Cartesian isotropic tensors for beginners

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In this paper, we show how to find the isotropic tensors from rank one to four and suggest a way to calculated higher orders following one of the methods exposed here. We describe two methods for calculating the isotropic tensors from rank one to four, almost step by step. An explicit representation of the components of the isotropic tensor from rank one to four is given.

Keywords: Fourth-rank; isotropic; tensor.

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1. Introduction

Tensors are very important in Physics and Mathematics. Originally, tensors appeared just as a logic structure and then they were used naturally in advanced theoretical fields of physics, such as Relativity, Electrodynamics and Field Theory, Nonlinear Optics, among others. In particular, Nonlinear Optics is just the application of Electromagnetic theory to a specific problem in which the nonlinear response of the material is described by the interaction between the electric fields and matter. The nonlinear response of a particular material is given by the susceptibility tensor, which have all the information of the structure of the material and the symmetries present in it. In the case, this material is a crystal, there are tables listing the tensors for their group of symmetry [1,2]. Thus, the physical properties of matter can be described by a tensor. Historically, Voigt was the first person to use the term tensor in reference to mechanical stress [3]. A tensor is an array of numbers representing a physical quantity for a specific set of coordinate axes [3], in our case, Cartesian axes. When this array of numbers is just one number, we call it a scalar or tensor of rank zero. Plain numbers or scalars have no subindex (subscript or suffix or sometimes just index) and just one component. First-rank tensors are commonly known under the name of vectors, which in $\mathbb{R}^3$ have three components with a short representation v_i ($i = 1, 2, 3$ or x, y and z directions), where i is the subindex and there is only one. A quantity with two subindices is a second-rank tensor m_{ij} (i, $j = 1, 2, 3$) and, as in general, people are is familiarized with a matrix, it is a common mistake to consider that a matrix and a second-rank tensor are the same thing; this is not strange because the written representation is the same. However, a matrix is just an array of numbers relating two sets of axes and a tensor is a physical quantity that, for one given set of axes, is represented by nine numbers [3]. Still sometimes, sloppily a tensor of rank two is called a matrix [2].

Of course, a third-rank tensor has three subindices d_{ijk} ($i, j, k = 1, 2, 3$) and in general has 27 components and could be imagined as an array of numbers of 3×3 with three layers, 3 cubes in length, 3 cubes of height and 3 cubes of width (see Fig. 1). This is what usually people call a tensor and superior ranks too. The n-rank tensor, always in $\mathbb{R}^3$, has n subindices $\chi_{x_1x_2x_3\ldots x_n}$ and 3^n components. Thus, the number of subindices is the rank of the tensor. A pictorial representation for the tensors from rank zero to three is shown in Fig. 1.

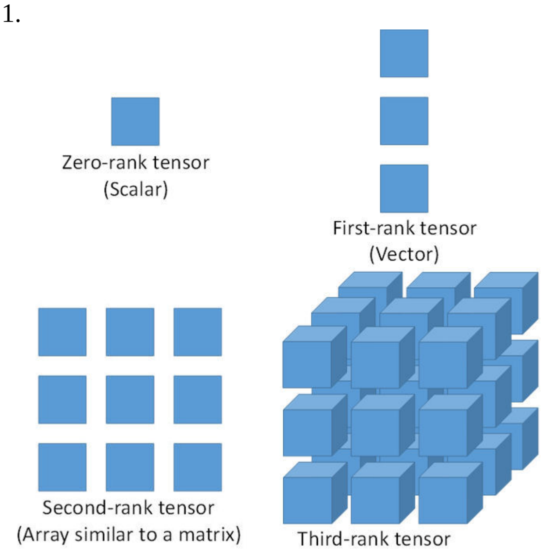

FIGURE 1. A pictorial representation of tensors from rank zero to third.

Up to this point, students start to have problems understanding the concepts. The second-rank tensor is easily understood by the students because they are familiarized with matrices and the representation of both entities is basically the same. But for third-rank tensors and beyond, they do not have a mental image of these mathematical entities. Also, it is common that the professors show the more compact and elegant notation since the beginning, working only with the more general properties and avoiding to show explicitly the elements of the tensors. In some way this approach is harder for the students who face this topic for the first time and darkened the concept and the procedure of calculating through tensors.

Finally, for closing this section, we resume the organization of this work. There is a section for each subsequent tensor from rank one to four (Secs. 2 to 5) and the conclusions are in Sec. 6 with a summary and comments. Also, in Appendix A we explain the path followed to calculate the isotropic tensor of rank four step by step and Appendix B is devoted to mention some applications for isotropic tensors and specific topics, where knowing the isotropic tensors of rank up to four is essential.

2. Isotropic first-rank tensor

Mathematically, all the tensors are defined by following a specific transformation rule. The components of a tensor describe some physical quantity associated with a particular set of orthogonal axes, but if we choose another set of orthogonal axes, the numerical value of the components change and this physical quantity remains the same. It happens that for a first-rank tensor the transformation rule is exactly the same than the one for transforming the coordinates of a point

$$v'_i = \sum_{j=1}^{3} R_{ij} v_j \; = R_{ij} v_j, \tag{1}$$

where in the last equality the summation is implied, following Einstein convention: when a letter subindex occurs twice in the same term, summation with respect to that subindex is to be automatically understood [3]. The subindex j in Eq. (1) is called dummy index and i is a free subindex. The subindex that has been summed (pictorially people called this "contracted") could be any letter $v'_i = R_{ij}v_j = R_{ir}v_r$, provided we do not use a letter that occurs elsewhere in the same term. Here, R_{ij} is a rotation matrix, relating two sets of Cartesian axes. The second-rank tensor follows the rule of transformation

$$m'_{ij} = R_{il} R_{jk} m_{lk}, \tag{2}$$

where now the summation is over subindices l and k, and this transformation rule is similar to the one of the product of two independent transformations of the coordinates of two points ($x'_i x'_j = R_{in}x_n R_{jm}x_m = R_{in}R_{jm}x_n x_m$). In general, an nth-rank tensor transforms by [3]

$$\chi'_{x_1 x_2 x_3 \ldots x_n} = R_{x_1 y_1} R_{x_2 y_2} R_{x_3 y_3} \ldots R_{x_n y_n} \chi_{y_1 y_2 y_3 \ldots y_n}, \tag{3}$$

note that in general, each index is "transformed" and then, we required to contract n-rotation matrices (this is in reality the same rotation matrix) with the nth-rank tensor. Now, something is isotropic when it does not change in function of the direction. In this sense, a tensor is isotropic when it is invariant under rotations. Therefore, all scalars are isotropic tensors, a zeroth-rank tensor is identical to itself because it is not associated to any system of coordinates, whereas there is no isotropic first-rank tensor (vector) [4]. Next, we are going to show this. An anticlockwise rotation around the z-axis by an angle ϕ_z transforms the elements of the vector (first-rank tensor) $\vec{v}$ in a new one $\vec{v}\,'$. Thus, the rotation matrix is

$$\mathbf{R}(\phi_z) = \begin{pmatrix} \cos\phi_z & \sin\phi_z & 0 \\ -\sin\phi_z & \cos\phi_z & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4}$$

where the subscript in the angle of rotation ϕ means that the axis of rotation is the z direction (it is true that the elements in the matrix $\mathbf{R}$ tell us already that, but later on we will see the necessity of labeling the angle of rotation when we change to a more compact notation). This is illustrated in Fig. 2a). Then, applying Eq. (1), there are three equations relating the components of the vectors:

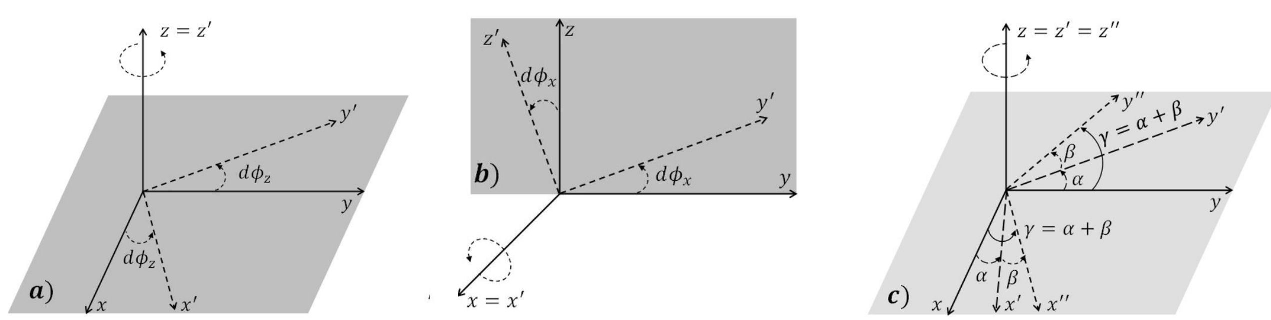

FIGURE 2. Graphical representation of a rotation around: a) z-axis by an angle $d\phi_z$, b) x-axis by an angle $d\phi_x$. c) Two successive rotations, first by an angle α and then by an angle β around the same axis, in this case the z-axis, have the additive result $\gamma = \alpha + \beta$.

$$v'_x = \cos\phi_z v_x + \sin\phi_z v_y, \tag{5}$$

$$v'_y = -\sin\phi_z v_x + \cos\phi_z v_y, \tag{6}$$

$$v'_z = v_z. \tag{7}$$

In the case of small angles, $\cos\phi_z \approx 1$ and $\sin\phi_z \approx d\phi_z$, therefore the last equations can be rewritten as

$$v'_x = v_x + d\phi_z v_y, \tag{8}$$

$$v'_y = -d\phi_z v_x + v_y, \tag{9}$$

$$v'_z = v_z. \tag{10}$$

However, as the tensor should be isotropic $\vec{v} = \vec{v}$, component to component

$$v_x = v_x + d\phi_z v_y, \tag{11}$$

$$v_y = -d\phi_z v_x + v_y, \tag{12}$$

$$v_z = v_z, \tag{13}$$

which reduces immediately to $v_x = v_y = 0$ and v_z is still undetermined. Now we can apply a second rotation around the x-axis, see Fig. 2b) to the resulting vector with only the v_z component. Thus, using the rotation matrix

$$\mathbf{R}(\phi_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_x & \sin\phi_x \\ 0 & -\sin\phi_x & \cos\phi_x \end{pmatrix}, \tag{14}$$

the resulting equation is

$$\begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_x & \sin\phi_x \\ 0 & -\sin\phi_x & \cos\phi_x \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix}, \tag{15}$$

and for the first row, after doing the inner product, we get identity $0 = 0$. The second and third rows yield respectively, the equations

$$0 = d\phi_x v_z, \tag{16}$$

$$v_z = v_z, \tag{17}$$

where, as before, in the case of small angles $\cos\phi_x \approx 1$ and $\sin\phi_x \approx d\phi_x$. In general, $d\phi_x$ is different from zero (or no rotation is applied) and therefore $v_z = 0$ from Eq. (16), whereas Eq. (17) is still consistent with this solution. We want to remember to the lector that successive small rotations around the same axis of rotation, just add and the result is the same that if only one rotation is applied for the total angle given by the combination of the small rotations, as can be seen in Fig. 2c). So, the last result should be valid too for no differential angles.

So, recapitulating, there is no isotropic first-rank tensor other than vector zero. The procedure carried up before, can be performed in any order, this means that it is possible to start doing a rotation around the y-axis and then rotate the resulting vector around the x-axis or the z-axis, and in the end the result should be the same. For high-rank tensors the three rotations around each of the coordinate axis are needed to establish the non-zero components of the isotropic tensor.

3. Isotropic second-rank tensor

As mentioned before, a second-rank tensor has nine components and two subindices $m_{ij}(i,j = 1,2,3)$, that we can represent in the same way that a 3×3 matrix:

$$\overleftrightarrow{m} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}. \tag{18}$$

A graphical representation of a second-rank tensor is given in Fig. 3. The first subindice represents the surface normal to that direction and the second subindice is associated with the

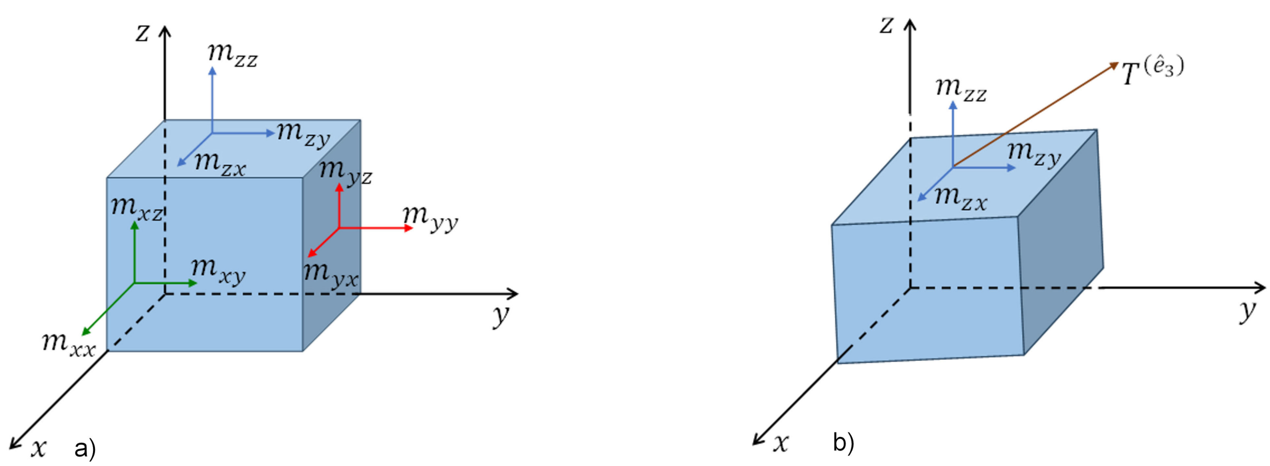

FIGURE 3. Graphical representation of a second-rank tensor. a) The different elements of the tensor. b) An stress vector deformed slightly the cube.

direction of that element of the tensor. For example, m_{zx} is an element in a plane normal to the z-axis and in the direction x. Also, it is possible to say that a second-rank tensor is formed for a basis of three sets of three orthogonal vectors, as can be seen at the top part of Fig. 3. This representation comes from topics of Continuum Mechanics [5], where a small deformation can be modeled using Cauchy Stress tensor σ_{ij} and the stress vector $T^{(\hat{e_i})}$, the latter is shown in the bottom of the Fig. 3, where the deformation is exaggerated for a better appreciation and it is calculated as $T^{(\hat{e_3})} = \sigma_{3j}\hat{e_j}$.

In order to look for the second-rank isotropic tensor, we need to apply the transformation expressed in Eq. (2), for doing this we can multiply directly the matrix given in Eq. (4), this is, we are going to apply a rotation around the z-axis. Then the resulting tensor is transposed and multiplied again for the same rotation matrix Eq. (4) and finally the resulting tensor is transposed again. All this is necessary because the way in which the subindices appear in Eq. (2) implies a specific order in the multiplication between the components of the matrices and the tensor. Then, mathematically

$$\overleftrightarrow{m} = \left[\mathbf{R} \cdot \left(\mathbf{R} \cdot \overleftrightarrow{m}\right)^T\right]^T = \left(\mathbf{R} \cdot \overleftrightarrow{m}\right) \cdot \mathbf{R}^T = \mathbf{R}\left(\phi\right) \cdot \overleftrightarrow{m} \cdot \mathbf{R}\left(-\phi\right), \tag{19}$$

this is because $\left(\mathbf{A} \cdot \mathbf{B}\right)^T = \mathbf{B}^T \cdot \mathbf{A}^T$ and that $\left[\mathbf{R}\left(\phi\right)\right]^T = \mathbf{R}\left(-\phi\right) = \left[\mathbf{R}\left(\phi\right)\right]^{-1}$, for the particular case of the rotation matrices around Cartesians axes x, y and z. Also, the superscript T is the transposition operation whereas the superscript "-1" is the inverse matrix. Combining Eq. (19) with Eqs. (4) and (18) for doing the transformation described by Eq. (2), yields

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} = \begin{pmatrix} 1 & d\phi_z & 0 \\ -d\phi_z & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} 1 & -d\phi_z & 0 \\ d\phi_z & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{20}$$

whereas as before we assume small angles and after doing all the products and just keeping terms until first order in $d\phi_z$, we arrive to

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} = \begin{pmatrix} m_{11} + m_{12} + d\phi_z m_{21} & m_{12} - d\phi_z\left(m_{11} - m_{22}\right) & m_{13} + d\phi_z m_{23} \\ m_{21} - d\phi_z\left(m_{11} - m_{22}\right) & m_{22} - d\phi_z\left(m_{12} + m_{21}\right) & m_{23} - d\phi_z m_{13} \\ m_{31} + d\phi_z m_{32} & m_{32} - d\phi_z m_{31} & m_{33} \end{pmatrix}. \tag{21}$$

From the last equality, Eq. (21), we can establish nine independent relations, with very direct solutions. For example, it is easy to see that $m_{33} = m_{33}$, whereas $m_{23} = m_{13} = m_{32} = m_{31} = 0$. After equaling to zero Eq. (21), in the resulting tensor, the component $(1, 2)$, gives $m_{11} = m_{22}$; and the component $(2, 2)$ yields $m_{12} = -m_{21}$. Finally, combining this very last equality with component $(1, 1)$, we get

$$\left(d\phi_z - 1\right) m_{21} = 0, \tag{22}$$

which immediately leads to $m_{12} = m_{21} = 0$. Now that all the relations between the components (for the transformation around the z-axis) were found, we need to do the same but rotate around a different axis. For the next transformation, we choose the x-axis with the rotation matrix given by Eq. (14) and already substituting the relations found previously. Thus

$$\begin{pmatrix} m_{11} & 0 & 0 \\ 0 & m_{11} & 0 \\ 0 & 0 & m_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & d\phi_x \\ 0 & -d\phi_x & 1 \end{pmatrix} \begin{pmatrix} m_{11} & 0 & 0 \\ 0 & m_{11} & 0 \\ 0 & 0 & m_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -d\phi_x \\ 0 & d\phi_x & 1 \end{pmatrix}, \tag{23}$$

where, as before, after doing the products and dismissing second-order terms in $d\phi_x$, the result is

$$\begin{pmatrix} m_{11} & 0 & 0 \\ 0 & m_{11} & 0 \\ 0 & 0 & m_{33} \end{pmatrix} = \begin{pmatrix} m_{11} & 0 & 0 \\ 0 & m_{11} & -d\phi_x\left(m_{11} - m_{33}\right) \\ 0 & -d\phi_x\left(m_{11} - m_{33}\right) & m_{33} \end{pmatrix}. \tag{24}$$

Therefore, the solution is $m_{33} = m_{11}$ and the second-rank isotropic tensor is

$$\overleftrightarrow{I}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{25}$$

with m_{11} taken as one. This tensor coincides with the identity matrix in its elements and this coincidence generates some confusion about the fact that the identity matrices for higher orders are also the isotropic tensors of that rank, but this is not true as we will see in the next section, $a_{iii} = 1$ is not the isotropic tensor of rank three. Moreover, higher-rank isotropic tensors sometimes have zeros in the "diagonal terms". To finish this section, it is worth mentioning that there is a compact form to represent the second-rank isotropic tensor, the Kronecker's delta δ_{ij} it takes the value one when $i = j$ and zero otherwise. In the next section, we will carry on the same procedure to determine the isotropic tensor form for third-rank.

4. Isotropic third-rank tensor

The number of components in a tensor escalates faster, for a third-rank tensor there are three subindices d_{ijk} $(i, j, k = 1, 2, 3)$ and in general 27 components. One representation of this tensor is as follows

$$\overleftrightarrow{d} = \begin{pmatrix} \begin{pmatrix} d_{111} & d_{121} & d_{131} \\ d_{112} & d_{122} & d_{132} \\ d_{113} & d_{123} & d_{133} \end{pmatrix} \\ \begin{pmatrix} d_{211} & d_{221} & d_{231} \\ d_{212} & d_{222} & d_{232} \\ d_{213} & d_{223} & d_{233} \end{pmatrix} \\ \begin{pmatrix} d_{311} & d_{321} & d_{331} \\ d_{312} & d_{322} & d_{332} \\ d_{313} & d_{323} & d_{333} \end{pmatrix} \end{pmatrix}, \tag{26}$$

where d_{ijk} can be understood in the following way. The first index "i" is related to the external column vector and determines a row in it, the second index "j" and the third one "k" are associated with the columns and rows in the internal 3×3 arrays. Then, the transformation for rotating around an axis is

$$d'_{ijk} = R_{il} R_{jm} R_{kn} d_{lmn}, \tag{27}$$

where one matrix of rotation is applied to each index. A similar procedure than the one described for the second-rank tensor should be done, first applying a rotation around the z-axis, then solving the equations and simplifying a second rotation around the x-axis and again the resulting equations should be solved. After that, a final rotation around the y-axis should be done, and one more time, it is needed to solve the set of equations to arrive at the isotropic third-rank tensor

$$\overleftrightarrow{I}_3 = d_{123} \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix}, \tag{28}$$

which is nothing else than the Levi-Civita tensor ε_{ijk} [4] when we choose $d_{123} = 1$. This result can be verified using Mathematica with the procedure explained in Ref. [6] applied to the tensor given in Eq. (26), where for anticlockwise rotations around all the Cartesian axes x, y and z the result is $d_{123}\varepsilon_{ijk}$ and in the case of clockwise rotation there is only a minus one multiplying the last result.

As can be seen from Eq. (28) when two subindices in the Levi-Civita tensor ε_{ijk} are equal the component is zero and when they are in the order 1-2-3 (et cyclic., even permutation) the component is one, otherwise it is minus one (1-3-2, odd permutation).

Now that we have introduced the isotropic third- and second-rank tensors, we are going to come back to the isotropic first-rank tensor and show that it is zero in a more elegant way. Also, this way of writing the transformation in terms of the isotropic third- and second-rank tensors will be useful to calculate higher order isotropic tensors.

The Eqs. (8)–(10) are writing explicitly for each Cartesian component but it is possible to combine all these expressions in a vectorial one

$$\vec{v}\,' = \vec{v} - d\vec{\phi} \times \vec{v}, \tag{29}$$

where $d\vec{\phi} = d\phi_x \hat{\imath} + d\phi_y \hat{\jmath} + d\phi_z \hat{k}$ and the only angle different from zero is the one that corresponds with the axis of rotation. Eq. (29) has a compact form for the components, which is

$$v'_i = v_i - \varepsilon_{ijk} d\phi_j v_k. \tag{30}$$

On the other hand, the isotropic second-rank tensor can be used to write

$$v_i = \delta_{ik} v_k, \tag{31}$$

which, after being combined with the Eq. (30) results in

$$v'_i = (\delta_{ik} - \varepsilon_{ijk} d\phi_j)\, v_k. \tag{32}$$

It is possible to interchange the index "j" with "k" in all the Eq. (32), but this is going to originate a change in the sign of the Levi-Civita tensor $\varepsilon_{ijk} = -\varepsilon_{ikj}$, then

$$v'_i = (\delta_{ij} + \varepsilon_{ikj} d\phi_k)\, v_j. \tag{33}$$

Finally, to arrive at a useful expression, we can interchange again the index "i" and "k" only in the Levi-Civita tensor $\varepsilon_{ikj} = -\varepsilon_{kij}$, with the result

$$v'_i = (\delta_{ij} - d\phi_k \varepsilon_{kij})\, v_j. \tag{34}$$

Equation (34) is a compact and elegant relation that is equivalent to Eqs. (8) to (10), but these three at the same time come from the rotation defined in Eq. (1). Thus, by direct comparison with this latter:

$$R_{ij} = \delta_{ij} - d\phi_k \varepsilon_{kij}. \tag{35}$$

So, the last equation is another way to calculate a rotation applied to a vector, just expressed in terms of the Kronecker's delta and the Levi-Civita tensor. Coming back to Eq. (34), if an isotropic first-rank tensor exists it must satisfy

$$v_i = (\delta_{ij} - d\phi_k \varepsilon_{kij})\, v_j, \tag{36}$$

that is rewritten as

$$\delta_{ij} v_j = (\delta_{ij} - d\phi_k \varepsilon_{kij})\, v_j, \tag{37}$$

which immediately yields

$$0 = d\phi_k \varepsilon_{kij} v_j. \tag{38}$$

The last equation for the particular case of a rotation around the z-axis ($d\vec{\phi} = d\phi_z \hat{k}$) takes us directly to the simplified version of Eqs. (11) to (13),

$$\begin{pmatrix} d\phi_z v_y \\ -d\phi_z v_x \\ 0 \end{pmatrix} = 0, \tag{39}$$

which immediately implies $v_x = v_y = 0$ and v_z is still undetermined as before. If, now, a second rotation around the x-axis ($d\vec{\phi} = d\phi_x \hat{\imath}$) is applied, then we get

$$\begin{pmatrix} 0 \\ d\phi_x v_z \\ 0 \end{pmatrix} = 0, \tag{40}$$

which is the same result given by the simplified version of Eqs. (16) and (17), with the final result that all the components of the vector $\vec{v}$ are zero and the only isotropic first-rank tensor is the null vector.

Moreover, with this new powerful definition of the rotation matrices Eq. (35), we can redo the procedure to find the second-rank isotropic tensor as follows: According to Eq. (2) and (35)

$$\begin{aligned} m'_{ij} &= R_{il} R_{jk} m_{lk} \\ &= (\delta_{il} - d\phi_p \varepsilon_{pil})(\delta_{jk} - d\phi_q \varepsilon_{qjk}) m_{lk}. \end{aligned} \tag{41}$$

So after doing the products, we arrive to

$$\begin{aligned} m_{ij} = (&\delta_{il}\delta_{jk} - \delta_{il} d\phi_q \varepsilon_{qjk} - d\phi_p \varepsilon_{pil} \delta_{jk} \\ &+ d\phi_p \varepsilon_{pil} d\phi_q \varepsilon_{qjk}) m_{lk}, \end{aligned} \tag{42}$$

where $m'_{ij} = m_{ij}$, because the isotropic condition that we are asking for, after the rotation, the original tensor and the transformed one must be the same. The last term in the parenthesis in Eq. (42) is going to be dropped because it is second order in $d\phi$. Thus, we can rewrite the last equation as

$$m_{ij} - \delta_{il}\delta_{jk} m_{lk} = -\left(d\phi_q \delta_{il} \varepsilon_{qjk} + d\phi_p \delta_{jk} \varepsilon_{pil}\right) m_{lk}, \tag{43}$$

but the left side of the equality is zero and in a similar fashion with Eq. (38), should not be dependence with the angle ϕ, therefore $d\phi_p = d\phi_q$, this means $p = q$, which can also can be justified because we are only doing a rotation around one axis; thus in one rotation, in particular, all the angles implied are the same around a particular fixed axis, x, y or z but not mixed. For this condition Eq. (43) changes to

$$0 = \delta_{il} \varepsilon_{qjk} m_{lk} + \delta_{jk} \varepsilon_{qil} m_{lk}. \tag{44}$$

Finally, we can contract the Kronecker's deltas with the $\overleftrightarrow{m}$ tensors and after some manipulation of the subindices it is possible to write it as

$$m_{ik} \varepsilon_{kqj} = \varepsilon_{iql} m_{lj}. \tag{45}$$

Now, something that we can do, is check directly in the tensors which are these conditions between the tensor components of $\overleftrightarrow{m}$. This is

$$\left(\begin{array}{c} \begin{pmatrix} 0 & -m_{13} & m_{12} \\ m_{13} & 0 & -m_{11} \\ -m_{12} & m_{11} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -m_{23} & m_{22} \\ m_{23} & 0 & -m_{21} \\ -m_{22} & m_{21} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -m_{33} & m_{32} \\ m_{33} & 0 & -m_{31} \\ -m_{32} & m_{31} & 0 \end{pmatrix} \end{array} \right) = \left(\begin{array}{c} \begin{pmatrix} 0 & m_{31} & -m_{21} \\ 0 & m_{32} & -m_{22} \\ 0 & m_{33} & -m_{23} \end{pmatrix} \\ \begin{pmatrix} -m_{31} & 0 & m_{11} \\ -m_{32} & 0 & m_{12} \\ -m_{33} & 0 & m_{13} \end{pmatrix} \\ \begin{pmatrix} m_{21} & -m_{11} & 0 \\ m_{22} & -m_{12} & 0 \\ m_{23} & -m_{13} & 0 \end{pmatrix} \end{array} \right), \tag{46}$$

and it is easy to establish directly that $m_{12} = m_{21} = m_{13} = m_{31} = m_{23} = m_{32} = 0$ and $m_{11} = m_{22} = m_{33}$. Therefore, as we already know from Eq. (25), the second-rank isotropic tensor will be

$$\overleftrightarrow{I}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{47}$$

after taking $m_{11} = 1$; which is nothing else that the Kronecker's delta δ_{ij}. This can be verified immediately from Eq. (45), changing $m_{ik} = \delta_{ik}$ and $m_{lj} = \delta_{lj}$. Thus

$$\delta_{ik} \varepsilon_{kqj} = \varepsilon_{iql} \delta_{lj} \rightarrow \varepsilon_{iqj} = \varepsilon_{iqj}. \tag{48}$$

There is a more elegant way of arriving at the second-rank isotropic tensor, we can multiply Eq. (45) by ε_{iqf} and using the identity

$$\varepsilon_{abc}\varepsilon_{ade} = \delta_{bd}\delta_{ce} - \delta_{be}\delta_{cd}, \tag{49}$$

then

$$m_{ik}\varepsilon_{kqj}\varepsilon_{iqf} = \varepsilon_{iqf}\varepsilon_{iql}m_{lj}, \tag{50}$$

changes to

$$m_{ik}\left(\delta_{ki}\delta_{jf} - \delta_{kf}\delta_{ji}\right) = \left(\delta_{qq}\delta_{fl} - \delta_{ql}\delta_{fq}\right)m_{lj}. \tag{51}$$

Please, note that before applying the identity Eq. (49), the subindices have to be reorganized $\varepsilon_{kqj}\varepsilon_{iqf} = \varepsilon_{qkj}\varepsilon_{qif}$. After multiplying and taking account that $\delta_{qq} = 3$ and $\delta_{ql}\delta_{fq} = \delta_{fl}$, we arrive at

$$m_{ii}\delta_{jf} - m_{jk}\delta_{kf} = 3m_{fj} - \delta_{fl}m_{lj}, \tag{52}$$

where some delta contractions were already done. Contracting the remaining deltas whatever it is possible

$$m_{ii}\delta_{jf} - m_{jf} = 3m_{fj} - m_{fj}, \tag{53}$$

finally

$$m_{ii}\delta_{jf} = 2m_{fj} + m_{jf}. \tag{54}$$

In this expression it is possible to interchange index j and f, arriving at

$$m_{ii}\delta_{fj} = 2m_{jf} + m_{fj}, \tag{55}$$

and subtracting Eq. (55) from Eq. (54)

$$m_{ii}\left(\delta_{jf} - \delta_{fj}\right) = m_{jf} - m_{fj}. \tag{56}$$

Therefore

$$m_{jf} = m_{fj}, \tag{57}$$

because $\delta_{jf} = \delta_{fj}$. Now coming back to Eq. (54) or (55), we get

$$m_{jf} = \frac{m_{ii}}{3}\delta_{jf}. \tag{58}$$

The last equation implies that

$$m_{ij} = a\delta_{ij}, \tag{59}$$

where a is a constant (scalar). This proves that the second-rank isotropic tensor is the Kronecker's delta.

Now, we wish to come back to Eq. (45), and rewrite it in a standard way that appears in the literature [7]:

$$m_{ik}\varepsilon_{kqj} - \varepsilon_{iql}m_{lj} = 0, \tag{60}$$

or

$$\varepsilon_{qjk}m_{ik} + \varepsilon_{qil}m_{lj} = 0, \tag{61}$$

where in the first term we interchanged twice the order of the indices, then the Levi-Civita tensor remains positive; and in the second term we interchanged one time the Levi-Civita tensor indices, which makes it negative. As a final step, the indices that are going to be contracted k and l, are dummy and can be renamed equal, this is

$$\varepsilon_{qjk}m_{ik} + \varepsilon_{qik}m_{kj} = 0. \tag{62}$$

This expression, Eq. (62), will be generalized for the higher-order tensors discussed in the paper. Then we will start with the next isotropic tensor, which is the third-rank tensor.

The procedure, which we started in Eq. (41) can be applied to look for the third-rank isotropic tensor but in this case making use of Eq. (27)

$$\begin{aligned} d'_{ijk} = R_{il}R_{jm}R_{kn}d_{lmn} = \left(\delta_{il} - d\phi_p\varepsilon_{pil}\right) \\ \times\left(\delta_{jm} - d\phi_q\varepsilon_{qjm}\right)\left(\delta_{kn} - d\phi_o\varepsilon_{okn}\right)d_{lmn}, \end{aligned} \tag{63}$$

for the first two factors we already knew that

$$d_{ijk} = R_{il}R_{jm}R_{kn}d_{lmn} = \left(\delta_{il}\delta_{jm} - \delta_{il}d\phi_q\varepsilon_{qjm} - d\phi_p\varepsilon_{pil}\delta_{jm}\right)\left(\delta_{kn} - d\phi_o\varepsilon_{okn}\right)d_{lmn}, \tag{64}$$

where, as before, we only keep terms to first order in $d\phi$ and because we are asking for isotropy the tensor to the left must be the same as the one to the right after the transformation. If the third factor is multiplied, yields

$$d_{ijk} = \left(\delta_{il}\delta_{jm}\delta_{kn} - \delta_{il}\delta_{jm}d\phi_o\varepsilon_{okn} - \delta_{il}d\phi_q\varepsilon_{qjm}\delta_{kn} - d\phi_p\varepsilon_{pil}\delta_{jm}\delta_{kn}\right)d_{lmn}, \tag{65}$$

in the last expression second-order terms $d\phi$ were neglected. Reorganizing the terms

$$d_{ijk} - \delta_{il}\delta_{jm}\delta_{kn}d_{lmn} = -\left(d\phi_o\delta_{il}\delta_{jm}\varepsilon_{okn} + d\phi_q\delta_{il}\delta_{kn}\varepsilon_{qjm} + d\phi_p\delta_{jm}\delta_{kn}\varepsilon_{pil}\right)d_{lmn}. \tag{66}$$

In Eq. (66) the left term of the equality is identically zero and for the right side, the same argumentation holds that for Eq. (43), this is $d\phi_q = d\phi_p = d\phi_o$ or equivalently $q = p = o$. Therefore

$$\delta_{il}\delta_{jm}\varepsilon_{qkn}d_{lmn} + \delta_{il}\delta_{kn}\varepsilon_{qjm}d_{lmn} + \delta_{jm}\delta_{kn}\varepsilon_{qil}d_{lmn} = 0. \tag{67}$$

Finally, it is possible to contract the Kronecker's deltas

$$\varepsilon_{qkn}d_{ijn}+\varepsilon_{qjm}d_{imk}+\varepsilon_{qil}d_{ljk}=0, \tag{68}$$

which is the generalization of the Eq. (62). In the same way, the indices that are going to be contracted n, m and l, are dummy and can be renamed equal, this is

$$\varepsilon_{qkn}d_{ijn}+\varepsilon_{qjn}d_{ink}+\varepsilon_{qin}d_{njk}=0. \tag{69}$$

Solving this equation is somehow cumbersome but possible. Now we are going to show how to do that.

We begin multiplying Eq. (69) by ε_{qkm}, yielding

$$\varepsilon_{qkm}\varepsilon_{qkn}d_{ijn}+\varepsilon_{qkm}\varepsilon_{qjn}d_{ink}+\varepsilon_{qkm}\varepsilon_{qin}d_{njk}=0, \tag{70}$$

and using the identity given in Eq. (49), the Eq. (70) can be simplified to

$$\left(\delta_{kk}\delta_{mn}-\delta_{kn}\delta_{mk}\right)d_{ijn}+\left(\delta_{kj}\delta_{mn}-\delta_{kn}\delta_{mj}\right)d_{ink}+\left(\delta_{ki}\delta_{mn}-\delta_{kn}\delta_{mi}\right)d_{njk}=0, \tag{71}$$

now we proceed to simplify the Kronecker's deltas

$$3\delta_{mn}d_{ijn}-\delta_{nm}d_{ijn}+\delta_{kj}\delta_{mn}d_{ink}-\delta_{kn}\delta_{mj}d_{ink}+\delta_{ki}\delta_{mn}d_{njk}-\delta_{kn}\delta_{mi}d_{njk}=0, \tag{72}$$

or

$$2d_{ijm}+d_{imj}-\delta_{mj}d_{inn}+d_{mji}-\delta_{mi}d_{njn}=0, \tag{73}$$

thus

$$2d_{ijm}+d_{imj}+d_{mji}=\delta_{mj}d_{inn}+\delta_{mi}d_{njn}, \tag{74}$$

and renaming the subscripts $m\rightarrow k$, we arrive at

$$2d_{ijk}+d_{ikj}+d_{kji}=\delta_{kj}d_{inn}+\delta_{ki}d_{njn}. \tag{75}$$

Coming back to Eq. (69) and doing a similar procedure, but now multiplying by ε_{qjm} in one case and by ε_{qim} in the second one, also changing at the end $m\rightarrow j$ in the first case and $m\rightarrow i$ in the second, we get

$$2d_{ijk}+d_{ikj}+d_{jik}=\delta_{ji}d_{nnk}+\delta_{jk}d_{inn}, \tag{76}$$

and

$$2d_{ijk}+d_{kji}+d_{jik}=\delta_{ik}d_{njn}+\delta_{ij}d_{nnk}, \tag{77}$$

respectively. For solving this system of equations, Eqs. (75) to (77), first we multiply Eq. (75) by δ_{ij}, therefore

$$2\delta_{ij}d_{ijk}+\delta_{ij}d_{ikj}+\delta_{ij}d_{kji}=\delta_{ij}\delta_{kj}d_{inn}+\delta_{ij}\delta_{ki}d_{njn}, \tag{78}$$

and simplifying

$$2d_{iik}+d_{iki}+d_{kii}=\delta_{ik}d_{inn}+\delta_{jk}d_{njn}, \tag{79}$$

finally, this is

$$2d_{iik}+d_{iki}+d_{kii}=d_{knn}+d_{nkn}, \tag{80}$$

for the case when $n=i$, immediately gives

$$d_{iik}=0. \tag{81}$$

In a similar way, if we start with the Eq. (76) and multiply it by δ_{ik}, after simplifying and doing $n=i$, we get

$$d_{iji}=0. \tag{82}$$

For the third equation, Eq. (77), the factor multiplying it, should be δ_{kj}; and after simplifying and taking $n=j$ in this case, yields

$$d_{ijj}=0. \tag{83}$$

Equations (81) to (83), imply that when two subindices are equal, the element is zero. With this, the system of equations, Eqs. (75) to (77), reduce to:

$$2d_{ijk}+d_{ikj}+d_{kji}=0, \tag{84}$$

$$2d_{ijk}+d_{ikj}+d_{jik}=0, \tag{85}$$

and

$$2d_{ijk}+d_{kji}+d_{jik}=0, \tag{86}$$

which is a system of three equations with four variables. For solving this system of equations, Eqs. (84) to (86), we can subtract Eq. (84) from Eq. (85), and then

$$d_{jik}=d_{kji}. \tag{87}$$

With this result, Eq. (87), substituted in Eq. (86), gives

$$d_{ijk}=-d_{jik}, \tag{88}$$

and by transitivity, from Eqs. (88) and (87):

$$d_{ijk}=-d_{kji}. \tag{89}$$

Also, the relation given in Eq. (88) can be substituted now in Eq. (85), for obtaining

$$d_{ijk} = -d_{ikj}. \tag{90}$$

Finally, relations from Eq. (88) to (90), mean

$$d_{ijk} = -d_{jik} = -d_{kji} = -d_{ikj}, \tag{91}$$

and together with the Eqs. (81) to (83), imply the definition of the third-rank isotropic tensor, this is Levi-Civita tensor ε_{ijk}, multiplied by some scalar constant b:

$$d_{ijk} = b\varepsilon_{ijk}. \tag{92}$$

This procedure can be replicated for the immediate high rank-tensor and we will show how to do that in the next section. But before that, we want to mention that the advantage to work directly with the elements of the tensors, at least for the ranks with a manageable number of elements (one, two and three) is that you fix the ideas of what are you doing when multiply and add the different contributions from each element of the tensor to the rotation around a particular axis. Clearly, this has the problem that it is a very cumbersome procedure, and it is needed to write big arrays of quantities. In contrast, the compact notation of the elements in terms of the subindices introduced in Sec. 4, is more elegant, quickly to manipulate and short to write but, as mentioned in the introduction, it is more abstract and for the students starting to study this subject it is hard to understand what it is being calculated and how it works. However, as we saw, once these concepts are dominated, the subindex notation is very powerful and allows to calculate high-order isotropic tensors without writing the 81 elements of, for example, the fourth-rank tensor or the 243 elements if we want to calculate the isotropic fifth-rank tensor.

5. Isotropic fourth-rank tensor

Like in previous sections, we start this one by giving a representation of a general fourth-rank tensor:

$$\chi^{(3)}_{ijkl} = \begin{pmatrix} \begin{pmatrix} \chi_{1111} & \chi_{1112} & \chi_{1113} \\ \chi_{1121} & \chi_{1122} & \chi_{1123} \\ \chi_{1131} & \chi_{1132} & \chi_{1133} \end{pmatrix} & \begin{pmatrix} \chi_{1211} & \chi_{1212} & \chi_{1213} \\ \chi_{1221} & \chi_{1222} & \chi_{1223} \\ \chi_{1231} & \chi_{1232} & \chi_{1233} \end{pmatrix} & \begin{pmatrix} \chi_{1311} & \chi_{1312} & \chi_{1313} \\ \chi_{1321} & \chi_{1322} & \chi_{1323} \\ \chi_{1331} & \chi_{1332} & \chi_{1333} \end{pmatrix} \\ \begin{pmatrix} \chi_{2111} & \chi_{2112} & \chi_{2113} \\ \chi_{2121} & \chi_{2122} & \chi_{2123} \\ \chi_{2131} & \chi_{2132} & \chi_{2133} \end{pmatrix} & \begin{pmatrix} \chi_{2211} & \chi_{2212} & \chi_{2213} \\ \chi_{2221} & \chi_{2222} & \chi_{2223} \\ \chi_{2231} & \chi_{2232} & \chi_{2233} \end{pmatrix} & \begin{pmatrix} \chi_{2311} & \chi_{2312} & \chi_{2313} \\ \chi_{2321} & \chi_{2322} & \chi_{2323} \\ \chi_{2331} & \chi_{2332} & \chi_{2333} \end{pmatrix} \\ \begin{pmatrix} \chi_{3111} & \chi_{3112} & \chi_{3113} \\ \chi_{3121} & \chi_{3122} & \chi_{3123} \\ \chi_{3131} & \chi_{3132} & \chi_{3133} \end{pmatrix} & \begin{pmatrix} \chi_{3211} & \chi_{3212} & \chi_{3213} \\ \chi_{3221} & \chi_{3222} & \chi_{3223} \\ \chi_{3231} & \chi_{3232} & \chi_{3233} \end{pmatrix} & \begin{pmatrix} \chi_{3311} & \chi_{3312} & \chi_{3313} \\ \chi_{3321} & \chi_{3322} & \chi_{3323} \\ \chi_{3331} & \chi_{3332} & \chi_{3333} \end{pmatrix} \end{pmatrix}. \tag{93}$$

In a sloppy language, we could say that a general fourth-rank tensor, can be represented as a 3×3 matrix, which also has a 3×3 matrices as elements [2]. This is, a second-rank tensor with elements that are also second-rank tensors. There are a total of 81 elements in one fourth-rank tensor, which are labeled as follows: The first index "i" in χ_{ijkl} corresponds to the rows and the second index "j" to the columns in the main array (the external one). In the same way, the indices "k" and "l" will correspond to the usual way of labeling a 3×3 matrix, namely, the rows and columns in the inner 3×3 array, respectively.

As the lector surely deduces in this section the calculation for determining the fourth-rank isotropic tensor will be just an extension of Eq. (41) and (63), as stated by Eq. (3):

$$s'_{ijkl} = R_{im}R_{jn}R_{ko}R_{lp}s_{mnop} = (\delta_{im} - d\phi_q\varepsilon_{qim})(\delta_{jn} - d\phi_r\varepsilon_{rjn})(\delta_{ko} - d\phi_s\varepsilon_{sko})(\delta_{lp} - d\phi_t\varepsilon_{tlp})\, s_{mnop}, \tag{94}$$

again, we can avoid doing some of the previous algebra, using the fact that for rotation matrices, to first order,

$$R_{ik}R_{jk} = (\delta_{ik} - d\phi_q\varepsilon_{qik})(\delta_{jk} - d\phi_r\varepsilon_{rjk}) = \delta_{ik}\delta_{jk} - d\phi_r\delta_{ik}\varepsilon_{rjk} - d\phi_q\delta_{jk}\varepsilon_{qik} = \delta_{ij} - d\phi_r\varepsilon_{rji} - d\phi_q\varepsilon_{qij}, \tag{95}$$

and interchanging the subindices "i" and "j" in the second term of the last equality, this is the first Levi-Civita tensor and also considering that $d\phi_r = d\phi_q$. Then

$$R_{ik}R_{jk} = \delta_{ij}, \tag{96}$$

in the same way $R_{ki}R_{kj} = \delta_{ij}$. Thus, going back to Eq. (94)

$$R_{jf}R_{ig}s_{ijkl} = R_{jf}R_{ig}R_{im}R_{jn}R_{ko}R_{lp}s_{mnop}, \tag{97}$$

where $s'_{ijkl} = s_{mnop}$, because as mentioned before, the isotropic condition that we are asking for, after the rotation, the original tensor and the transformed one must be the same. Therefore

$$(\delta_{jf} - d\phi_r\varepsilon_{rjf})(\delta_{ig} - d\phi_q\varepsilon_{qig})\, s_{ijkl} = \delta_{fn}\delta_{gm}(\delta_{ko} - d\phi_s\varepsilon_{sko})(\delta_{lp} - d\phi_t\varepsilon_{tlp})\, s_{mnop}. \tag{98}$$

We know from Eq. (43) that, in the first order approximation for $d\phi$, yields

$$\left(\delta_{if}\delta_{ig} - d\phi_q\delta_{jf}\varepsilon_{qig} - d\phi_r\delta_{ig}\varepsilon_{rjf}\right) s_{ijkl} = \left(\delta_{ko}\delta_{lp} - d\phi_t\delta_{ko}\varepsilon_{tlp} - d\phi_s\delta_{lp}\varepsilon_{sko}\right) s_{gfop}, \tag{99}$$

where after contracting the Kronecker's deltas on the first term on the left side and in the first term in the right side, they cancel each other. Also, as they should not be angular dependence and also because we are rotating around one axis at the time, this means that $d\phi_r = d\phi_q = d\phi_t = d\phi_s$ or equivalently $r = q = t = s$. Finally, at the same time, dividing by $d\phi_r$, because this quantity is small but different from zero. Thus

$$\delta_{jf}\varepsilon_{rig}s_{ijkl} + \delta_{ig}\varepsilon_{rjf}s_{ijkl} = \delta_{ko}\varepsilon_{rlp}s_{gfop} + \delta_{lp}\varepsilon_{rko}s_{gfop}, \tag{100}$$

and contracting the remaining Kronecker's deltas, the previous equation simplifies to

$$\varepsilon_{rlp}s_{gfkp} + \varepsilon_{rko}s_{gfol} - \varepsilon_{rig}s_{ifkl} - \varepsilon_{rjf}s_{gjkl} = 0. \tag{101}$$

The negative sign in the third and fourth terms can be absorbed in the Levi-Civita tensor interchanging one time the subindices:

$$\varepsilon_{rlp}s_{gfkp} + \varepsilon_{rko}s_{gfol} + \varepsilon_{rgi}s_{ifkl} + \varepsilon_{rfj}s_{gjkl} = 0. \tag{102}$$

In the same way, that with the tensors of rank two and three, the indices that are going to be contracted i, j, p and o, are dummy and can be renamed equal. This is, after doing the changes of indices and reorganizing the terms:

$$\varepsilon_{rgi}s_{ifkl} + \varepsilon_{rfi}s_{gikl} + \varepsilon_{rki}s_{gfil} + \varepsilon_{rli}s_{gfki} = 0. \tag{103}$$

The last relation is again, a generalization of Eqs. (62) and (69). Solving Eq. (103) is possible but requires a very large algebraic procedure. Instead, first we are going to show that the product of two Kronecker's deltas with different subindices is an isotropic four-rank tensor. If we return to Eq. (94) and conjecture that $s_{mnop} = \delta_{mn}\delta_{op}$, then

$$s_{ijkl} = R_{im}R_{jn}R_{ko}R_{lp}\delta_{mn}\delta_{op}, \tag{104}$$

and contracting the deltas with the rotation matrices

$$s_{ijkl} = R_{in}R_{jn}R_{kp}R_{lp}, \tag{105}$$

but according to Eq. (96), this is

$$s_{ijkl} = \delta_{ij}\delta_{kl}. \tag{106}$$

Therefore, the product of two Kronecker's deltas with different subindices is an isotropic four-rank tensor. However, there is still the freedom of choosing the order of the subindices in the deltas, which is a permutation of the four indices, that is $4! = 24$, but as the permutation of the subindices in a Kronecker's delta is the same delta $\delta_{ij} = \delta_{ji}$ and the same is true for the second one $\delta_{kl} = \delta_{lk}$ and also the product is counted twice because $\delta_{ij}\delta_{kl} = \delta_{kl}\delta_{ij}$. Then there are only three possibilities $(24/2^3)$. All these three different combinations of the four subindices are an isotropic four-rank tensor and thus the more general one is a linear combination of these possibilities:

$$I_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}, \tag{107}$$

where α, β and γ are arbitrary constants (scalars). An explicit representation with all the non-zero components is [4,7,8]

$$\overleftrightarrow{I}_4 = \begin{pmatrix} \begin{pmatrix} \alpha+\beta+\gamma & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} & \begin{pmatrix} 0 & \beta & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & \beta \\ 0 & 0 & 0 \\ \gamma & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \gamma & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha+\beta+\gamma & 0 \\ 0 & 0 & \alpha \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta \\ 0 & \gamma & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & \gamma \\ 0 & 0 & 0 \\ \beta & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \gamma \\ 0 & \beta & 0 \end{pmatrix} & \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha+\beta+\gamma \end{pmatrix} \end{pmatrix}. \tag{108}$$

As mentioned before, there is another way to derive Eq. (108), this can be done departing from Eq. (103), we are going to outline the principal steps here. In a similar way that for the third-rank isotropic tensor, we start multiplying Eq. (103) by ε_{rgj}, this is

$$\varepsilon_{rgj}\varepsilon_{rgi}s_{ifkl}+\varepsilon_{rgj}\varepsilon_{rfi}s_{gikl}+\varepsilon_{rgj}\varepsilon_{rki}s_{gfil}+\varepsilon_{rgj}\varepsilon_{rli}s_{gfki}=0, \tag{109}$$

which can be reexpressed, because Eq. (49), as

$$\left(\delta_{gg}\delta_{ji}-\delta_{gi}\delta_{jg}\right)s_{ifkl}+\left(\delta_{gf}\delta_{ji}-\delta_{gi}\delta_{jf}\right)s_{gikl}+\left(\delta_{gk}\delta_{ji}-\delta_{gi}\delta_{jk}\right)s_{gfil}+\left(\delta_{gl}\delta_{ji}-\delta_{gi}\delta_{jl}\right)s_{gfki}=0, \tag{110}$$

after simplifying and rearranging terms

$$2s_{jfkl}+s_{fjkl}+s_{kfjl}+s_{lfkj}=\delta_{jf}s_{iikl}+\delta_{jk}s_{ifil}+\delta_{jl}s_{ifki}, \tag{111}$$

and finally, if we want to keep easy tracking of the changes in the equations, we can rename the subindices as follows $j\rightarrow i$, $f\rightarrow j$ and $i\rightarrow m$. In this way, we get one of Hodge's equations [7]:

$$2s_{ijkl}+s_{jikl}+s_{kjil}+s_{ljki}=\delta_{ij}s_{mmkl}+\delta_{ik}s_{mjml}+\delta_{il}s_{mjkm}. \tag{112}$$

In order to get the other three equations that appears in Hodge's paper, we need to multiply again Eq. (103) but now by ε_{rfj} , ε_{rkj} and ε_{rlj} , after doing similar steps that with the previous equation, yield respectively,

$$2s_{ijkl}+s_{ikjl}+s_{jikl}+s_{ilkj}=\delta_{ji}s_{mmkl}+\delta_{jk}s_{imml}+\delta_{jl}s_{imkm}, \tag{113}$$

$$2s_{ijkl}+s_{ijlk}+s_{kjil}+s_{ikjl}=\delta_{kl}s_{ijmm}+\delta_{ki}s_{mjml}+\delta_{kj}s_{imml}, \tag{114}$$

and

$$2s_{ijkl}+s_{ljki}+s_{ilkj}+s_{ijlk}=\delta_{li}s_{mjkm}+\delta_{lj}s_{imkm}+\delta_{lk}s_{ijmm}. \tag{115}$$

A detailed explanation can be consulted at the end of this paper in the Appendix A. Now, for going further, we are going to conjecture the form of the terms with two subindices repeated. This is, in Eq. (112), s_{mmkl} is a fourth-rank tensor but $\delta_{ij}s_{mmkl}$ should be also a fourth-rank tensor. So, s_{mmkl} should be something that can be reduced to a second-rank tensor. Thus, a possible representation for this is

$$\delta_{ij}s_{mmkl}=A\delta_{ij}\delta_{lm}\delta_{mk}=A\delta_{ij}\delta_{lk}, \tag{116}$$

where A is a scalar constant and similar expressions apply for all the terms with two subindices m. Therefore, the new system of equations is:

$$2s_{ijkl}+s_{jikl}+s_{kjil}+s_{ljki}=A\delta_{ij}\delta_{kl}+B\delta_{ik}\delta_{jl}+C\delta_{il}\delta_{jk}, \tag{117}$$

$$2s_{ijkl}+s_{ikjl}+s_{jikl}+s_{ilkj}=A\delta_{ji}\delta_{kl}+B\delta_{jk}\delta_{il}+C\delta_{jl}\delta_{ik}, \tag{118}$$

$$2s_{ijkl}+s_{ijlk}+s_{kjil}+s_{ikjl}=A\delta_{kl}\delta_{ij}+B\delta_{ki}\delta_{jl}+C\delta_{kj}\delta_{il}, \tag{119}$$

$$2s_{ijkl}+s_{ljki}+s_{ilkj}+s_{ijlk}=A\delta_{li}\delta_{jk}+B\delta_{lj}\delta_{ik}+C\delta_{lk}\delta_{ij}, \tag{120}$$

where, as before, A, B and C are scalar constants. With these four equations, Eqs. (117) to (120), we can generate six more adding by pairs of the latter ones and from these we can get relations between the elements of the isotropic tensor that we are looking for. The explicit procedure is explained in Appendix A. Therefore, after solving the system of equations:

$$\alpha=\frac{4B-A-C}{10}, \tag{121}$$

$$\beta=\frac{4C-A-B}{10}, \tag{122}$$

and

$$\gamma=\frac{4A-B-C}{10}. \tag{123}$$

Hence, as stated by Eq. (107),

$$s_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}, \tag{124}$$

is an isotropic tensor of rank four. A different approach can be consulted in Jeffreys' book [8] and also a third way of showing which are the components of the isotropic fourth-rank tensor dividing them in five classes is given by Chandrasekharaiah and Debnath in their book [9]. These authors take explicit angles to apply the rotations (for example 90°C or 45°C). In the same way, Jeffreys [10] also took some specific angles to show how to calculate the isotropic fourth-rank tensor, but as in others mathematical classical papers [7,10,11] for experts in this subject, this is only outlined. Additionally, Hysa [12] calculates the isotropic second-rank tensor in general, doing the main steps and then for the isotropic fourth-rank tensor evaluates the rotation for 90°C about the z-axis. He also fixed specific values for the subindices in order to simplify the system of equations to solve. In contrast, in the procedure followed by us, we use a general rotation for an angle ϕ and somehow take a more general case, when we considered that $n = i$ in Eq. (80) and resulting in Eq. (81), for finding the isotropic third-rank tensor or in Eq. (116) when we conjecture the form of s_{mmkl}. Above this, we considered that our procedure is more general and detailed.

Before finishing this section, we want to mention that our conjecture of how to calculate the isotropic fourth-rank tensor, applied in Eq. (104), could work too for the isotropic fifth-rank tensor. This is because, for each subindex there is a rotation matrix transforming the tensor and contracting with one subindex. So, this is like "factorize" in a second-rank tensor multiplied by a third-rank tensor, that is

$$t_{ijklm} = R_{in}R_{jo}R_{kp}R_{lq}R_{mr}\delta_{no}\varepsilon_{pqr} = (R_{in}R_{jo}\delta_{no})\,(R_{kp}R_{lq}R_{mr}\varepsilon_{pqr})\,, \tag{125}$$

and therefore

$$t_{ijklm} = \delta_{ij}\varepsilon_{klm}, \tag{126}$$

t_{ijklm} should be an isotropic fifth-rank tensor. As before, there are other combinations of the five subindices that generate also isotropic fifth-rank tensors and the linear combination of these different isotropic fifth-rank tensors should be the more general isotropic fifth-rank tensor. It is known that the number of isotropic tensors of rank five is a total of six [11]. Then this idea could be extended to look for isotropic tensors of higher rank, the fifth-rank tensor analysis will be published elsewhere.

6. Conclusions

In summary, we have shown how to calculate isotropic tensors from rank one to four. The zero-rank isotropic tensor is all the scalars. The first-rank isotropic tensor is the vector zero, there is no other vector that fulfills the isotropic condition than that one. The isotropic tensor of rank two is the Kronecker's delta, whereas the third-rank tensor is the Levi-Civita third-rank tensor. Finally, the isotropic fourth-rank tensor is a linear combination of the product of all the combinations of two Kronecker's deltas with four independent subindices. The formal procedure shown here can be extended to look for isotropic tensors of higher rank.

Appendix A.

A detailed development of the procedure to solve the system of equations that generates the fourth-rank isotropic tensor is given in this appendix. Continuing with the calculation started with Eq. (109), as mentioned before, Eq. (103) is multiplied by ε_{rfj}, and after doing similar steps to the ones carried away to get Eq. (112), but only this time changing $g \to i$ and $i \to m$, Eq. (113) is obtained. The third equation to solve is the result of multiplying by ε_{rkj} the Eq. (103) and after simplifying and doing $g \to i$, $f \to j$, $j \to k$ and $i \to m$, we arrive at Eq. (114). Finally, the last equality for this system is obtained by multiplying again Eq. (103) but on this occasion by ε_{rlj}. Thus, after interchanging $g \to i$, $f \to j$, $j \to l$ and $i \to m$, leads to Eq. (115). After this system of four equations is complete, the next step is to use the conjecture introduced in Eq. (116) , then the Eqs. (117) to (120) are obtained. Now we can generate six more equations adding by pairs the latter ones. The addition of Eq. (117) with the Eq. (118) gives

$$4s_{ijkl} + 2s_{jikl} + s_{kjil} + s_{ljki} + s_{ikjl} + s_{ilkj} = 2\,(A\delta_{ij}\delta_{kl} + B\delta_{ik}\delta_{jl} + C\delta_{il}\delta_{jk})\,, \tag{A.1}$$

in the same way, Eq. (117) plus Eq. (119) results on

$$4s_{ijkl} + 2s_{kjil} + s_{jikl} + s_{ljki} + s_{ijlk} + s_{ikjl} = 2\,(A\delta_{ij}\delta_{kl} + B\delta_{ik}\delta_{jl} + C\delta_{il}\delta_{jk})\,, \tag{A.2}$$

whereas the pair Eq. (117) plus Eq. (120) yields

$$4s_{ijkl} + 2s_{ljki} + s_{jikl} + s_{kjil} + s_{ilkj} + s_{ijlk} = 2\left(A\delta_{ij}\delta_{kl} + B\delta_{ik}\delta_{jl} + C\delta_{il}\delta_{jk}\right). \tag{A.3}$$

Two more combinations come from Eq. (118) with Eq. (119)

$$4s_{ijkl} + 2s_{ikjl} + s_{jikl} + s_{ilkj} + s_{ijlk} + s_{kjil} = 2\left(A\delta_{ij}\delta_{kl} + B\delta_{ik}\delta_{jl} + C\delta_{il}\delta_{jk}\right), \tag{A.4}$$

and Eq. (118) with Eq. (120)

$$4s_{ijkl} + 2s_{ilkj} + s_{ikjl} + s_{jikl} + s_{ljki} + s_{ijlk} = 2\left(A\delta_{ij}\delta_{kl} + B\delta_{ik}\delta_{jl} + C\delta_{il}\delta_{jk}\right). \tag{A.5}$$

The final equation comes from the last combination possible, which is Eq. (119) plus Eq. (120):

$$4s_{ijkl} + 2s_{ijlk} + s_{kjil} + s_{ikjl} + s_{ljki} + s_{ilkj} = 2\left(A\delta_{ij}\delta_{kl} + B\delta_{ik}\delta_{jl} + C\delta_{il}\delta_{jk}\right). \tag{A.6}$$

From these six equations, we can get relations between the elements of the isotropic tensor that we are looking for. In particular, subtracting Eq. (A.6) from Eq. (A.1), leads to

$$s_{jikl} = s_{ijlk}, \tag{A.7}$$

also taking Eq. (A.2) minus Eq. (A.5) gives

$$s_{kjil} = s_{ilkj}, \tag{A.8}$$

and the last relation can be obtained from Eq. (A.3) minus Eq. (A.4):

$$s_{ljki} = s_{ikjl}. \tag{A.9}$$

Please, note that it is possible to generate additional relations, for example, interchanging or renaming the indices, if l is interchanged with k in Eq. (A.7), then

$$s_{jilk} = s_{ijkl}, \tag{A.10}$$

and so on. On the other hand, coming back to Eq. (118) and substituting Eq. (A.7), yields

$$2s_{ijkl} + s_{ikjl} + s_{ijlk} + s_{ilkj} = A\delta_{ji}\delta_{kl} + B\delta_{jk}\delta_{il} + C\delta_{jl}\delta_{ik}, \tag{A.11}$$

and proceeding as Hodge says [7], fixing subindex i but permutating in a cyclic way j, k and l, this generates, two more equations

$$2s_{iljk} + s_{ijlk} + s_{ilkj} + s_{ikjl} = A\delta_{li}\delta_{jk} + B\delta_{lj}\delta_{ik} + C\delta_{lk}\delta_{ij}, \tag{A.12}$$

and

$$2s_{iklj} + s_{ilkj} + s_{ikjl} + s_{ijlk} = A\delta_{ki}\delta_{lj} + B\delta_{kl}\delta_{ij} + C\delta_{kj}\delta_{il}. \tag{A.13}$$

Thus, adding these three equations, Eq. (A.11) to (A.13), leads to

$$2\left(s_{ijkl} + s_{iljk} + s_{iklj}\right) + 3\left(s_{ijlk} + s_{ilkj} + s_{ikjl}\right) = (A + B + C)\left(\delta_{ji}\delta_{kl} + \delta_{jk}\delta_{il} + \delta_{jl}\delta_{ik}\right). \tag{A.14}$$

Moreover, interchanging just l with k, gives

$$2\left(s_{ijlk} + s_{ikjl} + s_{ilkj}\right) + 3\left(s_{ijkl} + s_{iklj} + s_{iljk}\right) = (A + B + C)\left(\delta_{ji}\delta_{lk} + \delta_{jl}\delta_{ik} + \delta_{jk}\delta_{il}\right), \tag{A.15}$$

and multiplying Eq. (A.14) by 3 and subtracting two times Eq. (A.15), this yields

$$5\left(s_{ijlk} + s_{ilkj} + s_{ikjl}\right) = (A + B + C)\left(\delta_{ji}\delta_{kl} + \delta_{jk}\delta_{il} + \delta_{jl}\delta_{ik}\right), \tag{A.16}$$

or

$$s_{ijlk} + s_{ilkj} + s_{ikjl} = \frac{1}{5}(A + B + C)\left(\delta_{ji}\delta_{kl} + \delta_{jk}\delta_{il} + \delta_{jl}\delta_{ik}\right). \tag{A.17}$$

With this result, Eq. (A.17), we can go back to Eq. (A.11) and get

$$2s_{ijkl} + \frac{1}{5}(A + B + C)\left(\delta_{ji}\delta_{kl} + \delta_{jk}\delta_{il} + \delta_{jl}\delta_{ik}\right) = A\delta_{li}\delta_{jk} + B\delta_{lj}\delta_{ik} + C\delta_{lk}\delta_{ij}, \tag{A.18}$$

and finally, reorganizing the terms,

$$s_{ijkl} = \frac{1}{10}\left[(4B - A - C)\,\delta_{lk}\delta_{ij} + (4C - A - B)\,\delta_{lj}\delta_{ik} + (4A - B - C)\,\delta_{li}\delta_{jk}\right]. \tag{A1.19}$$

So, comparing with Eq. (107), it is direct to establish Eqs. (121) to (123).

Appendix B

This section is devoted to mentioning some applications of isotropic tensors in Continuum Mechanics and in Non-linear Optics second harmonic generation topics; for the latter case, it is essential to know the isotropic tensors for the ranks three and four. Additionally, the third-rank isotropic tensor is used to establish several relations in physics and mathematics in a compact way, as for example, the Electromagnetic tensor.

Elasticity

In elasticity there are four-rank tensors relating linearly the stress $\mathbf{T}$ and the infinitesimal strain $\mathbf{E}$, in the following way [5]

$$\mathbf{T} = \mathbb{S}\mathbf{E}, \tag{B.1}$$

or

$$\mathbf{E} = \mathbb{C}\mathbf{T}, \tag{B.2}$$

whereas $\mathbb{S}$ is the stiffness tensor and $\mathbb{C}$ is the compliance tensor, and they are the inverses of one another. The general elasticity tensor for an isotropic material is an isotropic fourth-rank tensor and can be represented as [13]

$$\mathbb{S}_{iso} = 3\kappa\mathbb{J} + 2\mu\mathbb{K}, \tag{B.3}$$

where $\mathbb{J}$ and $\mathbb{K}$ are two linearly independent symmetric fourth-rank tensors defined by

$$\mathbb{J} = \frac{1}{3}\overleftrightarrow{I}_2 \otimes \overleftrightarrow{I}_2, \tag{B.4}$$

and

$$\mathbb{K} = \mathbb{I} - \mathbb{J}, \tag{B.5}$$

where $\mathbb{I}$ denotes the fourth-rank identity tensor, which is defined in terms of its components as $I_{ijkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2$ and $\overleftrightarrow{I}_2$ is the second-rank isotropic tensor. Also, the constants κ and μ are positive and are named the bulk modulus and the shear modulus, respectively. With these quantities, it is possible to calculate the Young's modulus [14] for an isotropic material:

$$\langle E(\mathbf{n})\rangle = \frac{9\kappa\mu}{3\kappa + \mu}. \tag{B.6}$$

This is a very important measurable parameter for materials science and engineering.

Non-linear optics: Harmonic generation

In our case, a medium is nonlinear if the polarization P is not linearly proportional to the electric field E. Then, the nonlinear polarization can be expressed as a Taylor series in E [1]:

$$P_i = \chi_{ij}^{(1)}E_j + \chi_{ijk}^{(2)}E_jE_k + \chi_{ijkl}^{(3)}E_jE_kE_l + ... \tag{B.7}$$

The first term denotes the linear susceptibility with indices i and j, which corresponds to a second-rank tensor, resulting in $3 \times 3 = 9$ components. The next term is related to second-order nonlinear processes such as second harmonic generation (SHG) [15] and is characterized by a second-order susceptibility (third-rank tensor) with indices i, j and k, yielding $3 \times 3 \times 3 = 27$ components. On the other hand, the next term describes third-order nonlinear processes such as third harmonic generation (THG) [16] and consists of a fourth-order susceptibility tensor defined by indices i, j, k and l, and contains a total of $3 \times 3 \times 3 \times 3 = 81$ components.

On the other hand, a usual way of measuring the harmonic generation signal from a sample, is to rotate the sample around the normal to the surface, the direction of the normal is typically labeled the z-axis. These samples generally are crystals and the harmonic signal (intensity) changes with the azimuthal angle of rotation as a linear combination of a sinusoidal function and its harmonics. In this case, the main contribution to the signals comes from the anisotropic part of the susceptibility tensor, whereas the isotropic part of the susceptibility tensor contributes with a constant signal [17]. Therefore, the susceptibility tensor can be separated into two tensors, the isotropic part and the anisotropic part:

$$\overleftrightarrow{\chi} = \overleftrightarrow{\chi}_{iso} + \overleftrightarrow{\chi}_{ani}, \tag{B.8}$$

respectively. In this way, the isotropic tensors are not useful in this case but it is indispensable to know them for getting the anisotropic part of the susceptibility tensors.

Electromagnetic tensor

As we already saw in Eqs. (29) and (30), that the cross product can be written in terms of its components using the third-rank isotropic tensor, also known as the Levi-Civita tensor. In the same way exists a relation between the Electromagnetic tensor (also known as the field-strength tensor, Faraday tensor or Maxwell bivector) and the magnetic flux density $\mathbf{B}$, this is

$$B_i = -\frac{1}{2}\varepsilon_{ijk}F_{jk}, \tag{B.9}$$

in Cartesian coordinates. Here F_{jk} is the Electromagnetic tensor and the subindices i, j and k go from 1 to 4. This can be represented as follows

$$\overleftrightarrow{F} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{B.10}$$

Explaining further in this topic is beyond the scope of this work, but the interested reader can consult Jackson's classical book in Electromagnetic Theory [18].

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