Adding an Einsteinian motivation to key discussions in an electromagnetism course

L. E. Fuentes-Cobas[a] and M. E. Fuentes-Montero[b,*]

[a] *Centro de Investigación en Materiales Avanzados, Chihuahua, México.*
[b] *Facultad de Ciencias Químicas, Universidad Autónoma de Chihuahua, Chihuahua, México.*
[*] *e-mail: mfuentes@uach.mx*

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This paper aims to provide physics teachers with tools to help deepen the understanding of the laws of electromagnetism. The fundamental contributions of our proposal are: a) to use quotes from mythical characters in the history of science as a motivating educational resource; b) to promote the discussion of striking and fundamental topics; c) to mention diverse approaches and stimulate the search for correct answers to provocative questions. Citations from Einstein refer to principal contributions made by Newton, Maxwell and himself. Emphasis is placed on the cognitive value of differential (local, infinitesimal) analysis of fundamental concepts (field structure, causality, field relativistic transformations). The electromagnetism unity is analyzed from the point of view of special relativity. It is clarified that descriptions suggesting that the magnetic field is dispensable are contrary to the Einsteinian approach: they assume that, to describe the interaction between moving charges, there is a preferred coordinate system for each particular problem. An introductory presentation of the tensor form of Maxwell's equations is provided.

Keywords: Physics education; electromagnetic field; special relativity.

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1. Introduction

The nature of the electromagnetic field is one of the topics that attracts the most students and teachers. The charm of the subject is seasoned with a touch of indecipherability due to its intangible nature, and the requirement of relatively advanced mathematical tools for its satisfactory assimilation. The traditional development of electromagnetism courses, based on teacher presentations while the students take notes, does little to facilitate the understanding of the content [1, 2]. This paper provides physics teachers with resources to achieve a deeper knowledge of the laws of electromagnetism. The fundamental contributions of our proposal are: a) to take advantage of moments and quotes from mythical characters in the history of science as a motivating educational resource; b) to promote the discussion of striking and fundamental topics; c) to mention diverse approaches and stimulate the search for correct answers to the problems posed. The application of the aforementioned teaching resources should help to fulfil the purpose of the article. These tools have been identified as important both in classical works [3–5] and in recent programmatic documents of the European Physics Society [6] and the American Institute of Physics [7].

The present article considers the electromagnetic field's structure, within the framework of classical physics and special relativity.

The paper has been built from considerations by A. Einstein. It contains a selection of quotes from his book "Essays In Science" (EIS) [8]. We start with segments of the book in which the author of the Theory of Relativity comments about the contributions of Newton and Maxwell. Afterward, we develop our suggestion for the application of Einstein's considerations. The article plan covers the following topics:

- Stationary fields (Electrostatics and Magnetostatics).
- Time-varying fields and Electromagnetic Induction.
- Relativity of $\mathbf{E}$ and $\mathbf{B}$.
- Maxwell's equations in tensor form.

The consistency of unifying electric and magnetic components into the concept of electromagnetic field within Einstein's Relativity is reinforced. The educational importance of historical quotes and the cognitive value of detailed differential (local, infinitesimal) treatment of fundamental concepts (field structure, causality, field relativistic transformations) are highlighted.

Einstein about Newton and Maxwell.

We reproduce a quotation from the section on Newton in EIS [8]. Regarding Newton's work on Kepler's Laws, Einstein expresses his assessment of the scientific contribution of the author of *Principia*.

"Newton's object was to answer the question: Is there such a thing as a simple rule by which one can calculate the movements of the heavenly bodies in our planetary system completely, when the state of motion of all these bodies at one moment is known? Kepler's empirical laws of planetary movement, deduced from Tycho Brahe's observations, confronted him, and demanded explanation. (...) The most important point, however, is this: these laws are concerned with the movement as a whole, and not with the question of how the state of motion of a system gives rise to that which immediately follows it in time; they are, as we should say now, integral and not differential laws. The differential law is the only form that completely satisfies the modern physicist's demand for causality. The clear conception of the differential law is one of Newton's greatest intellectual achievements."

The independent variable in Newton's Second Law, $d\mathbf{p} = \mathbf{F}dt$, is time. Einstein highlights the concept of causality as the characterization of movement generation at the level of infinitesimal increments of time.

We now transcribe a segment of Einstein's analysis of Maxwell's contribution to knowledge. This consideration is related to the first applications of partial differential equations to study deformable bodies and the supposed waves of ether in light (EIS, pp. 41 and following).

"Neglecting the important individual results which Clerk Maxwell life-work produced in important departments of physics, and concentrating on the changes wrought by him in our conception of the nature of physical reality, we may say this: Before Clerk Maxwell people conceived of physical reality– in so far as it is supposed to represent events in nature–as material points, whose changes consist exclusively of motions, which are subject to partial differential equations. After Maxwell, they conceived physical reality as represented by continuous fields, not mechanically explicable, which are subject to partial differential equations. This change in the conception of reality is the most profound and fruitful one that has come to physics since Newton..."

In Maxwell's equations, the independent variables are spatial coordinates and time. The dependent ones are the $\mathbf{E}$ and $\mathbf{B}$ fields. In the electro- and magnetostatic cases ($\partial/\partial t = 0$), Maxwell's equations characterize the spatial structure of the fields. In the general electrodynamic case ($\partial/\partial t \neq 0$), Maxwell's equations describe the structure and causality of the electromagnetic field at the infinitesimal level [9].

Bohr on causality

Causality in electromagnetism, in particular the relationship of Maxwellian electromagnetism to Newtonian mechanics, has been referred to by Niels Bohr. In his article on causality and complementarity [10], he wrote: "In classical mechanics, the forces between bodies were assumed to depend simply on the instantaneous positions and velocities; but the discovery of the retardation of electromagnetic effects made it necessary to consider force fields as an essential part of a physical system, and to include in the description of the state of the system at a given time the specification of these fields at every point of space. Yet, as is well known, the establishment of the differential equations connecting the rate of variation of electromagnetic intensities in space and time has made possible a description of electromagnetic phenomena in complete analogy to causal analysis in mechanics".

Expressing physical relationships at the level of differentials of time, space, electronic density, electromagnetic field, among others, is to reach the level of detail desired by the current researcher in dynamic systems, fluids, electromagnetism, quantum physics, etc. One purpose of this article is to highlight the importance of the differential approach.

In the sections that follow, based on the above Einsteinian quotes (with Bohr's final touch), we suggest a path for discussing the electromagnetic field concept, in vacuum, that tracks the Newton → Maxwell → Einstein's development of ideas.

2. The structure of electro- and magnetostatic fields

Table I summarizes the Maxwell equations for time-invariant electromagnetic fields.

The difference between the differential and integral forms of Maxwell's equations is that the former describes what happens to the fields in the vicinity of a given point, and the latter refers to what happens in a finite region of space. The global structure of the fields (integral equations) is generated by infinitesimal concatenations characterized by the differential equations. Newton, Maxwell and Einstein gave priority to differential equations. In the electromagnetic theory, the divergence and curl (rotational) differential operations are of significant interest. The divergence describes the local variations of a vector field along the direction in which it points. The curl characterizes the spatial variations of the field associated with displacements perpendicular to the vector. The qualitative interpretation of the divergence and rotational operators in the Purcell-Morin textbook [11] is illustrative.

Studying electrostatics and magnetostatics in parallel has methodological advantages, as discussed in the text [12]. Maxwell equations for electrostatics and magnetostatics are "inter-crossed". The flux of $\mathbf{B}$ is always zero, and the enclosed charge determines that of $\mathbf{E}$. The circulation of electrostatic $\mathbf{E}$ is always null, while for $\mathbf{B}$, it depends on the enclosed current.

The differences between these equations lead to different solutions and to qualitatively different fields. The general solution of Maxwell equations for the electrostatic case is

TABLE I. Maxwell equations for stationary conditions ($\partial\mathbf{E}/\partial t = \partial\mathbf{B}/\partial t = 0$)

Case →	Electrostatics		Magnetostatics	
Equation ↓	Differential	Integral	Diferential	Integral
a) Gauss's laws for $\mathbf{E}$ and $\mathbf{B}$	$\nabla\cdot\mathbf{E} = \rho/\varepsilon_0$	$\oint \mathbf{E}\cdot d\mathbf{a} = q/\varepsilon_0$	$\nabla\cdot\mathbf{B} = 0$	$\oint \mathbf{B}\cdot d\mathbf{a} = 0$
b) Static $\mathbf{E}$ is conservative. Ampere's law for $\mathbf{B}$	$\nabla\times\mathbf{E} = 0$	$\oint \mathbf{E}\cdot d\mathbf{l} = 0$	$\nabla\times\mathbf{B} = \mu_0\mathbf{J}$	$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I$

Coulomb's law, expressed in terms of $\mathbf{E}$. For the field associated with a differential charge dq, which is part of a continuous object, this law is written as in Eq. (1):

$$d\mathbf{E} = \frac{dq\widehat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}, \tag{1}$$

where $\mathbf{E}$ is a polar vector whose lines of force are born in (emerge from) the positive charges, and die in the negative ones. In the magnetostatic case, the solution of the equations for the differential of $\mathbf{B}$, corresponding to a current element $Id\mathbf{l}$, is given by the Biot-Savart law, Eq. (2):

$$d\mathbf{B} = \frac{\mu_0 I d\mathbf{l} \times \widehat{\mathbf{r}}}{4\pi r^2}. \tag{2}$$

where $\mathbf{B}$ is an axial vector (or "pseudovector") whose lines of force revolve around the currents, following the well-known "right-hand rule", without being born (or dying) anywhere. The axial nature of $\mathbf{B}$ is expressed by the "×" operator in the Ampere law (Table I) and in Eq. (2). The behaviours of $\mathbf{E}$ and $\mathbf{B}$ under those symmetry transformations, including the inversion, are different. This topic is discussed, for example, in Ref. [13].

Figure 1 schematically compares the electrostatic and magnetostatic cases with simple cylindrical symmetries. We use these examples to discuss some ideas. Maxwell equations, coupled with symmetry considerations, lead to the solutions of the $\mathbf{E}$ and $\mathbf{B}$ fields of Fig. 1. For all space, it follows that E is perpendicular to the charged rod and $\mathbf{B}$ is circular about the current. The modules of these vectors are given by Eqs. (3) and (4).

$$E(r) = \left\{ \begin{matrix} \frac{\rho r}{2\varepsilon_0} & r < R \\ \frac{\lambda}{2\pi\varepsilon_0 r} & r > R \end{matrix} \right\}, \tag{3}$$

$$B(r) = \left\{ \begin{matrix} \frac{\mu_0 J r}{2\varepsilon_0} & r < R \\ \frac{\mu_0 I}{2\pi r} & r > R \end{matrix} \right\}. \tag{4}$$

In free space, the geometries of the $\mathbf{E}$ and $\mathbf{B}$ fields are different ("radial" versus "circular"), but both are irrotational ($\nabla \times \mathbf{E} = \nabla \times \mathbf{B} = 0$). For example, the vector $\mathbf{E}$ satisfies the condition

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0. \tag{5}$$

At the differential scale, a variation of one component of the electric field under displacements in one direction implies a variation of another component of the same field under displacements in another direction. This is a point worth highlighting regarding the *structure* of the electrostatic field.

We launch our series of invitations to discuss issues of interest through questions related to Maxwell's equations for static fields.

Introductory question

In the integral form of Gauss's Law of Electricity ($\varepsilon_0 \oint \mathbf{E} \cdot d\mathbf{a} = q$), is $\mathbf{E}$ attributable to q?

Topics for discussion

- Is the expression "the field associated with a charge" better than the popular "the field produced by a charge? This question has to do with causality. It is worth considering it in the discussion of any Maxwell equation. The search for a correct answer can lead us to Quantum Electrodynamics [14].

- What kind of new physical phenomena would happen if Eq. (5) was satisfied by an integrated electromagnetic field in a combined space-time?

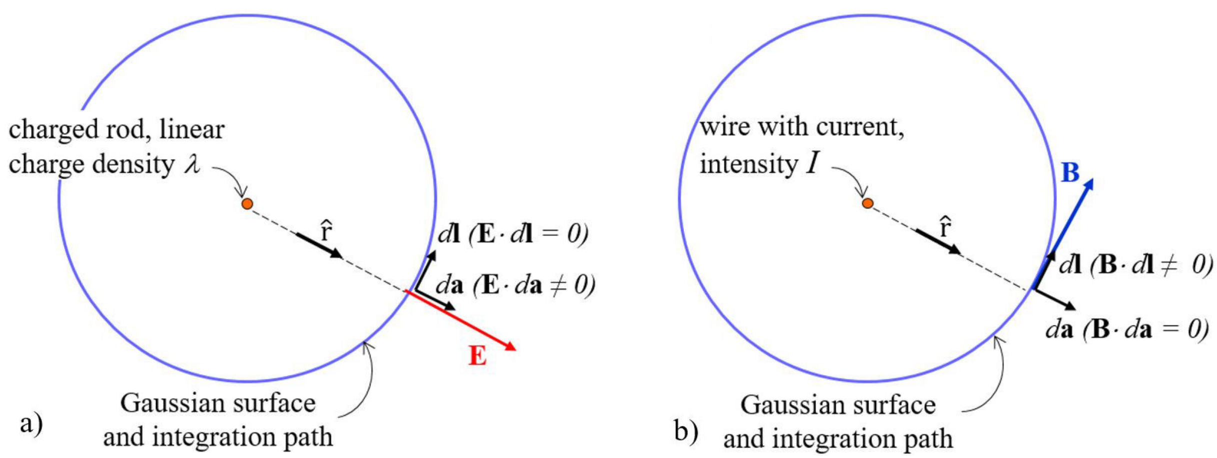

FIGURE 1. Electrostatic a) and magnetostatic b) fields associated to infinite rods, of charge and current. The rods come out perpendicularly to the drawing. Linear λ and volumetric ρ charge densities are related by $\lambda = \rho\pi R^2$. The intensity I and current density J are related by $I = J\pi R^2$. R is the radius of the rods (one insulating, the other conducting). Unit vectors $\hat{\mathbf{r}}$ point from the charge or current elements to the observation points. The (dark blue) circles represent, for Gauss's laws, the intersections of cylindrical Gaussian surfaces with the diagram. For line integrals, these circles are integration paths.

TABLE II. Faraday and Ampere-Maxwell equations.

Equation	Differential form	Integral form
Faraday	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint \mathbf{E} \cdot d\mathbf{l} = -\dfrac{d\phi_B}{dt}$
Ampere-Maxwell	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\varepsilon_0 \dfrac{d\mathbf{E}}{dt}$	$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I + \mu_0\varepsilon_0 \dfrac{d\phi_E}{dt}$

3. Time-varying fields. The laws of Faraday and Ampere-Maxwell

Faraday's remarkable proposition of the concept of lines of force bridging the space between interacting charges, currents, and magnets established the starting point for the transcendental work done by Maxwell. Maxwell mathematized this idea and built a unified theory of electromagnetism. The concept of the electromagnetic field, including the novel but essential displacement current ($\mathbf{J_D} = \mu_0\varepsilon_0[\partial\mathbf{E}/\partial t]$), is central to Maxwell's theory.

The general laws of Electromagnetism, including time-dependent interactions, are formed by Gauss's Laws (presented in Table I, valid throughout the whole of Electrodynamics), plus Faraday's Law for induction and Ampere's Law (generalized by Maxwell for $\mathbf{B}$ induced by a changing $\mathbf{E}$). Table II presents the time dependent Maxwell's Equations.

A century and a half of experiments, theory, teaching and applications (cellular phones, electric cars, synchrotrons,) of electromagnetism are based on the equations in Tables I and II.

Electric and magnetic fields ($\mathbf{E}$, $\mathbf{B}$) associated with time variations of ($\mathbf{B}$, $\mathbf{E}$) respectively, are commonly denoted "induced" fields. Figure 2 represents induced fields corresponding to cylindrical symmetries.

The induced $\mathbf{E}$ and $\mathbf{B}$ both run circularly, counter clockwise, according to the respective associated ("inducing") fields $\partial\mathbf{B}/\partial t$ and $\partial\mathbf{E}/\partial t$. Equations (6) and (7) represent the modules of these vectors as functions of the distances to the symmetry axes of each configuration.

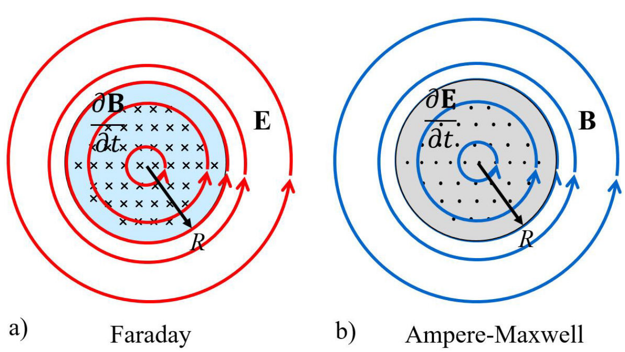

FIGURE 2. Electromagnetic induction in configurations of cylindrical symmetry.

$$E(r) = \begin{Bmatrix} \frac{r}{2}\frac{dB}{dt} & r < R \\ \frac{R^2}{2r}\frac{dB}{dt} & r > R \end{Bmatrix}, \tag{6}$$

$$B(r) = \begin{Bmatrix} \frac{\mu_0\varepsilon_0 r}{2}\frac{dE}{dt} & r < R \\ \frac{\mu_0\varepsilon_0 R^2}{2r}\frac{dE}{dt} & r > R \end{Bmatrix}. \tag{7}$$

A couple of conceptual comments and a discussion topic are worth introducing:

- As described by Eq. (6), an electric field is found in the region $r > R$, where $B = 0$. This field is linked with a changing magnetic field in *another* region of space ($r < R$). Analogously, a varying electric field in $r < R$ induces a magnetic field, even for $r > R$. The local relationship between the components of vectors $\mathbf{E}$ and $\mathbf{B}$ shapes the global structure of the electromagnetic field.

- *Electromagnetic induction* is the answer to the question, related to Eq. (5), and posed in the previous Section.

Topic for discussion

We question the cause-effect relationship in induction phenomena. Take, for example, Faraday's Law. How do we interpret the mathematical expression of this law? In its usual form, as shown in Table II, the general interpretation is that a time-changing magnetic field induces an electric field. If we write this equation in a way that reminds us of Newton's 2nd Law ($d\mathbf{p}/dt = \mathbf{F}$), namely $\partial\mathbf{B}/\partial t = -\nabla \times \mathbf{E}$, it is valid to interpret that the temporal variation of the magnetic field is determined by certain local spatial variations of the electric field. This discussion is proposed by Hill [15, 16], applying an approach consistent with Jefimenko's alternative formulation of electromagnetism [17].

4. Relativity of E and B

The following is a fragment of the already cited EIS given by Einstein at Columbia University, New York ([8] page 113):

"The special aim which I have constantly kept before me is logical unification in the field of physics. To start with, it disturbed me that electro dynamics should pick out one state of motion in preference to others, without any experimental justification for this preferential treatment. Thus arose

the special theory of relativity, which, moreover, welded together into comprehensible unities the electrical and magnetic fields, as well as mass and energy, or momentum and energy, as the case may be".

Also of interest is the renowned textbook "The Feynman Lectures on Physics", which was published some forty years later. It presents, at the undergraduate level, among several clarifying explanations, the relativistic relationship between electricity and magnetism [18]. In Section 13-6, Prof. Feynman shows how the magnetic action of a current-carrying wire on a mobile charge may be predicted on the basis of charge invariance, Gauss's law for electricity and Lorentz contraction. Today, a broad spectrum of levels and treatments for this subject can be found. The following contributions typify from introductory [19–21] to advanced [22, 23] available presentations.

Below is a compact analysis of the topic, focusing on the interaction between an electric current and a moving charge. In Fig. 3 the electric current distribution of Fig. 1 is detailed in side view. The conductor contains positive (cations) and negative (electrons) charges.

Figure 3a) shows the description from the "Laboratory" frame (K). The cations are at rest and the electrons are moving to the right at a speed v_0. Linear charge densities are equal to $\pm\lambda_0$. The conductor has no net charge but carries a current $I = \lambda_0 v_0$ to the left [off the drawing in Fig. 1b)]. The positive charge q, at a distance r from the conductor, is moving to the right with a speed v. Even though $\mathbf{E} = 0$, we shall now "discover" the existence of a force acting on q, by purely electrical and relativistic arguments.

We call K' the "Mobile charge" frame, as shown in Fig. 3b). The tasks to be accomplished are: Find how K' sees the charge distribution, find $\mathbf{E}'$ and also $\mathbf{F}' = q\mathbf{E}'$. Finally, transform $\mathbf{F}'$ to $\mathbf{F}$.

When we stand in the K' system and look at the charges in this configuration, we can see a significant difference from what is observed at K. The conductor is no longer electrically neutral, it is charged. This happens as a combined result of the charge invariance and the Lorentz contraction.

For K', the negative charges in the configuration move more slowly than for K, so the distance between them appears greater than for the observer in the laboratory. On the other hand, K' sees positive charges moving, so it sees them closer together. Uniting these considerations of relative distance with that of relativistic invariance of the charge, it is concluded that, for the mobile charge (K'), the "conductor" has a positive charge. Performing the calculation of the net charge density observed from K', it is found:

$$\lambda' = \frac{\gamma v \lambda_0 v_0}{c^2} = \frac{\gamma v I}{c^2}, \tag{8}$$

where

$$\gamma = \frac{1}{\sqrt{1-(v^2/c^2)}} = \frac{1}{\sqrt{1-\beta^2}}; \quad \beta = \frac{v}{c}, \tag{9}$$

and c is the speed of light in vacuum.

We already know how K' sees the configuration: it sees it as a positively charged infinite wire, with a linear density λ' given by (8). The field E' in K' is sought by applying Gauss's Law, Eq. (3):

$$E' = \frac{\lambda'}{2\pi\varepsilon_0 r'} = \frac{Iv\gamma}{2\pi\varepsilon_0 c^2 r}. \tag{10}$$

From here we obtain that q (positive) will be repelled by the current, with a force of module:

$$F' = qE' = \frac{Iqv\gamma}{2\pi\varepsilon_0 c^2 r}, \tag{11}$$

as measured by K'.

To find the force on q, measured from K, the relativistic force transformation law is applied. The result is the following:

$$F = \frac{1}{\gamma}F' = \frac{Iqv}{2\pi\epsilon_0 c^2 r} = \frac{\mu_0 Iqv}{2\pi r}; c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}. \tag{12}$$

Equation (12) coincides with the one that would have been obtained by applying the magnetic force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, with the vector $\mathbf{B}$ given by Eq. (4). The curious thing about the case is that result (12) was obtained *without* using the idea of a magnetic field.

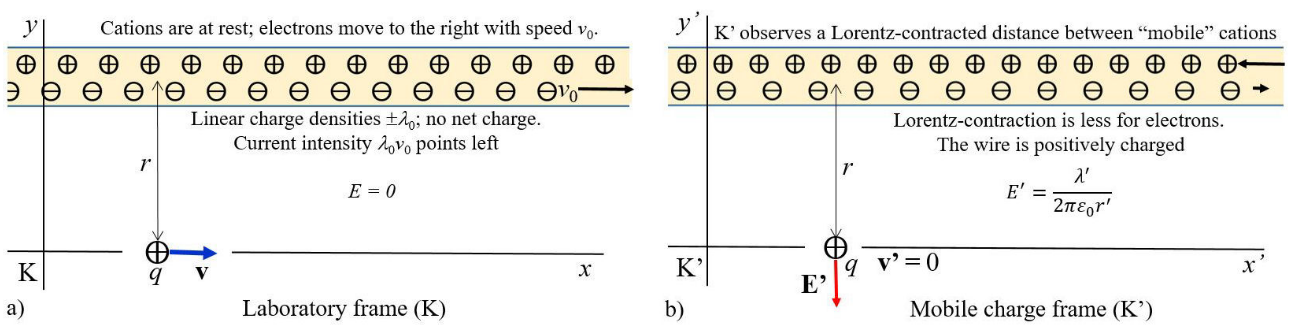

FIGURE 3. Interaction of a moving charge with an electric current.

The following question is pertinent: Is $\mathbf{B}$ introduced for convenience, to avoid looking for a system with q at rest? In other words, would the concept of a magnetic field be dispensable?

No. Einstein's words quoted at the beginning of this section make this clear. A fundamental problem of electrodynamics is that of the structure of the field, which enables the interaction between charges. Thus, the accurate field must be one that correctly describes the interaction between charges, in any dynamic situation and from any inertial frame. Only the combination of $\mathbf{E}$ and $\mathbf{B}$ satisfies this condition.

In our particular case, observing from the laboratory system, there is $\mathbf{B}$, but $\mathbf{E} = 0$. In the mobile charge system, $\mathbf{E}$ and $\mathbf{B}$ exist (are different from zero). The latter does not act only because the charge is stationary.

Topic for discussion

Some authors summarize what was analyzed here with the scheme [Electricity + Relativity → Magnetism] [24,25]. On the other hand, [26] demonstrates the mathematical validity of the opposite hypothesis [Magnetism + Relativity → Electricity]. Who is right?

We present formulas (13). They are the relativistic transformation equations of the electromagnetic fields between two inertial reference systems.

$$\left.\begin{array}{l} E'_x = E_x \\ E'_y = \gamma\left(E_y - vB_z\right) \\ E'_z = \gamma\left(E_z + vB_y\right) \\ B'_x = B_x \\ B'_y = \gamma\left(B_y + (v/c^2)E_z\right) \\ B'_z = \gamma\left(B_z - (v/c^2)E_y\right) \end{array}\right\}. \tag{13}$$

The Eqs. (13) have a high degree of generality, which is consistent with the field concept. If the components of ($\mathbf{E}$, $\mathbf{B}$) are known, at a point in space-time, measured with respect to an inertial reference system, these equations allow the components of ($\mathbf{E}'$, $\mathbf{B}'$) to be calculated at the same point of space-time, measured from another inertial system, which moves with speed v with respect to the first. The application of (13) does not require *any* knowledge about the distant charges or currents that gave rise to these fields. This property–the local character of the relativistic transformation laws–is not inherent to $\mathbf{E}$ or $\mathbf{B}$ separately.

The approximate transformation equations for the case of low velocities ($\beta = v/c \ll 1; \gamma \to 1$), are:

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B},$$

$$\mathbf{B}' = \mathbf{B}. \tag{14}$$

The practical applicability of (14) extends to the approximate limit $\beta \sim 0.1$.

Topic for discussion

It is worth comparing the interpretations of the transformation equations given in different texts [11,23]. If it is considered that the relativistic transformation relations of a physical field must have a local character, is the electromagnetic field a physical field? And the electric one?

The integration as a concept of the electric and magnetic fields, and their interactions, is achieved compactly and elegantly via the relativistic tensor representation of the laws of electromagnetism. An introductory exposition of this formulation is presented next. We follow the lectures by Tong [27] and Hughes [28].

5. Maxwell equations in tensor form

The space of relativity is space-time, four-dimensional. The 4-vector x^η denotes the position of a point in this space:

$$x^\eta = (ct, x, y, z). \tag{15}$$

Greek indexes = 0, 1, 2, 3.

Therefore, the electromagnetic field, a unified entity, is characterized by the electromagnetic field tensor:

$$F^{\eta\xi} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}. \tag{16}$$

The sources of the electromagnetic field are also described by a unified magnitude, the 4-vector current density J^η:

$$J^\eta = (c\rho, J_x, J_y, J_z). \tag{17}$$

A variant of $F^{\eta\xi}$, named its dual and denoted $\widetilde{F}^{\eta\xi}$ or $*F^{\eta\xi}$, is also functional. $\widetilde{F}^{\eta\xi}$ is obtained from $F^{\eta\xi}$ by means of the substitutions $E \to cB$ and $B \to -E/c$:

$$F^{\eta\xi} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & /E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix}. \tag{18}$$

In the tensor formulation, the Maxwell equations are expressed, in progressively compact notations, as follows:

$$\sum_\eta \frac{\partial F^{\eta\xi}}{\partial x^\eta} = \frac{\partial F^{\eta\xi}}{\partial x^\eta} = \partial_\eta F^{\eta\xi} = \mu_0 J^\xi, \tag{19}$$

$$\sum_\eta \frac{\partial \widetilde{F}^{\eta\xi}}{\partial x^\eta} = \frac{\partial \widetilde{F}^{\eta\xi}}{\partial x^\eta} = \partial_\eta \widetilde{F}^{\eta\xi} = 0. \tag{20}$$

Accordingly, Maxwell's equations in their usual form are obtained by expanding Eqs. (19) and (20).

The Gauss Law for electricity is deduced from (19). Calculating the case $\xi = 0$:

$$\frac{\partial E_x}{c\partial x} + \frac{\partial E_y}{c\partial y} + \frac{\partial E_z}{c\partial z} = \mu_0 c\rho \to \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}. \tag{21}$$

The Ampere-Maxwell Law corresponds to the cases $\xi = 1, 2, 3$. We expand $\xi = 1$:

$$-\frac{\partial E_x}{c^2\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x$$
$$\downarrow$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \quad (22)$$

The Gauss Law for magnetism and Faraday's Law are derived from the Eq. (20).

$\xi = 0:$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \rightarrow \nabla \cdot \mathbf{B} = 0, \quad (23)$$

$\xi = 1:$

$$-\frac{\partial B_x}{c\partial x} - \frac{\partial E_z}{c\partial y} + \frac{\partial E_y}{c\partial z} = 0 \rightarrow \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \quad (24)$$

6. Conclusions

This article proposes several tools to help improve students' understanding of some fundamental aspects of the laws of electromagnetism. Included are quotes from Einstein and Bohr on the topics analyzed, the raising of questions open to discussion and numerous citations to historical and current contributions of interest. The class discussion of the proposed questions will probably promote research and deeper learning. The citations (the majority with effective links) may form a guide to further explorations.

We suggest that the following concepts should be highlighted in the development of the electromagnetic course:

- Since Newton, a fundamental goal of physics has been to establish descriptions, at the infinitesimal level of detail, of the cause-effect relationships and the structure of the objects of the material world. The proper mathematical framework to fulfill this purpose comprises differential equations.

- The electromagnetic theory forms a conceptual body consistent with Einstein's special relativity. In this framework, the physical field has electromagnetic nature. Its structure, at the local level in space-time, is described by Maxwell's equations. The infinitesimal variations of the electromagnetic field generate its structure on a global, macroscopic scale.

A couple of cases, representative of the relevance of the local character of the field properties, are:

- The information of what happens in the vicinity of a point in space-time characterizes the interaction between the field components. This point is worth mentioning from the analysis of time-independent fields, but reveals itself in a colorful manner in the study of electromagnetic induction.

- The components of the field are transformed relativistically. The relativistic transformation law (considering both the electrical and magnetic components) has a local character. The interaction via electromagnetic field is correctly characterized from any inertial system.

- Maxwell's equations acquire their most encompassing expression in the tensor formalism of Einstein's relativity.

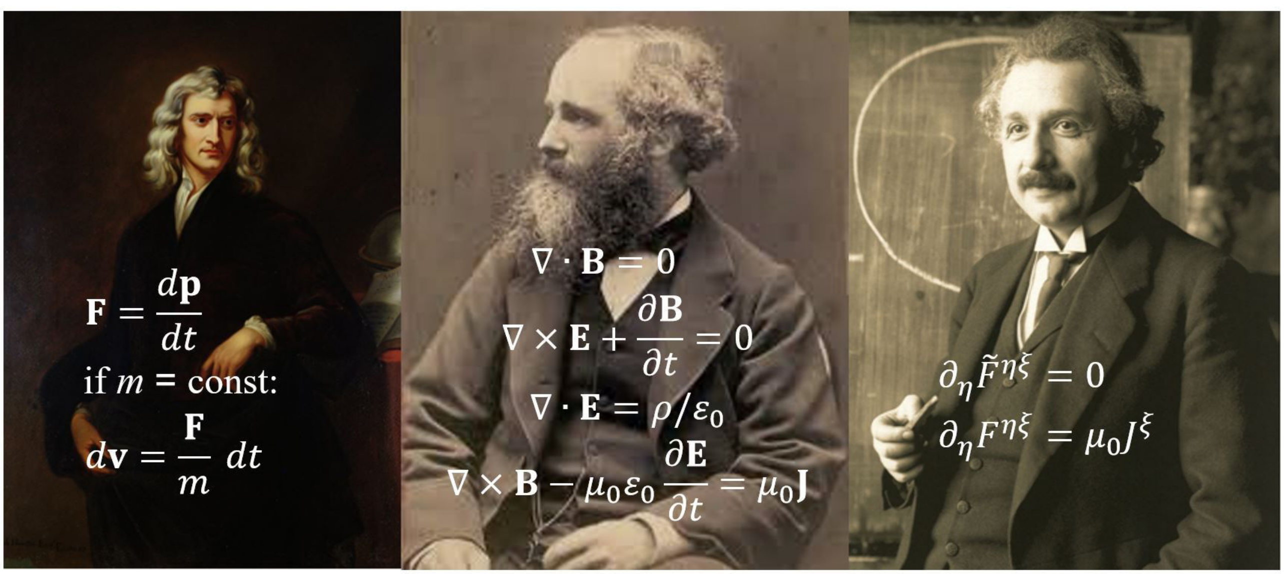

FIGURE 4. Newton, Maxwell, Einstein and a selection of fundamental equations of physics.

Supplementary information

The Appendix A consists of a pictorial tribute to Newton, Maxwell and Einstein for some of their fundamental contributions, expressed in mathematical language, highlighted in the present article. Appendix B shows the conversion of several electromagnetic magnitudes from Maxwell's notation to current symbology.

Appendix A.

Newton, Maxwell, Einstein and a selection of fundamental equations of physics, associated with their contributions, commented in the present article (see Fig. 4).

Appendix B.

The original and present day forms of electromagnetic magnitudes

The objective of the present Appendix is to facilitate the reading of transcendental segments in Maxwell's original writings. In his landmark paper [29], Maxwell characterized fields through their components. As an example, we briefly compare the original symbols in the (currently denoted) Ampere-Maxwell equation with their present-day notation. Maxwell represented it by means of expressions of the type of the Eqs. (B.1):

$$\left.\begin{array}{l} \frac{d\gamma}{dy} - \frac{d\beta}{dz} = 4\pi p' \\ \frac{d\alpha}{dz} - \frac{d\gamma}{dx} = 4\left(\pi q'\right) \\ \frac{d\beta}{dx} - \frac{d\alpha}{dy} = 4\left(\pi r'\right) \end{array}\right\} . \tag{B.1}$$

The magnitudes in Eq. (B.1), in Maxwell's article, have the following meaning:

$\alpha, \beta, \gamma \rightarrow$ components of the intensity of the magnetic field (nowadays H_1, H_2, H_3).

$p' = p + df/dt$.

$p, q, r \rightarrow$ electric current density components (currently j_1, j_2, j_3).

$f, g, h \rightarrow$ components of electric displacement (today D_1, D_2, D_3).

In Table III shows the conversion to the current nomenclature (in the International System of Units) of a selection of quantities presented in the Maxwell's originals. The contemporary notation for Maxwell theoretical ideas is due to Heaviside [30, 31]. The way Faraday and Ampere-Maxwell Laws appear in Heaviside's book (p. 130) is as follows:

$$-curl\mathbf{E} = \mu\dot{\mathbf{H}}, \tag{B.2}$$

$$curl\mathbf{H} = k\mathbf{E} + c\dot{\mathbf{E}}. \tag{B.3}$$

In the present-day symbology, μ is the magnetic permeability, k is the electrical conductivity and c can be related to the product of the permittivity and the permeability.

In Table III shows the equivalence between several symbols used by Maxwell and the ones of current usage.

TABLE III. Electromagnetic magnitudes in Maxwell's notation and in current symbology.

Maxwell works		Current nomenclature (SI)	
Description	Symbols	Description	Symbols
Quantity of Free Electricity	e	Electric Charge Density	$\rho = dq/dV$
Electric Potential	ψ	Electric Potential	ϕ, φ
Electromotive Force	P, Q, R	Electric field	$\mathbf{E} = [E_1, E_2, E_3]$
Electric Displacement	f, g, h	Electric Displacement	$\mathbf{D} = [D_1, D_2, D_3] \quad = \varepsilon_0 \mathbf{E} + \mathbf{P}$
Electric Elasticity	$k \rightarrow P = kf,$	Electric	$\mathbf{D} = \varepsilon \mathbf{E}$
	$Q = kg, R = kh$	Permittivity	(linear isotropic) [$\varepsilon = l/k$(Maxwell)]
Electric Current	p, q, r	Electric Current Density	$\mathbf{J} = [j_1, j_2, j_3]$
Total Electric Current	$p' = p + df/dt,$	Total Electric Current Density	$\mathbf{J}_T = \mathbf{J} + d\mathbf{D}/dt$
	similar for q', r'		
Magnetic Intensity =			
Magnetic Force	α, β, γ	Magnetic Intensity	$\mathbf{H} = [H_1, H_2, H_3]$
Magnetic Induction	$\mu\alpha, \mu\beta, \mu\gamma$	Magnetic Induction	$\mathbf{B} = [B_1, B_2, B_3] \ = \mu_0(\mathbf{H} + \mathbf{M})$
Coefficient of Magnetic Induction	μ	Magnetic permeability	$\mathbf{B} = \mu \mathbf{H}$ (linear-isotropic)
Equations of Currents	$\frac{d\gamma}{dy} - \frac{d\beta}{dz} = 4\pi p',$	Ampere-Maxwell Equation	$\nabla \times \mathbf{H} = J_T = \mathbf{J} + \frac{d\mathbf{D}}{dt}$
	similar for q', r'		
Electric Resistance	$P = -\rho p, Q = -\rho q,$	Electric Resistivity	$\mathbf{E} = \rho \mathbf{J}$
	$R = -\rho r$		
Equation of Free Electricity	$e + \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0$	Gauss Law for Electricity	$\nabla \cdot \mathbf{D} = \rho$
Equation of Continuity	$\frac{de}{dt} + \frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz} = 0$	Equation of Continuity	$\frac{d\rho}{dt} + \nabla \cdot \mathbf{J} = 0$

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