Using Desmos for visualizing Fourier series in mathematical a physics course

E. Pratidhina

Department of Physics Education, Widya Mandala Surabaya Catholic University, Indonesia.

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The Fourier series has been used widely to model various physical phenomena. It becomes a core concept taught in mathematical physics courses. To help students understand the concept and interpretation of the Fourier series, we propose the utilization of the `Desmos` application in the classroom. `Desmos` has a feature of a graphing calculator that can be used for visualizing a function without programming. With an appropriate teaching approach, the Fourier series can be understood more easily.

Keywords: `Desmos`; Fourier series; visualization; mathematical physics.

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1. Introduction

Mathematics and physics have a close relationship. Mathematics has an important role in physics [1,2]. Mathematics is used as a tool for modeling physical phenomena, conceptualizing physical processes, making predictions, and solving physical problems [3]. Hence, in an undergraduate program, a mathematical physics course is given to early-year students majoring in physics. In this course, students are introduced to basic mathematical tools that are usually used in basic and advanced physics courses. Some of them are differential equations, linear algebra, vector analysis, tensor analysis, Fourier series, and Fourier transform.

Fourier series is a fundamental mathematical concept that is useful for modeling various physical phenomena, such as electrical signals [4], sound waves [5], electromagnetic waves (4), heat transfer mechanisms [6], communication systems [7], fluids [8], properties of crystals [9], etc. Because it is a fundamental concept, the Fourier series is usually taught during the early years of college for physics or engineering majors. Early-year students often face difficulty when trying to relate the Fourier series concept with its applications. Visualization may help students interpret the meaning of the summation of each term in the Fourier series and relate it with the actual physical phenomena.

Visualization of the Fourier series can be assisted using free applications like `Desmos`. `Desmos` is an online application developed to help everyone learn math, love math, and grow with math [10]. One of the main available tools is Graphing Calculator and it is free. Compared to other applications used for visualizing equations, the `Desmos` Graphing Calculator is simple to use, it does not require any coding or specific data entry. `Desmos` has potency as a powerful pedagogical tool for math subjects in high school, especially in graphical representation [11-13]. A study shows that incorporating `Desmos` in class has a positive impact on students' general understanding of function concepts and students' ability to analyze function [14]. `Desmos` has also been used as a medium to teach the concept of limit [15].

In this paper, we would like to describe an alternative learning activity that involves observation of physical phenomena, analytical modeling using the Fourier series formula, visualizing using Demos, and interpretations.

2. Theory

A periodic function can be expanded into a series of sines and cosines. Suppose a function $f(t)$ has a period of 2π, we can write $f(t)$ as:

$$f(t) = \frac{1}{2}a_0 + a_1\cos t + a_2\cos 2t + a_3\cos 3t + ...$$
$$+ b_1\sin t + b_2\sin 2t + b_3\sin 3t + ...,$$
$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nt + b_n\sin nt. \tag{1}$$

The coefficients a_n and b_n can be determined through the formula:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt, \tag{2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\, dt. \tag{3}$$

3. Method

For visualizing Fourier series expansion that has been yielded through analytical methods, the feature of the Graphic Calculator in `Desmos` is used. How to use the Graphic Calculator in `Desmos` is very straightforward. It can be accessed through https://www.desmos.com/calculator. Figure 1 shows the program layout. There is a box for typing the Fourier series equations. The range of the x-axis and y-axis can be set using the panel on the right side (see Fig. 1). The graph will be shown in the graph panel. The program can visualize two or more equations simultaneously by adding more equations.

For implementing `Desmos` in a mathematical physics course, a scenario presented in Fig. 2 can be used. At first, the teacher may explain the Fourier series concept to students.

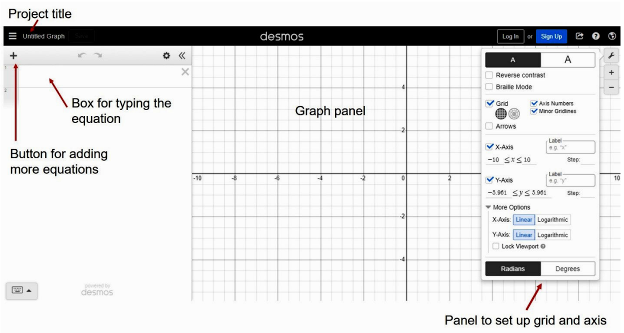

FIGURE 1. The layout of `Desmos`.

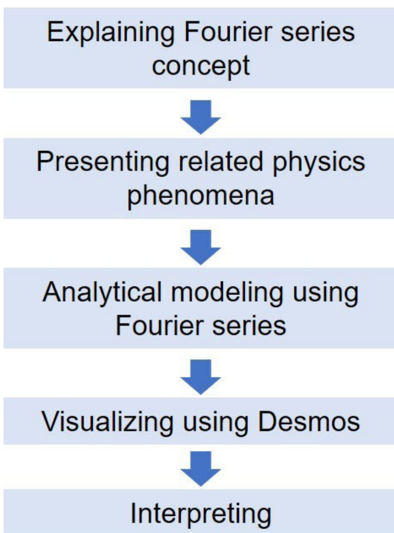

FIGURE 2. Teaching scenarios.

After that, the teacher can present related physics phenomena to students. Then, students are asked to model the phenomena and calculate the Fourier series coefficient in Eq. (2) and (3). After the Fourier series equation is constructed, students can use `Desmos` to visualize the Fourier series expansion with various numbers of terms included. At the end, students are stimulated to interpret the visualization results from `Desmos`. Through this activity, students are expected to be able to calculate the Fourier series coefficient, understand the contributions of each term, and be aware of the Fourier series applications in physics.

4. Results and discussions

Five examples of Fourier series expansion will be discussed, *i.e.*, full wave-rectified sine wave, half wave-rectified sine wave, sawtooth wave, square wave, and rectified sawtooth. The teaching approach proposed includes presenting the physics phenomenon, analytical modeling using the Fourier series, visualization using `Desmos` and interpretation.

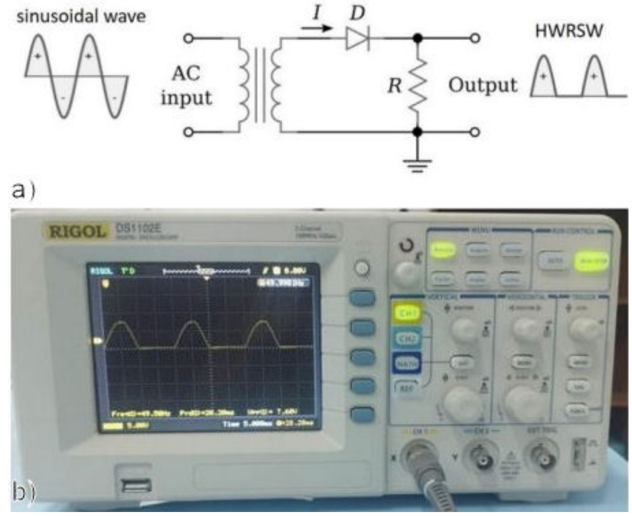

FIGURE 3. a) Half-rectified circuit with AC input, b) HWRSW voltage observed using an oscilloscope.

4.1. Half wave-rectified sine wave

One example of a half wave-rectified sine wave (HWRSW) is the electrical voltage produced by a half-rectified circuit with alternating current (AC) input. The half-rectified circuit consists of a diode. The diode has the property of passing current in only one direction. When AC voltage is going into a diode, the diode only passes the positive alternation of voltage. Figure 3a) shows an example of a circuit diagram to produce HWRSW electrical voltage. Figure 3b) presents the HWRSW electrical voltage measured by an oscilloscope.

HWRSW with an amplitude of $A = 1$ and period of 2π can be modeled with the function $f(t)$.

$$f(t) = \begin{cases} 0\,; & -\pi < t < 0, \\ \sin t\,; & 0 < t < \pi. \end{cases} \tag{4}$$

The coefficients a_n and b_n are evaluated using Eq. (2) and (3), giving expansion in the Fourier series as presented below:

$$\begin{aligned} f(t) = {} & \frac{1}{\pi} + \frac{1}{2}\sin t - \frac{4}{6\pi}\cos 2t \\ & - \frac{2}{15\pi}\cos 4t - \frac{2}{35\pi}\cos 6t - \frac{2}{63\pi}\cos 8t - \ldots \end{aligned} \tag{5}$$

Figure 4 shows the visualization of HWRSW and the Fourier series expansions. We have evaluated the coefficients a_n and b_n up to $n = 16$. According to the calculations, $b_n = 0$ for $n = 2, 3, 4, \ldots$. Hence, the sine term vanishes for that n. Meanwhile, $a_n = 0$ for $n = odd\ numbers$, hence the cosine terms vanish for odd numbers of n. The summations up to terms $n = 2$ are visualized. The form is similar to HWRSW, but it has slightly higher peaks and it has some ripples in the flat part. As we include more terms n in the partial summations, the peak attains its exact value $(A = 1)$, and the ripple at the flat part eases off (see Fig. 4). The equations input to the `Desmos` to produce graphs are presented in Fig. S1 in the supplementary material. Using visualizations

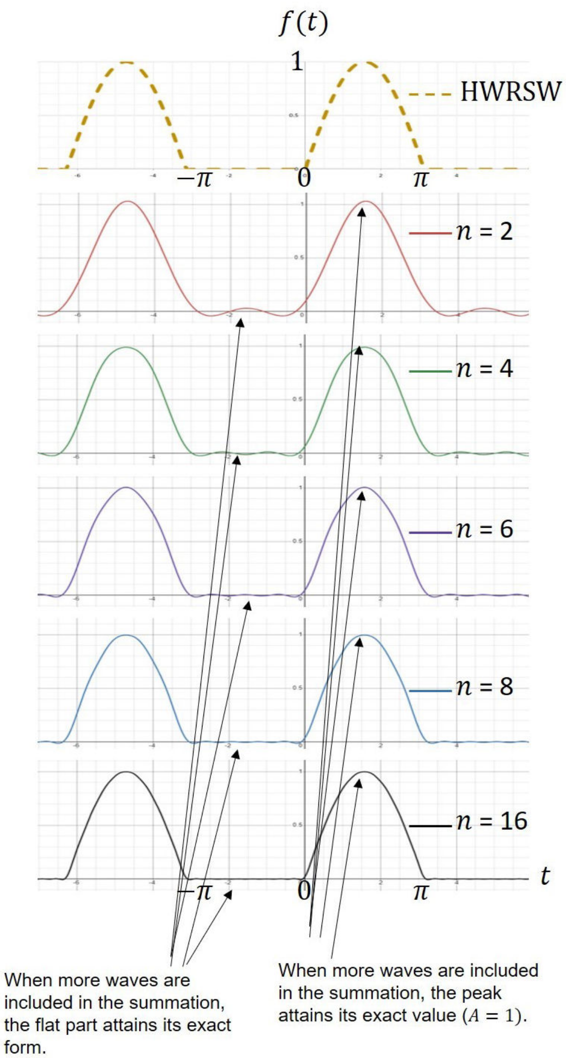

FIGURE 4. Visualization of HWRSW and the Fourier series expansion with $n = 2, 4, 5, 6$, and 16.

in Desmos, students can relate the results of the Fourier series expansions with the periodic function being represented. Students can also identify which terms have significant contributions to the representation.

4.2. Full wave-rectified sine wave

An example of a full wave-rectified sine wave (FWRSW) can also be found in electrical signals. Voltage in FWRSW form can be produced by passing AC voltage in a rectifier circuit as illustrated in Fig. 5a). The full-wave rectifier uses two diodes. The full-wave rectifier has more benefits than half-wave rectifiers, such as higher output voltage and fewer ripples. It has higher efficiency than the half-wave rectifier [16].

FWRSW with amplitude $A = 1$ can be modeled with the function $f(t)$:

$$f(t) = \begin{cases} -\sin t\,; & -\pi < t < 0, \\ \sin t\,; & 0 < t < \pi. \end{cases} \tag{6}$$

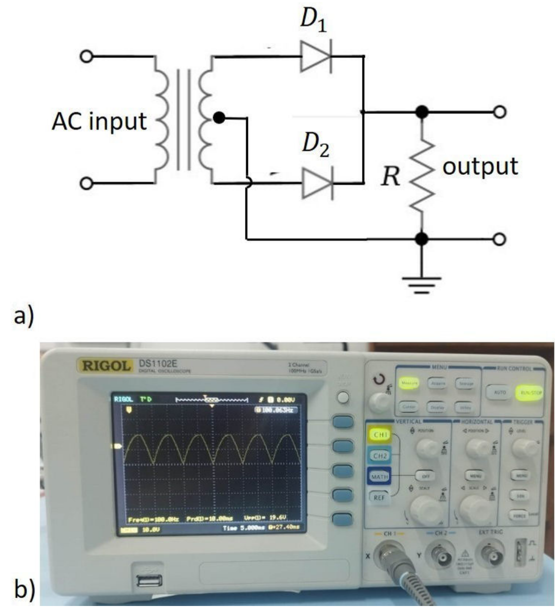

FIGURE 5. a) Full-rectified circuit with AC input, b) FWRSW voltage observed using an oscilloscope.

The coefficients a_n and b_n are evaluated using Eq. (2) and (3), giving expansion in the Fourier series as presented below:

$$f(t) = \frac{2}{\pi} - \frac{4}{3\pi}\cos 2t - \frac{4}{15\pi}\cos 4t - \frac{4}{35\pi}\cos 6t - \frac{4}{63\pi}\cos 8t - \ldots \tag{7}$$

The visualization of FWRSW and Fourier series expansion using Desmos is presented in Fig. 6. Meanwhile, the equations used in Desmos are presented in Fig. S2 in the supplementary material. With only $n = 2$ included in the partial summation, the produced wave has a peak and convex part that is slightly above the actual value. As more terms are included in the partial summation, the convex and the peak approach their actual values. By using visualization, students are stimulated to be aware of the role of each term in the Fourier series.

4.3. Square wave

For introducing square waves to students, we can present the square wave signals generated by a frequency generator (see Fig. 7). Square waves are usually applied in digital switching circuits and binary logic devices. They are utilized as clock signals to trigger circuits and data transmission timing sequence control [17].

Mathematically, a square wave with a period of can be represented by functions:

$$f(t) = \begin{cases} -1\,; & -\pi < t < 0, \\ 1\,; & 0 < t < \pi. \end{cases} \tag{8}$$

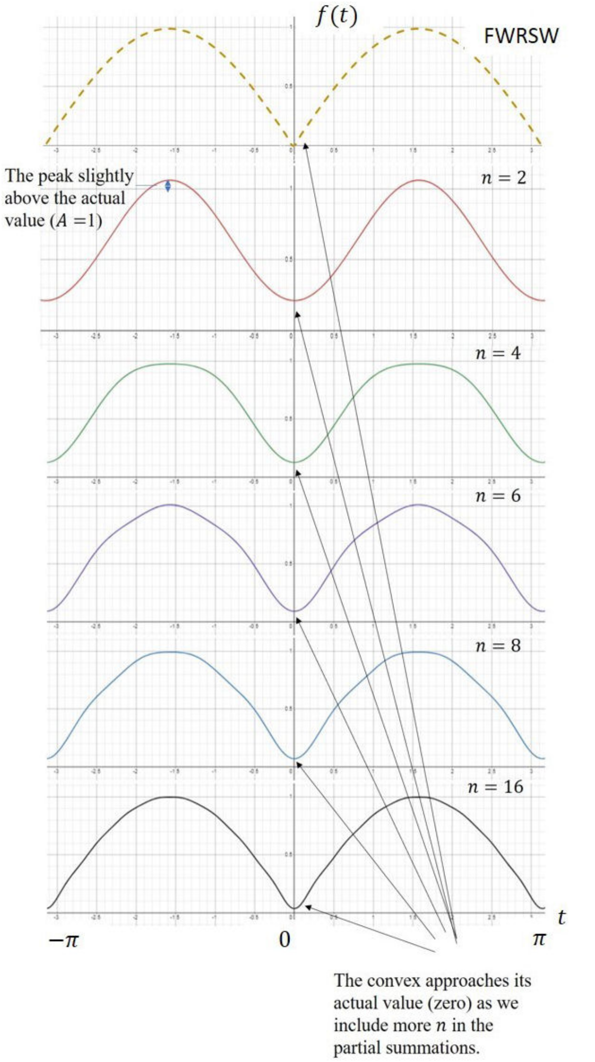

FIGURE 6. Visualization of FWRSW and the Fourier series expansion with $n = 2, 4, 6, 8$, and 16.

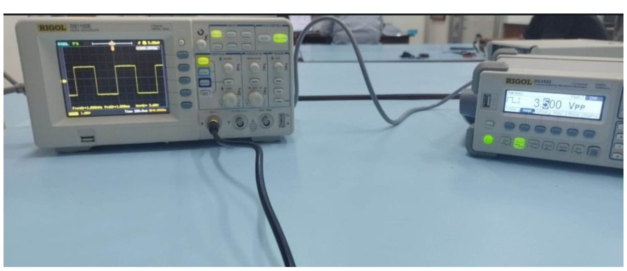

FIGURE 7. A square wave is generated by a signal generator and observed by using an oscilloscope.

The coefficients a_n and b_n are evaluated using Eq. (2) and (3), giving expansion in the Fourier series as presented below:

$$f(t) = \frac{4}{\pi}\sin t + \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\sin 5t + \frac{4}{7\pi}\sin 7t + ... \quad (9)$$

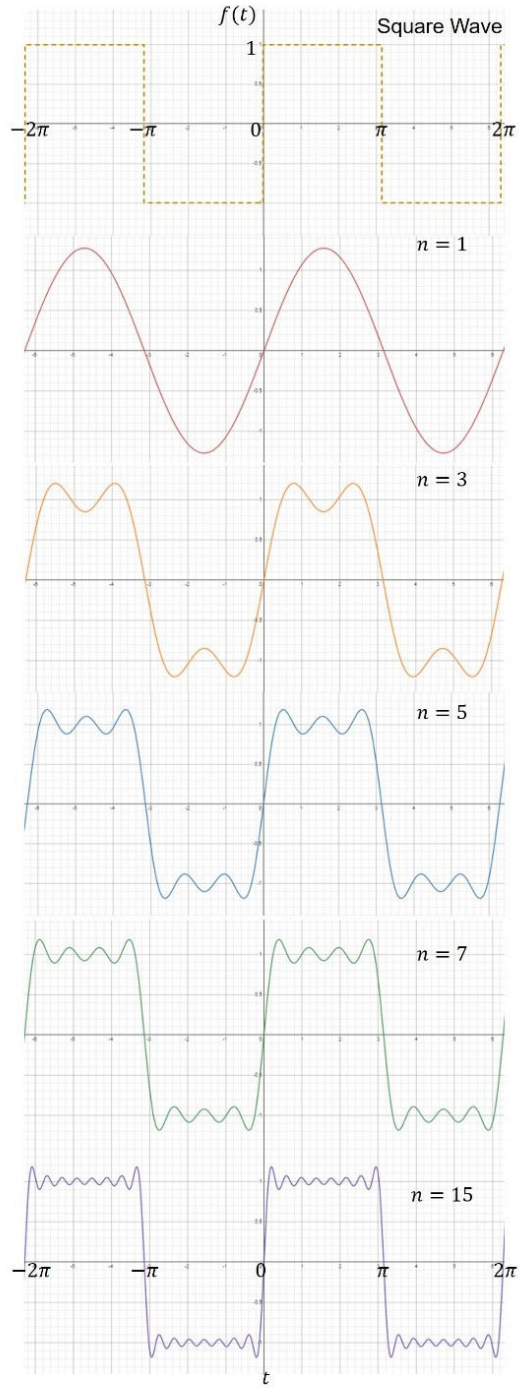

FIGURE 8. Visualization of square wave and the Fourier series expansion with $n = 1, 3, 5, 7$ and 15.

Figure 8 shows the visualization of the Fourier series expansion of a square wave with $n = 1, 3, 5, 7$, and 15 in Desmos. Figure S3 in the supplementary material presents the equations used in Desmos. When only $n = 1$ is included in the partial summation, the function becomes $f(t) = 1/2 + (2/\pi)\sin t$. As depicted in Fig. 8, the Fourier expansion does not resemble the square wave, it is just a sinusoidal function. When we include terms up to $n = 3$ in the summation, the waveform becomes more similar to the square wave but there are some significant ripples. As more n is included in the summations, the waveform becomes closer to the square wave, and the ripples in the flat part become smoother.

4.4. Triangular wave

Another type of common periodic function is a triangular wave. Triangular waveform has many applications in optics, such as optical signal conversion, pulse compression, signal copying, and optical frequency conversion. It has advantages because a triangular waveform has a rising and falling linear edge in optical intensity [18]. Figure 9 shows an example of a triangular wave generated by a signal generator. Let a triangular wave with an amplitude of 1 and a period of 2π can be described by the following function.

$$f(t) = \begin{cases} -\frac{t}{\pi}\ ; & -\pi < t < 0, \\ \frac{t}{\pi}\ ; & 0 < t < \pi. \end{cases} \tag{10}$$

The coefficients a_n and b_n are evaluated using Eq. (2) and (3), giving expansion in the Fourier series as presented below:

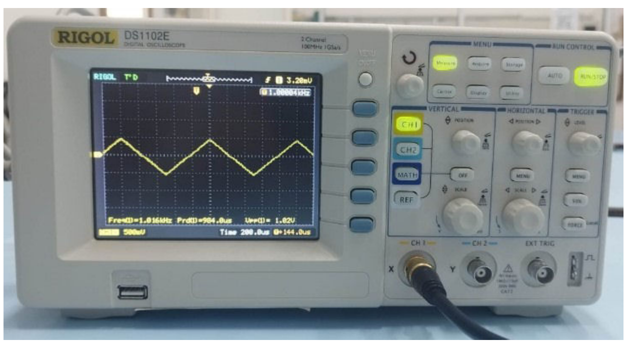

FIGURE 9. A triangular wave is generated by a signal generator and observed by using an oscilloscope.

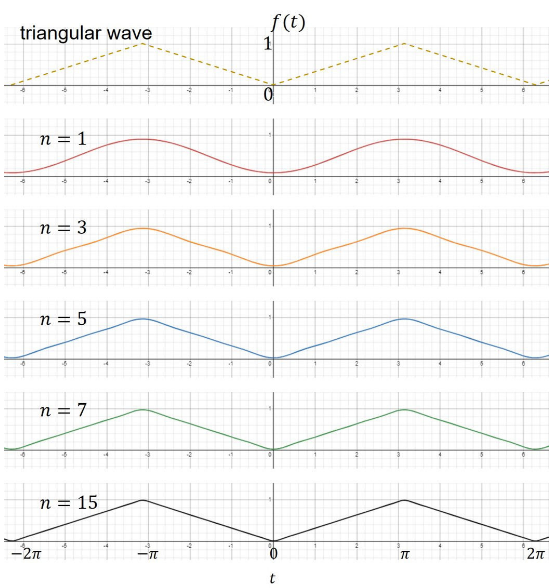

FIGURE 10. Visualization of the triangular wave and the Fourier series expansion with $n = 1, 3, 5, 7$, and 15.

$$f(t) = \frac{1}{2} - \frac{4}{\pi^2}\cos t - \frac{4}{9\pi^2}\cos 3t - \frac{4}{25\pi^2}\cos 5t - \frac{4}{49\pi^2}\cos 7t - ... \tag{11}$$

Figure 10 visualizes a triangular wave and Fourier series expansion of a triangular wave, which includes the summation of terms up to $n = 1, 3, 5, 7$ and 15, while the equations are presented in Fig. S4 in the supplementary material. With $n = 1$, the waveform does not resemble a triangular wave. With $n = 3$, the waveform approaches the triangular waveform. However, the peaks have not reached $A = 1$. As we include more n, the waveform attains its exact form and the peak attains its exact value $A = 1$.

4.5. Sawtooth wave

Figure 11 presents a sawtooth wave. It is similar to a triangular wave, but there is a part where it falls down rapidly. One of the important applications of sawtooth waves is for switching regulators [19]. Mathematically, a sawtooth wave with an amplitude of 1 and a period of 2π can be represented by the function:

$$f(t) = \frac{t}{\pi}; \quad -\pi < t < \pi. \tag{12}$$

Using Eq. (2) and (3), the Fourier coefficient can be determined. For the function above, a_n vanishes for $n \geq 0$ Meanwhile, b_n coefficient is

$$b_n = (-1)^{n+1}\left(\frac{2}{n\pi}\right); \quad n = 1, 2, 3, ... \tag{13}$$

Hence, using the method of Fourier, the series representation of the sawtooth wave is yielded as:

$$f(t) = \frac{2}{\pi}\sin t - \frac{2}{2\pi}\sin 2t + \frac{2}{3\pi}\sin 3t - \frac{1}{2\pi}\sin 4t + \frac{2}{5\pi}\sin 5t - ... \tag{14}$$

Figure 12 shows the sawtooth waveform and the partial summations of the Fourier series of sawtooth waves, which

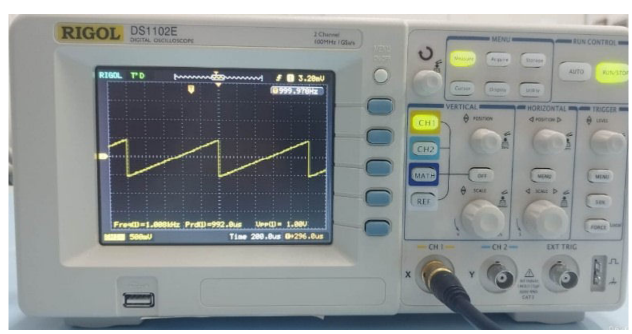

FIGURE 11. A sawtooth wave generated by a signal generator.

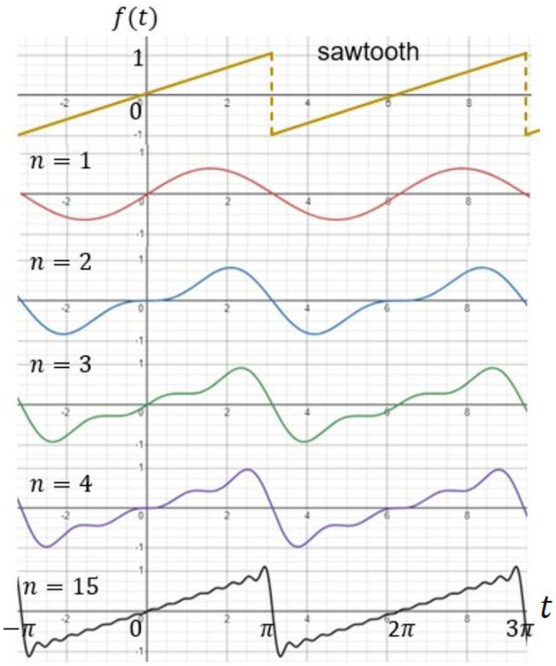

FIGURE 12. Sawtooth waveform and partial summations of the Fourier series of sawtooth waves (with $n = 1, 2, 3, 4$, and 15).

include terms with $n = 1, 2, 3, 4$ and 15. The equations used in `Desmos` are presented in Fig. S5 in the supplementary material. For greater n, the waveform becomes more resembling the actual sawtooth waveform in Fig. 11. We can see that when we include up to $n = 15$, the ripples on the sloppy part become smaller.

The teaching approach proposed in this paper includes an introduction to related physics phenomena, theoretical analysis, and visualization using `Desmos`. It is expected to help students relate the analytical method of Fourier series expansion that they learn in mathematical physics class with the real physics phenomena represented. `Desmos` has an important role in bridging the mathematical equation with the physics phenomena since it can provide precise graphical plotting of Fourier series equations yielded by the students. `Desmos` is very practical for students, it does not require programming or data entry that may add another cognitive load or burden to students during the learning process. It can also be accessed through mobile phones so that all students can use it in the classroom.

5. Conclusions

In this paper, we have demonstrated one application of Desmos in a mathematical physics course. Desmos can be utilized in the classroom to visualize the Fourier series. By typing some of the terms in the Fourier series that are yielded from analytical expansions, the graph can be shown by Desmos. With visualizations, students can relate the analytical equations with the periodic functions that are being represented. Students also can interpret the coefficients in the Fourier series more easily. Visualization of the Fourier series can also be done using a spreadsheet program such as described in Ref. [20]. We chose Desmos simply because it can be accessed online through a computer or mobile device easier and students just need to input the expressions.

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