sinh w'\,\mathbf{e}_0 + (\cosh w'\cos\theta' + \mathrm{i}\sin\theta')\mathbf{e}_1\\ &+ (\cosh w'\sin\theta' - \mathrm{i}\cos\theta')\mathbf{e}_2\big],\end{aligned} \tag{18}$$

respectively, where w and w' are the rapidities of S and S′ with respect to S_0. Hence,

$$\begin{aligned}g(\overline{\mathbf{M}},\mathbf{M}') = \mathrm{e}^{\mathrm{i}(\theta-\theta')}\big[&(\cosh w\cosh w' + 1)\cos(\theta-\theta')\\ &- \sinh w\sinh w' - \mathrm{i}(\cosh w + \cosh w')\sin(\theta-\theta')\big].\end{aligned} \tag{19}$$

One can verify that the last expression can be abbreviated as

$$2\big[\cosh w/2\cosh w'/2 - \mathrm{e}^{\mathrm{i}(\theta-\theta')}\sinh w/2\sinh w'/2\big]^2.$$

According to Eq. (16), the right-hand side of (19) must be equal to $\mathrm{e}^{\mathrm{i}\tilde{\phi}}(\cosh\tilde{w}+1)$, where $\tilde{\phi}$ is the angle formed by the spatial axes of S′ with respect to those of S (that is, the Wigner angle) and $\tilde{w}$ is the rapidity of S′ with respect to S, which amounts to the abbreviated expression

$$\begin{aligned}&\mathrm{e}^{\mathrm{i}\tilde{\phi}/2}\cosh\tilde{w}/2\\ &= \cosh w/2\cosh w'/2 - \mathrm{e}^{\mathrm{i}(\theta-\theta')}\sinh w/2\sinh w'/2.\end{aligned}$$

In a similar manner, according to Eq. (15), $g(\mathbf{M},\mathbf{M}')$ is equal to $\mathrm{e}^{\mathrm{i}(\tilde{\phi}-2\tilde{\theta})}(\cosh\tilde{w}-1)$, where $\tilde{\theta}$ is the angle between the velocity of S′ and the x-axis of S.

4. Doppler effect and aberration of light

As another application of the representation of inertial frames by complex vectors we shall give a simple derivation of the formulas for the relativistic Doppler effect and the aberration of light.

The usual wave four-vector of an electromagnetic plane wave with angular frequency ω, propagating in a direction in the xy-plane making an angle α with the x-axis has components

$$k^\mu = \frac{\omega}{c}(1,\cos\alpha,\sin\alpha,0),$$

or, forgetting the z-component,

$$\mathbf{k} = \frac{\omega}{c}(\mathbf{e}_0 + \cos\alpha\,\mathbf{e}_1 + \sin\alpha\,\mathbf{e}_2).$$

Hence, the Lorentz invariant $g(\mathbf{M},\mathbf{k})$ is the complex number [see Eq. (14)]

$$g(\mathbf{M},\mathbf{k}) = \frac{\omega}{c}\mathrm{e}^{-\mathrm{i}\alpha}$$

and if the frame S′ is in the standard configuration with S, that is, S′ is represented by $\mathbf{M}' = (\sinh w)\,\mathbf{e}_0 + (\cosh w)\,\mathbf{e}_1 - \mathrm{i}\,\mathbf{e}_2$ we have

$$\begin{aligned}g(\mathbf{M}',\mathbf{k}) &= \frac{\omega}{c}\left(\mathrm{e}^{-\mathrm{i}\alpha/2}\cosh w/2 - \mathrm{e}^{\mathrm{i}\alpha/2}\sinh w/2\right)^2\\ &= \frac{\omega}{c}\left(\mathrm{e}^{-w/2}\cos\alpha/2 - \mathrm{i}\,\mathrm{e}^{w/2}\sin\alpha/2\right)^2,\end{aligned}$$

which must be equal to $\frac{\omega'}{c}\mathrm{e}^{-\mathrm{i}\alpha'}$, where ω' is the angular frequency of the wave measured in S′ and α' is the angle between the x'-axis and the direction of propagation of the wave measured in S′. This leads to

$$\tan\frac{\alpha'}{2} = \mathrm{e}^w\tan\frac{\alpha}{2} = \sqrt{\frac{1+v/c}{1-v/c}}\,\tan\frac{\alpha}{2}.$$

5. Concluding remarks

Even though the product (defined by the metric tensor g) between any pair of space-time vectors is invariant under the Lorentz transformations, not every product gives something relevant. As we have shown, the fact that a single complex space-time vector represents an inertial frame of the class considered here (or, equivalently, a Lorentz transformations that preserves the z-axis) allows us to obtain easily, among other things, the Wigner angle.

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