Electromagnetic fields with symmetry. II

G. F. Torres del Castillo
Instituto de Ciencias, Benemérita Universidad Autónoma de Puebla,
72570 Puebla, Pue., México,
e-mail: gtorres@fcfm.buap.mx

Received 17 August 2024; accepted 19 July 2025

We show that in the framework of special or general relativity, the invariance of an electromagnetic field (with or without sources) under a one-parameter family of space-time transformations leads to a constant of motion for a charged test particle if the transformations also leave the space-time metric invariant. We also show that if the electromagnetic field and the space-time metric are invariant under a one-parameter family of space-time transformations, then one can find four-potentials that are also invariant under this family of transformations.

Keywords: Electromagnetic fields; symmetries; constants of motion; relativistic mechanics; Lorentz transformations.

DOI: https://doi.org/10.31349/RevMexFisE.23.020215

1. Introduction

In a recent paper [1] it was shown that if an electromagnetic field is invariant under a one-parameter family of translations or rotations (in the Euclidean three-dimensional space), then one can find a constant of motion associated with this symmetry for a charged test particle in this field and that this constant of motion can be determined without having to choose electromagnetic potentials for the field. Furthermore, one can find electromagnetic potentials with the symmetry of the field.

The aim of this paper is to show, in a covariant manner, without having to adapt the coordinates to the symmetry, that if an electromagnetic field is invariant under a one-parameter family of transformations that also leave invariant the (flat or curved) space-time metric, one can find a conserved quantity for the equations of motion a charged test particle, which does not require the specification of a four-potential for the field. We also show that, under the conditions mentioned above, one can find a four-potential possessing the same symmetry as the electromagnetic field and the metric.

In Sec. 2 we show that one half of the components of an electromagnetic field, with or without sources, invariant under a one-parameter family of space-time transformations can be expressed in terms of a real-valued, gauge-independent function and in Sec. 3 we show that, if the transformations also leave the metric invariant, this function takes part in the constant of motion associated with this symmetry for a charged particle subjected to the electromagnetic field. In Sec. 4 we show that if the electromagnetic field is invariant under a family of space-time transformations then one can find four-potentials for this field sharing the same symmetry.

Throughout this paper it will be assumed that the reader is acquainted with the basic notions and the standard notation of special and general relativity (see, *e.g.*, Ref. [2]). The lower case Greek indices run from 0 to 3 and there is sum over each repeated index appearing once as a subscript and once as a superscript.

2. Definition of symmetry

At first sight, it would seem that the definition of the invariance of the electromagnetic field is a straightforward matter; however, one can see that there are three different natural definitions, depending on what type of tensor field is employed to represent the electromagnetic field. As is well known, the electromagnetic field can be represented by three different tensor fields, whose components are usually denoted as $F_{\alpha\beta}$, $F^{\alpha}{}_{\beta}$ and $F^{\alpha\beta}$, and these components are related by

$$F_{\alpha\beta} = g_{\alpha\gamma} F^{\gamma}{}_{\beta}, \qquad F^{\alpha}{}_{\beta} = g_{\beta\gamma} F^{\alpha\gamma},$$

where $g_{\alpha\beta}$ are the components of the space-time metric. (Additionally, one can consider the dual of $F_{\alpha\beta}$.) Owing to the differences in the rules of transformation of these tensors, a transformation leaving invariant one of these tensors may not leave invariant another, except in the case where the transformation also leaves invariant the metric tensor. (This difference was not relevant in the examples treated in Ref. [1] because the only transformations considered there were ordinary rotations and translations, which leave invariant the space-time metric.)

As in most cases encountered in physics, the more relevant symmetries are those belonging to families of transformations depending on one or more continuous parameters and in what follows we shall consider a family of space-time transformations depending on one parameter s (for instance, spatial rotations, translations, and boosts in the Minkowski space-time),

$$X^{\alpha} = X^{\alpha}(x^{\beta}, s), \tag{1}$$

and we shall assume that for $s = 0$, the transformation is the identity: $X^{\alpha}(x^{\beta}, 0) = x^{\alpha}$. As our basic definition, we shall say that an electromagnetic field is invariant under the transformations (1) if

$$F_{\alpha\beta}(x^{\gamma}) = \frac{\partial X^{\mu}}{\partial x^{\alpha}} \frac{\partial X^{\nu}}{\partial x^{\beta}} F_{\mu\nu}(X^{\rho}), \tag{2}$$

for all values of s. (By contrast with the Lorentz transformations, which are usually considered as passive transformations (that is, the phenomena are described from different reference frames), here we are considering *active* transformations, x^α and X^α are coordinates with respect to a unique coordinate system of two possibly different points.)

Taking the partial derivative of both sides of Eq. (2) with respect to s, at $s = 0$, making use of the definition

$$\xi^\alpha \equiv \left.\frac{\partial X^\alpha}{\partial s}\right|_{s=0}, \tag{3}$$

we obtain the basic equation

$$0 = \xi^\rho \frac{\partial F_{\alpha\beta}}{\partial x^\rho} + \frac{\partial \xi^\nu}{\partial x^\beta} F_{\alpha\nu} + \frac{\partial \xi^\mu}{\partial x^\alpha} F_{\mu\beta}. \tag{4}$$

(The vector field with components ξ^α is usually called the infinitesimal generator of the family of transformations (1). The expressions on the right-hand side of Eq. (4) are the components of the Lie derivative of the differential form $F_{\alpha\beta}\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta$ with respect to the vector field $\xi^\alpha\, \partial/\partial x^\alpha$, see, *e.g.* Refs. [2–4].)

Equation (4) can be rewritten as

$$0 = \xi^\rho \frac{\partial F_{\alpha\beta}}{\partial x^\rho} + \frac{\partial \xi^\nu}{\partial x^\beta} F_{\alpha\nu} + \frac{\partial (\xi^\mu F_{\mu\beta})}{\partial x^\alpha} - \xi^\mu \frac{\partial F_{\mu\beta}}{\partial x^\alpha}.$$

Then, making use of the homogeneous Maxwell equations

$$\frac{\partial F_{\alpha\beta}}{\partial x^\rho} + \frac{\partial F_{\rho\alpha}}{\partial x^\beta} + \frac{\partial F_{\beta\rho}}{\partial x^\alpha} = 0 \tag{5}$$

and the antisymmetry $F_{\alpha\beta} = -F_{\beta\alpha}$, we obtain

$$0 = \frac{\partial (\xi^\mu F_{\mu\beta})}{\partial x^\alpha} - \frac{\partial (\xi^\mu F_{\mu\alpha})}{\partial x^\beta},$$

which is locally equivalent to the existence of a function Π such that

$$\xi^\mu F_{\mu\alpha} = -\frac{\partial \Pi}{\partial x^\alpha}. \tag{6}$$

(The minus sign is included in order to get agreement with the expressions given in Ref. [1].) (In Ref. [2], sec. 33.6, the existence of a scalar potential equivalent to Π is obtained assuming that ξ^α is simultaneously the generator of symmetries of the metric, and without realizing the role of this scalar potential in the constant of motion for a charged particle subjected to the field, see Sec. 3.)

3. Constant of motion

All the equations obtained in the previous section are valid in a flat or curved space-time and in any coordinate system. Now we shall consider the special case where the space-time is the Minkowski space-time and the coordinates x^α are Cartesian, that is, the metric tensor has components

$$(g_{\alpha\beta}) = \mathrm{diag}\,(-1, 1, 1, 1).$$

Then, the equations of motion of a particle with rest-mass m and electric charge q in a given electromagnetic field are

$$m\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} = \frac{q}{c} F^\alpha{}_\beta \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}, \tag{7}$$

where τ is the proper time of the particle.

Assuming that the electromagnetic field is invariant under the family of transformations generated by ξ^α, contracting Eq. (7) with ξ_α, making use of Eq. (6) we have

$$m\xi_\alpha \frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} = -\frac{q}{c}\frac{\partial \Pi}{\partial x^\beta}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = -\frac{q}{c}\frac{\mathrm{d}\Pi}{\mathrm{d}\tau},$$

which amounts to

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(m\xi_\alpha \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} + \frac{q}{c}\Pi\right) = m\frac{\partial \xi_\alpha}{\partial x^\beta}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}. \tag{8}$$

The right-hand side of Eq. (8) vanishes for all possible four-velocities of the particle if and only if

$$\frac{\partial \xi_\alpha}{\partial x^\beta} + \frac{\partial \xi_\beta}{\partial x^\alpha} = 0. \tag{9}$$

These last equations amount to say that ξ^α generates symmetries of the space-time metric (a Killing vector) and one finds that their only solutions correspond to Lorentz transformations (see, *e.g.*, Ref. [2], sec. 33.3, or Ref. [4], sec. 6.1). Thus, in order to obtain a constant of motion, the useful symmetries of the electromagnetic field are also symmetries of the metric tensor and therefore the three different definitions of symmetry of the electromagnetic field mentioned above coincide.

Summarizing, for a charged test particle in a given electromagnetic field (with or without sources) invariant under a one-parameter family of Lorentz transformations with infinitesimal generator ξ^α, the function

$$\xi_\alpha p^\alpha + \frac{q}{c}\Pi, \tag{10}$$

where p^α are the components of the four-momentum of the particle and Π is defined by Eq. (6), is a constant of motion (*cf.* Eqs. (12) and (14) of Ref. [1]). It may be remarked that Π [and, hence, the constant of motion (10)] is defined directly from the electromagnetic field and therefore is gauge-independent.

3.1. An example

It may be illustrative to give an electromagnetic field invariant under a family of Lorentz transformations, different from the usual spatial translations and rotations. As is well known, for a boost along the x-axis, the x-components of the electric and the magnetic fields are invariant. Hence, an electromagnetic field with E_x and B_x constant, and all the other components equal to zero, must be invariant under the one-parameter family of Lorentz transformations

$$\begin{aligned} X^0 &= x^0 \cosh s + x^1 \sinh s, \\ X^1 &= x^0 \sinh s + x^1 \cosh s, \end{aligned} \tag{11}$$

and $X^2 = x^2$, $X^3 = x^3$. A convenient way of proving this assertion consists in verifying the existence of a function Π satisfying Eq. (6). Taking the electromagnetic field given by $F_{10} = E_0 = -F_{01}$, $F_{23} = B_0 = -F_{32}$, where E_0 and B_0 are constants, with all the other components $F_{\alpha\beta}$ equal to zero (uniform electric and magnetic fields pointing along the x-axis), from Eqs. (3) and (11) we find $\xi^0 = x^1$, $\xi^1 = x^0$ and $\xi^2 = \xi^3 = 0$. Then, Eq. (6) gives

$$\frac{\partial \Pi}{\partial x^0} = -\xi^\mu F_{\mu 0} = -E_0 x^0, \quad \frac{\partial \Pi}{\partial x^1} = -\xi^\mu F_{\mu 1} = E_0 x^1,$$

and, similarly, $\partial\Pi/\partial x^2 = 0 = \partial\Pi/\partial x^3$, which are satisfied with

$$\Pi = \frac{E_0}{2}\left[(x^1)^2 - (x^0)^2\right].$$

(Note that Π does not depend on B_0.)

According to the discussion above, for a charged test particle in this electromagnetic field, owing to the invariance of the field under the one-parameter family of boosts (11) (in fact, one-parameter group), there is a constant of motion given by [see Eq. (10)]

$$-x^1 p^0 + x^0 p^1 + \frac{qE_0}{2c}\left[(x^1)^2 - (x^0)^2\right].$$

(The field is also invariant under translations along the four coordinate axes as well as under spatial rotations about the x^1-axis, and therefore there are five additional constants of motion.)

3.2. Curved space-time

The results obtained above remain valid if the coordinates are not Cartesian or if the space-time is curved, by simply replacing the partial derivatives by covariant derivatives. In particular, the condition (9) becomes

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0, \tag{12}$$

where ∇_α denotes the covariant derivative compatible with the metric. Equations (12) are the Killing equations corresponding to the metric tensor $g_{\alpha\beta}$, which determine the symmetries of the metric (see, *e.g.*, Refs. [2–4]).

4. The four-potential

In the preceding sections, the four-potential of the electromagnetic field has not been mentioned and the constant of motion associated with a one-parameter family of transformations leaving invariant the electromagnetic field has been expressed directly in terms of the components $F_{\mu\nu}$ [see Eqs. (6) and (10)]. In this section we shall show that if the electromagnetic field is invariant under the one-parameter family of transformations (1) then one can find a four-potential for this field that is also invariant under the transformations (1).

As is well known, the Maxwell equations (5) are locally equivalent to the existence of a four-potential, A_α, such that

$$F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}.$$

We shall say that the four-potential is invariant under the transformations (1) if

$$A_\alpha(x^\gamma) = \frac{\partial X^\mu}{\partial x^\alpha} A_\mu(X^\rho) \tag{13}$$

[*cf.* Eq. (2)]. Taking the partial derivative with respect to s, at $s = 0$, of both sides of Eq. (13) one finds

$$0 = \xi^\rho \frac{\partial A_\alpha}{\partial x^\rho} + \frac{\partial \xi^\mu}{\partial x^\alpha} A_\mu \tag{14}$$

[*cf.* Eq. (4)]. (Owing to the fact that we have two different tensor fields corresponding to the four-potential, A^α and A_α, we have two different definitions of invariance under the transformations (1), but they coincide when the transformations leave the metric tensor invariant. The expressions on the right-hand side of Eq. (14) are the components of the Lie derivative with respect to the vector field $\xi^\alpha\partial/\partial x^\alpha$ of the 1-form $A_\mu \mathrm{d}x^\mu$.)

A straightforward computation shows that the symmetry condition (4) can be rewritten in terms of the four-potential in the form

$$\begin{aligned} 0 \;=\; & \frac{\partial}{\partial x^\alpha}\left(\xi^\mu \frac{\partial A_\beta}{\partial x^\mu} + \frac{\partial \xi^\mu}{\partial x^\beta} A_\mu\right) \\ & - \frac{\partial}{\partial x^\beta}\left(\xi^\mu \frac{\partial A_\alpha}{\partial x^\mu} + \frac{\partial \xi^\mu}{\partial x^\alpha} A_\mu\right) \end{aligned} \tag{15}$$

and therefore if the four-potential is invariant under the family of transformations generated by ξ^α, then the electromagnetic field is also invariant under those transformations, but the converse is not necessarily true, that is, Eq. (15) does not imply Eq. (14). However, Eq. (15) is equivalent to the local existence of a function, G, such that

$$\xi^\mu \frac{\partial A_\alpha}{\partial x^\mu} + \frac{\partial \xi^\mu}{\partial x^\alpha} A_\mu = \frac{\partial G}{\partial x^\alpha}. \tag{16}$$

On the other hand, for a given electromagnetic field, the four-potential is not unique but it is defined up to a gauge transformation: $A'_\alpha = A_\alpha + \partial\Lambda/\partial x^\alpha$, where Λ is an arbitrary function. Under this transformation the left-hand side of Eq. (16) transforms into

$$\begin{aligned} & \xi^\mu \frac{\partial A_\alpha}{\partial x^\mu} + \xi^\mu \frac{\partial^2 \Lambda}{\partial x^\mu \partial x^\alpha} + \frac{\partial \xi^\mu}{\partial x^\alpha} A_\mu + \frac{\partial \xi^\mu}{\partial x^\alpha}\frac{\partial \Lambda}{\partial x^\mu} \\ & \qquad = \frac{\partial G}{\partial x^\alpha} + \frac{\partial}{\partial x^\alpha}\left(\xi^\mu \frac{\partial \Lambda}{\partial x^\mu}\right). \end{aligned}$$

Hence, if Λ is chosen in such a way that

$$\xi^\mu \frac{\partial \Lambda}{\partial x^\mu} = -G, \tag{17}$$

the four-potential A'_α is invariant under the family of transformations generated by ξ^α. (Equation (17) is a linear partial differential equation for Λ, which is always solvable. See the example below.) (In Ref. [2], sec. 33.6, Eq. (14) is employed to define the symmetry of the electromagnetic field, restricting the discussion to transformations that leave the metric invariant; apart from this unnecessary restriction, Eq. (14) has the inconvenience of being gauge-dependent.)

It may be noticed that the right-hand side of Eq. (14) can be rewritten in the form

$$\begin{aligned} \xi^\rho \frac{\partial A_\alpha}{\partial x^\rho} + \frac{\partial \xi^\mu}{\partial x^\alpha} A_\mu &= \xi^\mu \left(\frac{\partial A_\alpha}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\alpha} \right) + \frac{\partial (\xi^\mu A_\mu)}{\partial x^\alpha} \\ &= \xi^\mu F_{\mu\alpha} + \frac{\partial (\xi^\mu A_\mu)}{\partial x^\alpha}, \end{aligned}$$

therefore, if the four-potential A_α satisfies the symmetry condition (14) then there exists the function Π, defined up to an additive constant by Eq. (6), and we can assume that $\xi^\mu A_\mu = \Pi$. This implies that the constant of motion (10) is given by

$$\xi_\alpha \left(p^\alpha + \frac{q}{c} A^\alpha \right). \tag{18}$$

4.1. Example

In order to illustrate the results of this section, we shall find a four-potential for the electromagnetic field considered in Sec. 3.1, invariant under the transformations (11). One can verify that $A_\alpha = (0, -E_0 x^0, 0, B_0 x^2)$ is a four-potential for the uniform fields employed in Sec. 3.1, but Eqs. (14) are not satisfied. In fact (recalling that $\xi^\alpha = (x^1, x^0, 0, 0)$) one finds that

$$\begin{aligned} \xi^\rho \frac{\partial A_0}{\partial x^\rho} + \frac{\partial \xi^\mu}{\partial x^0} A_\mu &= -E_0 x^0, \\ \xi^\rho \frac{\partial A_1}{\partial x^\rho} + \frac{\partial \xi^\mu}{\partial x^1} A_\mu &= -E_0 x^1, \\ \xi^\rho \frac{\partial A_2}{\partial x^\rho} + \frac{\partial \xi^\mu}{\partial x^2} A_\mu &= 0, \\ \xi^\rho \frac{\partial A_3}{\partial x^\rho} + \frac{\partial \xi^\mu}{\partial x^3} A_\mu &= 0, \end{aligned}$$

which are of the form (16) with

$$G = -\frac{E_0}{2} \left[(x^0)^2 + (x^1)^2 \right].$$

Hence, the linear partial differential equation (17) takes the form

$$x^1 \frac{\partial \Lambda}{\partial x^0} + x^0 \frac{\partial \Lambda}{\partial x^1} = \frac{E_0}{2} \left[(x^0)^2 + (x^1)^2 \right]$$

and, by inspection, one finds the particular solution $\Lambda = (E_0/2)\, x^0 x^1$. Finally, a four-potential invariant under the transformations (11) is given by

$$A'_\alpha = (E_0 x^1/2, -E_0 x^0/2, 0, B_0 x^2).$$

5. Concluding remarks

If the electromagnetic field is invariant under the transformations generated by several Killing vectors, we can still obtain the corresponding constants of motion as shown in Sec. 3, but a four-potential simultaneously invariant under all the Killing vectors may not exist. A simple example is given by a static, uniform field in the Minkowski space-time, which is invariant under the translations along the four coordinate axes, but a four-potential invariant under all these translations would have constant Cartesian components, and the electromagnetic field would be equal to zero.

---

1. G.F. Torres del Castillo, Electromagnetic fields with symmetry, *Rev. Mex. Fís. E* **21** (2024) 020201. `https://doi.org/10.31349/RevMexFisE.21.020201`.
2. H. Stephani, Relativity, An Introduction to Special and General Relativity, 3rd ed. (Cambridge University Press, Cambridge, 2004).
3. B.F. Schutz, Geometrical Methods of Mathematical Physics (Cambridge University Press, Cambridge, 1982). `https://doi.org/10.1017/CBO9781139171540`.
4. G.F. Torres del Castillo, Differentiable Manifolds, A Theoretical Physics Approach, 2nd ed. (Springer, Cham, 2020). `https://doi.org/10.1007/978-3-030-45193-6`.