Euler's number: a new experimental estimation

M. S. Kovačević[a], and D. Tosi[b]

[a]*Faculty of Science, University of Kragujevac, Kragujevac, Serbia,*
[b]*School of Engineering and Digital Science, Nazarbayev University, Astana, Kazakhstan.*

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Euler's number e is one of the most well-known numbers in mathematics. The base of the natural logarithm is represented by the number e, often known as Neper's number in books. In the work of distinguished mathematician and physicist Jacob Bernoulli, the number e appears as the limit value of a number sequence that Bernoulli studied dealing with the issue of interest. Although it was primarily used for financial calculations this remarkable number quickly began to be applied in a wide range of natural phenomena and scientific laws of physics, biology, and chemistry. Students in high schools who are nearing the end of their schooling are taught that $\lim_{n\to\infty}(1+[1/n])^n$ is equal to the number $e = 2.718\ldots$. This study reports on a new experiment in physics using communicating vessels, where the number e appears indirectly. For example, if in the described experiment, a vessel with an area of 100 cm^2 is divided into $N = 100$ smaller vessels with an area of 1 cm^2, we will theoretically reproduce the number e with an accuracy of 0.5 %. It is also emphasized that Euler's number is currently used more frequently and may be found in a wide range of scientific fields as well as daily life.

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1. The origins and significance of the number e

One of the most important mathematical constants is the number e, also referred to as Euler's number or Naper's constant. In actuality, though, it is the result of the combined efforts of three eminent mathematicians - John Napier, Jacob Bernoulli, and Leonard Euler - who each made a significant contribution to its discovery. This important number, despite bearing Euler's name, is far older in mathematics, having first been used in mid-16th-century calculations [1–5]. John Napier, a Scottish polymath, asked whether there was a quick, algorithmic method for calculating the product of very large numbers - more especially, exponents - in the 17th century. Although Napier was unable to find the number *e*, he did manage to generate a list of logarithms that he had inadvertently computed using the constant. By publishing his book, *Mirifici Logarithmorum Canonis Descriptio*, in 1614, he became the first person of Greek ancestry to coin the term "logarithm." Essentially, the goal of logarithms was to replace the tedious process of multiplying two numbers with the easier operation of adding two other numbers together. Each number was assigned a corresponding value, initially called a "logarithm" by Napier, which had the unique property that the product of two logarithms could be used to calculate the product of the two original numbers.

It is said that in 1683 while considering problems involving continuous compound interest, Jacob Bernoulli made the discovery of *e*. He saw that the quantity of money would converge towards a limit that was eventually determined to be one of the representations of *e* (see Appendix A) as the compounding period got smaller and smaller and more and more periods were taken into consideration. The number e was initially given a name in the 1700s by the Swiss-German mathematician Leonhard Euler, but Napier had already suggested it existed in 1614 when he was researching logarithms and bases. The notation "*e*" first appeared in a letter written by Euler to Goldbach in 1731, and it is attributed to him for popularizing the number *e*. In the years that followed, he made a number of discoveries about *e*, but he did not fully address the concepts underlying *e* until 1748, when Euler wrote *Introductio in Analysin infinitorum*. Furthermore, by extending it into a convergent infinite series of factorials, he demonstrated its irrationality. An irrational number is a real number that cannot be expressed as the ratio of two integers. In other words, it cannot be written as a fraction a/b, where a and b are integers and $b \neq 0$. Irrational numbers have non-repeating, non-terminating decimal expansions. These numbers cannot be precisely represented by a simple fraction or a finite decimal. Since then, the use of Euler's number has become more widespread and now it appears in many branches of science and in everyday life [6–11]. Even outside school or university, we cannot escape Euler's number. The number *e* appears in a variety of natural and mathematical contexts, especially in biology, physics, engineering, and several other fields. More details about this can be found in Appendix B.

2. A review of definitions of the number e

Euler's definition: Select a number $t \geqslant 1$ and measure the area under the hyperbola $y = 1/x$, above the x-axis, and between the lines $x = 1$ to $x = t$. Using integral calculus notation, this area is equal to "the definite integral of function $f(x) = 1/x$ between $x = 1$ and $x = t$", denoted by $\int_1^t (1/x)dx$. This area is a function of t, and so we give this function a name, the "natural logarithm" of t, denoted

$$\ln(t) = \int_1^t \frac{1}{x}\,dx. \tag{1}$$

When t is close to 1, this area is small, and upon closer inspection, we can see that for really large t, the area is eventually as large as you like. For t approximately 2.7, this area equals 1. The number e is defined so the area under $y = 1/x$ between $x = 1$ and $x = e$ is precisely 1. To put it another way, e is defined to be the number such that $\ln(e) = 1$, or $e = \ln -1(1)$ in the case of inverse function notation.

Napier's definition: In terms of exponential functions, let us consider the following examples, which are given by 2^x or 10^x. In these instances, the number 2 or 10 is called a base and, of course, x is the exponent. We will use b for the base, so an exponential function is expressed as $f(x) = bx$, where b is a positive constant. When $b > 1$, these functions increase quite quickly, and if graphed, they are climbing when they cross the y-axis, and get even steeper to the right. According to many calculus textbooks, define e to be the unique base b so that the slope of the graph of $y = bx$ as it intersects the y-axis, its value is equal to 1. They use this as a definition, since they must first discuss slopes (derivatives) before they discuss areas (integrals). For those having seen some calculus, this definition says that if $f(x) = e^x$, then $f'(0) = 1$. Some books use the notation "$\exp(x)$" before they use e^x, to remind us of exponentiation, and so many think that this is where the "e" comes from. One can then later prove that these two definitions for e (one from ln, and one from slopes) define the same number.

For those who have studied logarithms, it is known that logarithms and exponentiation are inverse operations. For example, $\log_{10}(x) = y$ is the same as $x = 10^y$. The natural logarithm of t, which was defined above in terms of area, can be shown to be the same as $\log_e(x)$, logarithm to the base e, hence the name natural "logarithm" (and "ln"). When the base is e, the two statements $\ln(x) = y$ and $x = e^y$ are the same.

A mathematical definition of e: Students first discovered that e is defined as an infinite sum in high school. Newton did publish this sum in 1669, but he never referred to it as e. So

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ... \tag{2}$$

What does it mean to have an unlimited number of additions? The infinite sum is said to converge if, as we sum them up from the beginning, we maintain a running total (also known as a partial sum) if this cumulative total approaches a finite value. Although we won't demonstrate it, you can persuade yourself that this sum converges by looking at the first five terms and obtaining

$$1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots = 2.5 + 0.166666... + 0.04166666... = 2.709333... \tag{3}$$

One additional term gives 2.7166666..., providing proof that the infinite sum does, in fact, converge to the previously stated value. In theory, we could calculate e to as many decimal places as we wanted by applying Newton's formula. Ten decimal places added to the number e result in $e = 2.7182818285$. One can define e in a variety of ways. As an illustration,

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n, \tag{4}$$

is frequently used as a definition. With $n = 2$, the expression in the limit is $(3/2)^2 = 9/4 = 2.25$, and with $n = 3$, one gets $64/27 \approx 2.37$, only a little closer to e. Some give

$$e = \lim_{h\to 0^+}(1 + h)^{\frac{1}{h}}, \tag{5}$$

an equivalent definition to the last, using $h = 1/n$.

Euler unveiled several remarkable properties of the number e, which is why it is fittingly associated with his name. Euler's identity is known

$$e^{i\pi} + 1 = 0. \tag{6}$$

This formula is one of the most famous in mathematics because it combines the numbers e, π, 0 and 1. The number e is an irrational number, meaning it cannot be expressed as the ratio of two integers. Its decimal representation is infinite and non-repeating. Finally, let us state that, apart from being irrational, the number e is also a transcendental number, which means that it is not the root of any (non-zero) polynomial with integer coefficients [4, 8].

Obtaining the number e without using an approximation, series, or method like limits or integrals can be quite tricky, as e is inherently defined through such approaches. However, in a broader sense, there are a few conceptual ways we can think about understanding e without directly using standard approximations or series. In summary, while it's difficult to obtain the number e without relying on approximations or methods like limits or series, there are ways to conceptually understand e through its applications in continuous growth processes, logarithmic relationships, and deeper mathematical identities like Euler's formula. These approaches give insight into what e represents, though they often still involve implicit forms of approximation or limit processes.

3. A novel experiment to estimate number e

Two communicating vessels with the same cross-section S, A and B, are depicted in Fig. 1. The pipe that connects the two vessels can be opened and closed using valve K. Water is present in vessel A up to level H.

The two vessels' liquid levels will equalize if we open valve K, and both vessels' new liquid levels will be

$$h = \frac{H}{2}. \tag{7}$$

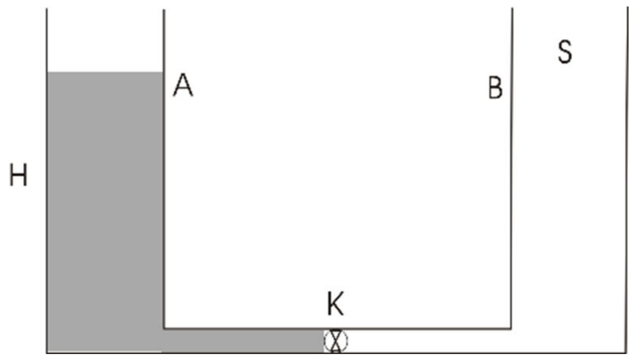

FIGURE 1. A thin pipe connecting two communicating vessels.

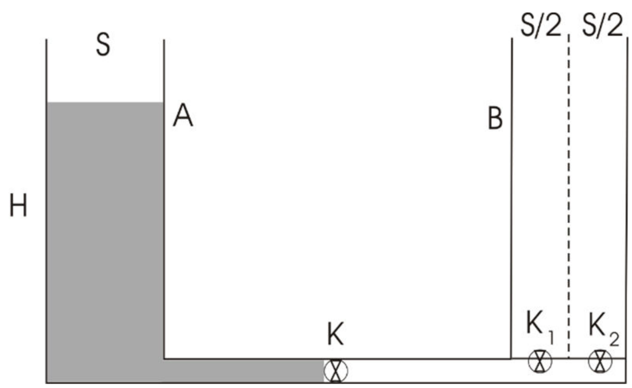

FIGURE 2. Two vessels that are in communication: vessel A has a cross section of S, while vessel B is split into two identical smaller vessels with cross sections of $S/2$ by a vertical barrier.

It raises an intriguing question: is it possible to raise the liquid level in vessel B above that of vessel A? The short answer is that it is not possible to do so without a pump since the law of communicating vessels forbids it. Here, however, we suggest a method to actually achieve this "impossible" circumstance. Assume that a vertical barrier separates vessel B into two equal sections that can each be filled separately with liquid from vessel A. We will refer to those sections as B_1 and B_2 (Fig. 2).

The process of merging vessels A and B_1 leads to level h_1, which we calculate from the relation

$$\rho\, S\,(H - h_1) = \rho\, \frac{S}{2}\, h_1, \tag{8}$$

where ρ is the density of liquid. This is how we get results $h_1 = 2H/3$. Now, with part B_1 isolated, we fill part B_2 of vessel B. In vessel A, the liquid level drops according to the relation

$$\rho\, S\,(h_1 - h_{\mathrm{A}}) = \rho\, \frac{S}{2}\, h_{\mathrm{A}}. \tag{9}$$

So, we get that the new height of the liquid is in the vessel A will be $h_{\mathrm{A}} = 4H/9$ (in the vessel B_2 is the same $h_2 = h_{\mathrm{A}}$). Let's now close the tap K (Fig. 1) and remove the barrier from vessel B. The liquid level in vessel B will be

$$h_B = \frac{h_1 + h_2}{2} = \frac{5H}{9}. \tag{10}$$

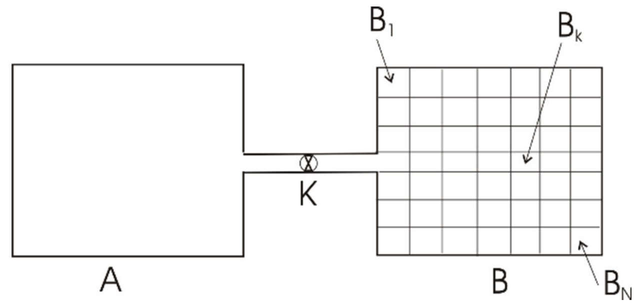

FIGURE 3. Communicated vessels A and B: viewed from above, vessel B is partitioned into N equal cells with a square cross-section by vertical barriers.

Since in vessel A, after the double discharge of liquid, the level remained $h_A = 4H/9$, we see that there is more liquid in vessel B because, indeed, $h_{\mathrm{A}} < h_{\mathrm{B}}$. Thus, with a few simple laboratory operations, but without the use of pumps, we "bypassed" the law of communicating vessels. Let us now try to generalize this procedure. Let's divide vessel B not into two, but into a larger number of equal parts with some vertical barriers, as shown in Fig. 3 (view is from above). Let's mark with N the number of cells in vessel B and mark them in order, B_1, B_2, ..., B_k, ..., B_N. Each vessel cell can be connected to vessel A independently of all other cells of vessel B. For practical reasons, the vessels are assumed to be of square cross-section. The cross-sectional area of vessels A and B are the same and mark them as S. In accordance with this the cross-sectional area of the vessel cell B_k is equal to S/N. The liquid filling process is the same as the one described for the two-component system, only now N such operations must be performed.

Let's mark the liquid level in cell B_N with h_k. Using a series of equations of the type 9, it is obtained

$$h_1 = \frac{N}{N+1} H, \tag{11a}$$

$$h_2 = \left(\frac{N}{N+1}\right)^2 H, \tag{11b}$$

$$\vdots$$

$$h_k = \left(\frac{N}{N+1}\right)^k H. \tag{11c}$$

After the last vessel B_N has been filled, in the vessel A there is liquid left up to level h_{A}:

$$h_{\mathrm{A}} = h_N = \left(\frac{N}{N+1}\right)^N H. \tag{12}$$

At the end, if we remove the grid of vertical barriers in vessel B, the liquid will remain in the integral vessel B at the height that is easiest to find from the relation

$$\rho S\,(H - h_{\mathrm{A}}) = \rho S h_{\mathrm{B}}. \tag{13}$$

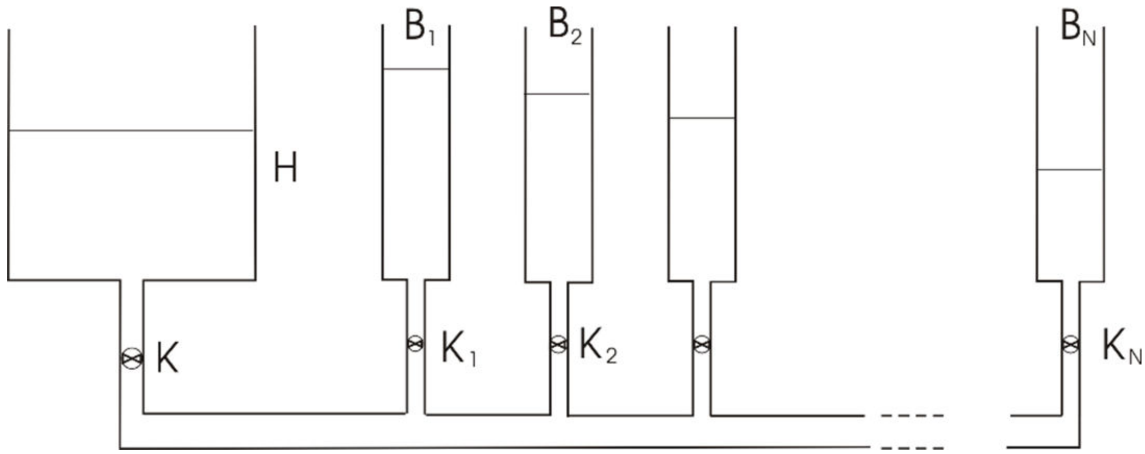

FIGURE 4. Connected vessels A and B. Vessel B is divided into N equal vessels with the same square cross-section (front view).

It follows from here that $h_\mathrm{B} = H(1 - [N/\{N+1\}]^N)$. Let's introduce the symbol $e_N = (1 + 1/N)^N$ so we can write the following expressions for the liquid levels in vessels A and B, respectively:

$$h_\mathrm{A} = \frac{H}{e_N}, \tag{14}$$

$$h_\mathrm{B} = H\left(1 - \frac{1}{e_N}\right). \tag{15}$$

Let's take a look at some specific examples. For $N = 10$ we get $h_\mathrm{A} = 0.3855H$ and $h_\mathrm{B} = 0.6144H$. If $N = 25$, then $h_\mathrm{A} = 0.3751H$ and $h_\mathrm{B} = 0.6249H$. If $N = 50$ we find $h_\mathrm{A} = 0.3715H$ and $h_\mathrm{B} = 0.6285H$. In fact, we can draw the graph of $h_\mathrm{A} = f_\mathrm{A}(N)$ and $h_\mathrm{B} = f_\mathrm{B}(N)$. Using these graphs, we conclude that h_A cannot be less than $H/e = 0.36789H$, and h_B cannot be greater than $H(1-1/e) = 0.63212H$. The students in the high school learn that $\lim_{x\to\infty}(1+1/x)^x$ is equal to the famous number $e = 2.718\ldots$.

In this way, we conclude that the level difference of liquid in vessels A and B $(h_\mathrm{B} - h_\mathrm{A})$, increases when we take smaller and smaller divisions of vessel B, but that we can never reduce the level in vessel A arbitrarily much. In the laboratory, this fact could be demonstrated like this. Let the main tap K be open, and the tap K_1 be open (Fig. 4). Then the levels in vessels A and B1 are equalized. Then we close K1 and open K2, and so on, until the last faucet KN is also opened. Finally, the levels in vessels A and BN are the same in amount, if the number N is not too small. In other words, in the described experiment, we can "measure" the number e using a meter. We have the following formula

$$e = \frac{\text{liquid level in the vessel at the beginning of the experiment}}{\text{liquid level in the vessel at the end of the experiment}}. \tag{16}$$

For example, if the cross section of the vessels A (and B) is $S = 100$ cm^2, and we make $N = 100$ smaller vessels of the vessel B, each of them with a cross section of 1 cm^2, we will reproduce, theoretically, the number e with an accuracy of 0.5%.

4. Conclusion

A relatively straightforward experimental method to estimate the numerical value of e has been introduced in this study. The theoretical foundation of the proposed experiment is based on the fundamental law of two connected vessels. The unique result has been obtained that allows us to "measure" the number e. Note that the capillary phenomena should also be considered, which will certainly occur if we have not properly designed the sections and selected appropriate materials.

If the readers were interested in this example of connected vessels, here are two questions to think about: 1) Is it true that the maximum difference $(h_\mathrm{B} - h_\mathrm{A})$ is obtained for an equal division of the area S?, 2) Can some other interesting division be realized, which will lead to a different threshold value, instead of e.

It is noted that Euler's number is currently used more frequently and can be found in a wide range of scientific fields as well as daily life. For instance, Euler's number appears in probability theory. Euler's formula is extensively used in quantum mechanics and relativity. In electrical engineering, signals that vary periodically over time are often represented as a combination of sine and cosine functions, which are more conveniently expressed as the real part of exponential functions with imaginary exponents, utilizing Euler's formula. The Fourier series is based on it. In fluid dynamics, Euler's formula is used to describe potential flow. In finance, e is also the amount of money if you deposit 1 Swiss Frank at a continuously compounded interest rate of 100%: the terminal value of an interest compounded m times per annum is $(1 + 100\%/m)^m$ and it tends to $e = 2.71828...$ if $m \to \infty$. Number e is one of the most important numbers in all of mathematics, alongside coincidentally with 0, 1, i, and π. The number also appears in bell curves, which are found in statistics about random variables that add up. Patterns like the logarithmic spiral are commonly found in nature, from the curvature of flowers and seashells to the rotations of tornadoes and galaxies. Another instance where the constant e appears in nature is in the rate of decay of natural radioactive material. Additionally, bacterial growth follows an exponential equation, which remains valid as long as sufficient food is available for the bacteria. At the end, we highlight the most important, the special case of Euler's identity with $x = \pi$ is the famous Euler's formula $e^{i\pi}$ that was described by Richard Feynman as "[...] the most remarkable formula in mathematics [...], our jewel".

Apéndice

A. The Bernoulli's experiment

He conducted the following thought experiment:

1. Assume the interest rate of a bank is 100% and the interest is paid once a year. If we store $1 in the bank, we will get $2 at the end of the year.

2. If the ratio doesn't change, but the payment way will change every half year, then the interest rate will change to 50% in half year. So, when we get $0.5 in the middle of the one year, we will store $1.5 into bank and get $\$1.5 \cdot 50\% = \$0.75$ in the next half year. Furthermore, at the end of this year, we will get $2.25 in total.

3. If the interest is paid every four months, store the money in the same way as before, we will get $2.37 at the end of the year.

4. In the same spirit, if the bank gives back the interest every day the capitalization of the interest will give you approximately $2.71456748202 at the end of the year; in turn, paying interest every second, 31536000 seconds in a year, you will end up having $2.7182817813.

5. If we keep up the same tendency, decreasing the length of the period between two consecutive interest payments (and proportionally the interest for every period) towards infinity, we get the number e.

B. Some examples where e arises

There are many other processes in which the number e appears in biology, for example: random processes and biology, probability of no events, mutation rates, enzyme kinetics. Exponential models are used to study the survival time of organisms or systems, drug absorption and elimination in the body, the spread of infectious diseases often follows exponential growth or decay in the early stages of an outbreak, heart rate variability and biological rhythms and many others. In physics, the number e is found in the equations governing radioactive decay, electrical circuits, and heat transfer. In engineering, particularly in control systems and signal processing, e is essential in solving differential equations that describe dynamic systems. For example, in mechanical and electrical systems, the response of a system to a disturbance or input often involves e, especially when analyzing the behavior of systems with exponential rise or decay, such as in damping and resonance phenomena. In computer science, e is used in algorithms related to data analysis, such as in machine learning, where it helps to describe the behavior of certain optimization algorithms. The natural logarithm, which is based on e, is frequently used in calculating probabilities and entropy in information theory. Apart from financial calculations, e is involved in growth models, such as compound interest or in the study of economic systems evolving over time. Economic processes that involve continuous change or growth, like inflation, can be modeled using exponential functions involving e. Here's a breakdown of some examples where e arises, along with how it can sometimes be related to series or processes.

1. *Compound Interest*
   In finance, e naturally arises in the context of continuous compounding. If P is the principal amount, r is the interest rate, and t is time, the value of the investment after continuous compounding is:
   $$A = Pe^{rt}. \tag{B.1}$$

2. *Population Growth*
   Exponential growth models in biology use the number e to describe populations that grow without resource limitations. If $N(t)$ is the population at time t, and N_0 is the initial population, the growth equation is:
   $$N = N_0 e^{rt}. \tag{B.2}$$
   Here, r is the growth rate.

3. *Radioactive Decay*
   The decay of radioactive substances is governed by exponential decay. If $N(t)$ represents the quantity of the substance at time t, and λ is the decay constant:
   $$N(t) = N_0 e^{-\lambda t}. \tag{B.3}$$

4. *Probability and Statistics*
   Normal distribution: The Gaussian function, which describes the normal distribution, involves e in its formula:
   $$f(x) = \frac{1}{\sqrt{2\pi gma^2}} e^{-\frac{(x-\mu)^2}{2gma^2}}. \tag{B.4}$$

5. *Complex Numbers*
   Euler's formula: A profound relationship in mathematics relates e, i, and trigonometric functions:

$$e^{i\theta} = \cos\theta + in\theta, \tag{B.5}$$

   which forms the foundation of many applications in physics and engineering. Setting $\theta = \pi$ gives the elegant equation $e^{i\pi} + 1 = 0$.

6. *Diffusion and Transport Processes*
   Biological diffusion, such as oxygen or nutrient transport within tissues, can be modeled using equations that involve e. For example, solutions to Fick's law for diffusion are exponential:

$$C(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}. \tag{B.6}$$

In summary, e is used across diverse disciplines to describe processes of growth, decay, and rate of change, often in situations where changes occur continuously or in proportion to their current state.

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