R-separable solutions of the Schrödinger equation

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We present two examples where the Schrödinger equation admits R-separable solutions. In one of them (a particle in a uniform force field) the Schrödinger equation admits separable and R-separable solutions.

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1. Introduction

The standard method employed in the solution of the partial differential equations encountered in physics is that of separation of variables. Even though this method does not always work, its application is straightforward and elementary. There is a class of solutions of the partial differential equations closely related to the separable solutions, rarely mentioned, called R-separable solutions, which involve an appropriate function that mixes two or more of the independent variables. An illustrative example is given by the Laplace equation in toroidal coordinates (α, β, ϕ):

$$\frac{\partial}{\partial\alpha}\left(\frac{\sinh\alpha}{\cosh\alpha-\cos\beta}\frac{\partial u}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{\sinh\alpha}{\cosh\alpha-\cos\beta}\frac{\partial u}{\partial\beta}\right)+\frac{1}{(\cosh\alpha-\cos\beta)\sinh\alpha}\frac{\partial^2 u}{\partial\phi^2}=0.$$

One can verify that this equation does not admit (multiplicatively) separable solutions, but it admits solutions of the form

$$u=\sqrt{\cosh\alpha-\cos\beta}\,A(\alpha)B(\beta)\Phi(\phi),$$

where A, B, Φ are functions of one variable (see, *e.g.*, Refs. [1, 2]). These solutions are called R-separable.

The concept of R-separability also appears in connection with the Hamilton–Jacobi equation; in this case, an R-separable solution is the sum of some function mixing two or more of the variables appearing in the equation plus one-variable functions depending on each of the coordinates and the time (see, *e.g.*, Ref. [3]).

The aim of this paper is to call the attention to the R-separable solutions of the Schrödinger equation, showing its usefulness in elementary problems. In Sec. 2 we consider the elementary problem of a particle in a uniform force field and we show that besides the well-known separable solutions (involving the Airy functions), there are also R-separable solutions that do not require special functions. In Sec. 3 we find R-separable solutions of the Schrödinger equation for a particle in a uniform force field whose strength changes linearly in time.

2. Particle in a uniform force field

The Schrödinger equation for a particle of mass m in a uniform force field is given by

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+max\Psi=\mathrm{i}\hbar\frac{\partial\Psi}{\partial t}, \tag{1}$$

where a is a constant (the negative of the acceleration of the particle in the corresponding problem in classical mechanics). Since the potential does not depend on the time, this equation admits separable solutions of the form $\Psi = \psi(x)\exp(-\mathrm{i}Et/\hbar)$, where E is a separation constant (representing the energy of the state), and $\psi(x)$ must satisfy the second-order ordinary differential equation

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}+max\psi=E\psi. \tag{2}$$

Despite its innocent appearance, the solution of this equation requires special functions (the Airy functions or the Bessel functions of order $1/3$) (see, *e.g.*, Refs. [4, 5]).

On the other hand, noting that

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t}-max\Psi=\exp\left(-\frac{\mathrm{i}maxt}{\hbar}\right)\times\mathrm{i}\hbar\frac{\partial}{\partial t}\left[\Psi\exp\left(\frac{\mathrm{i}maxt}{\hbar}\right)\right],$$

we find it convenient to express Ψ in the form

$$\Psi=\exp(-\mathrm{i}maxt/\hbar)\,\tilde{\Psi}. \tag{3}$$

Then, substituting (3) into Eq. (1) we obtain

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\tilde{\Psi}}{\partial x^2}-\frac{2\mathrm{i}mat}{\hbar}\frac{\partial\tilde{\Psi}}{\partial x}-\frac{m^2a^2t^2}{\hbar^2}\tilde{\Psi}\right)=\mathrm{i}\hbar\frac{\partial\tilde{\Psi}}{\partial t},$$

which does not contain the coordinate x explicitly and therefore it admits separable solutions of the form $\tilde{\Psi} = F(t)\exp(\mathrm{i}kx)$, where k is a constant and the function F must satisfy the separated equation

$$-\frac{\hbar^2}{2m}\left(-k^2+\frac{2kmat}{\hbar}-\frac{m^2a^2t^2}{\hbar^2}\right)F=\mathrm{i}\hbar\frac{\mathrm{d}F}{\mathrm{d}t}.$$

By contrast with Eq. (2), this last equation is of first order and its solution only involves elementary functions; in fact, up to a multiplicative constant,

$$F = \exp\left[\frac{\mathrm{i}\hbar}{2m}\left(-k^2 t + \frac{kmat^2}{\hbar} - \frac{m^2a^2t^3}{3\hbar^2}\right)\right].$$

Thus, we conclude that the Schrödinger equation (1) also admits R-separable solutions of the form

$$\Psi^{(k)}(x,t) = \frac{1}{\sqrt{2\pi}}\exp\left[\frac{\mathrm{i}}{\hbar}\left(-maxt + \hbar kx - \frac{\hbar^2k^2}{2m}t + \frac{\hbar kat^2}{2} - \frac{ma^2t^3}{6}\right)\right], \tag{4}$$

where the superscript (k) has been included in order to emphasize that the wave function (4) depends on the parameter k. The wave function $\Psi^{(k)}$ is an eigenfunction of the Hermitean operator $\hat{p} + mat$, which is conserved, with eigenvalue $\hbar k$. The constant factor $1/\sqrt{2\pi}$ is a normalization factor:

$$\int_{-\infty}^{\infty} \overline{\Psi^{(k)}(x,t)}\, \Psi^{(k')}(x,t)\,\mathrm{d}x = \delta(k-k'). \tag{5}$$

Even though the wave functions $\Psi^{(k)}$ are not stationary states, any solution of the Schrödinger equation (1) can be expressed in the form

$$\Psi(x,t) = \int_{-\infty}^{\infty} \phi(k)\,\Psi^{(k)}(x,t)\,\mathrm{d}k, \tag{6}$$

where $\phi(k)$ is a time-independent function. As a consequence of (5), the function $\phi(k)$ is given by

$$\phi(k) = \int_{-\infty}^{\infty} \overline{\Psi^{(k)}(x,t)}\,\Psi(x,t)\,\mathrm{d}x \tag{7}$$

In particular, the stationary solutions of Eq. (1), characterized by a time-dependence of the form $\exp(-\mathrm{i}Et/\hbar)$, for some constant value E, must be of the form (6). In fact, substituting the explicit expression (4) into the right-hand side of Eq. (6) we have

$$\begin{aligned}&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)\,\exp\left[\frac{\mathrm{i}}{\hbar}\left(-maxt + \hbar kx - \frac{\hbar^2k^2}{2m}t + \frac{\hbar kat^2}{2} - \frac{ma^2t^3}{6}\right)\right]\mathrm{d}k\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)\,\exp\left\{\frac{\mathrm{i}}{\hbar}\left[(\hbar k - mat)x + \frac{1}{6m^2a}(\hbar k - mat)^3 - \frac{\hbar^3k^3}{6m^2a}\right]\right\}\mathrm{d}k\end{aligned} \tag{8}$$

and in order to have the desired time-dependence we take

$$\phi(k) = \frac{1}{\sqrt{2\pi}}\left(\frac{\hbar^2}{2m^2a}\right)^{1/3}\exp\left(\frac{\mathrm{i}\hbar^2k^3}{6m^2a} - \frac{\mathrm{i}kE}{ma}\right)$$

so that (8) becomes (with the change of variable $s \equiv \left(\frac{\hbar^2}{2m^2a}\right)^{1/3}(k - mat/\hbar)$)

$$\begin{aligned}&\frac{1}{2\pi}\left(\frac{\hbar^2}{2m^2a}\right)^{1/3}\int_{-\infty}^{\infty}\exp\left\{\frac{\mathrm{i}}{\hbar}\left[(\hbar k - mat)\left(x - \frac{E}{ma}\right) + \frac{1}{6m^2a}(\hbar k - mat)^3\right]\right\}\mathrm{d}k\ \exp(-\mathrm{i}Et/\hbar)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(\frac{\mathrm{i}s^3}{3} + \mathrm{i}s(x - E/ma)\right)\mathrm{d}s\ \exp(-\mathrm{i}Et/\hbar).\end{aligned}$$

The last expression is the product of a function of x (in fact an Airy function [4, 5]) multiplied by a function of t.

2.1. The propagator

As pointed out above, any solution of the Schrödinger equation (1) can be written in the form (6) for some function $\phi(k)$, which can be determined by means of (7) in terms of the wave function at some initial time t_{i}:

$$\phi(k) = \int_{-\infty}^{\infty}\overline{\Psi^{(k)}(x_{\mathrm{i}},t_{\mathrm{i}})}\,\Psi(x_{\mathrm{i}},t_{\mathrm{i}})\,\mathrm{d}x_{\mathrm{i}}.$$

Substituting this last expression into Eq. (6) we find that for arbitrary final values of x and t,

$$\Psi(x_{\mathrm{f}},t_{\mathrm{f}}) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\overline{\Psi^{(k)}(x_{\mathrm{i}},t_{\mathrm{i}})}\,\Psi(x_{\mathrm{i}},t_{\mathrm{i}})\,\mathrm{d}x_{\mathrm{i}}\right]\Psi^{(k)}(x_{\mathrm{f}},t_{\mathrm{f}})\,\mathrm{d}k = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\overline{\Psi^{(k)}(x_{\mathrm{i}},t_{\mathrm{i}})}\,\Psi^{(k)}(x_{\mathrm{f}},t_{\mathrm{f}})\,\mathrm{d}k\right]\Psi(x_{\mathrm{i}},t_{\mathrm{i}})\,\mathrm{d}x_{\mathrm{i}}.$$

The expression inside the brackets is the so-called propagator and it is sometimes denoted as $K(x_{\rm f}, t_{\rm f}; x_{\rm i}, t_{\rm i})$. Thus, we have

$$K(x_{\rm f}, t_{\rm f}; x_{\rm i}, t_{\rm i}) = \int_{-\infty}^{\infty} \overline{\Psi^{(k)}(x_{\rm i}, t_{\rm i})}\, \Psi^{(k)}(x_{\rm f}, t_{\rm f})\, {\rm d}k \tag{9}$$

and, making use of Eq. (4) and the formula

$$\int_{-\infty}^{\infty} \exp(-{\rm i}\alpha k^2 + {\rm i}\beta k)\, {\rm d}k = \sqrt{\frac{\pi}{{\rm i}\alpha}}\, \exp\left(\frac{{\rm i}\beta^2}{4\alpha}\right),$$

we readily obtain

$$\begin{aligned} K(x_{\rm f}, t_{\rm f}; x_{\rm i}, t_{\rm i}) &= \frac{1}{2\pi} \exp\left\{\frac{\rm i}{\hbar}\left[-\frac{ma^2}{6}({t_{\rm f}}^3 - {t_{\rm i}}^3) - ma(x_{\rm f}t_{\rm f} - x_{\rm i}t_{\rm i})\right]\right\} \\ &\quad \times \int_{-\infty}^{\infty} \exp\left[-{\rm i}\frac{\hbar(t_{\rm f} - t_{\rm i})}{2m}k^2 + {\rm i}\left(x_{\rm f} - x_{\rm i} + \frac{a({t_{\rm f}}^2 - {t_{\rm i}}^2)}{2}\right)k\right] {\rm d}k \\ &= \sqrt{\frac{m}{2\pi{\rm i}\hbar(t_{\rm f} - t_{\rm i})}}\, \exp\left\{\frac{{\rm i}m}{2\hbar}\left[\frac{(x_{\rm f} - x_{\rm i})^2}{t_{\rm f} - t_{\rm i}} - a(t_{\rm f} - t_{\rm i})(x_{\rm i} + x_{\rm f}) - \frac{a^2}{12}(t_{\rm f} - t_{\rm i})^3\right]\right\}. \end{aligned} \tag{10}$$

The derivation of the propagator (10) is very similar to that usually given in the case of a free particle and contrasts with those obtained making use of other, more elaborated, approaches (see, *e.g.*, Refs. [8,9]).

The solutions (4) have the following peculiarity: the expression inside the parenthesis on the right-hand side of (4),

$$S(x, P, t) \equiv -maxt + Px - \frac{P^2}{2m}t + \frac{Pat^2}{2} - \frac{ma^2t^3}{6},$$

where we have replaced $\hbar k$ by the parameter P, is an exact solution of the Hamilton–Jacobi equation for the Hamiltonian $H = p^2/2m + mgx$ [see Eq. (1)] and it is also (additively) R-separable [6,7].

3. Particle in a uniform force field with constant growth

In this section we shall consider the Schrödinger equation for a particle in the time-dependent potential $-Atx$, where A is a constant,

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} - Atx\Psi = {\rm i}\hbar\frac{\partial\Psi}{\partial t}. \tag{11}$$

Owing to the form of the potential, this equation cannot be solved by separation of variables; however, noting that

$${\rm i}\hbar\frac{\partial\Psi}{\partial t} + Atx\Psi = \exp\left(\frac{{\rm i}At^2x}{2\hbar}\right){\rm i}\hbar\frac{\partial}{\partial t}\left[\Psi\exp\left(-\frac{{\rm i}At^2x}{2\hbar}\right)\right]$$

we define $\tilde{\Psi}$ by

$$\Psi = \exp\left(\frac{{\rm i}At^2x}{2\hbar}\right)\tilde{\Psi}$$

and from (11) we obtain the partial differential equation

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\tilde{\Psi}}{\partial x^2} + \frac{{\rm i}At^2}{\hbar}\frac{\partial\tilde{\Psi}}{\partial x} - \frac{A^2t^4}{4\hbar^2}\tilde{\Psi}\right) = {\rm i}\hbar\frac{\partial\tilde{\Psi}}{\partial t}$$

that does not contain x explicitly and therefore admits separable solutions of the form $\tilde{\Psi} = F(t)\exp({\rm i}kx)$, where k is a constant and $F(t)$ is a function of t only that must satisfy the first-order ordinary differential equation

$$\frac{1}{2m}\left(\hbar^2k^2 + At^2\hbar k + \frac{A^2t^4}{4}\right)F = {\rm i}\hbar\frac{{\rm d}F}{{\rm d}t}.$$

Hence, the Schrödinger equation (11) admits the R-separable solutions

$$\begin{aligned} \Psi^{(k)}(x, t) = \frac{1}{\sqrt{2\pi}} \exp\Bigg[\frac{\rm i}{\hbar}\Bigg(&\frac{At^2x}{2} \\ &+ \hbar kx - \frac{\hbar^2k^2}{2m}t - \frac{At^3\hbar k}{6m} - \frac{A^2t^5}{40m}\Bigg)\Bigg], \end{aligned} \tag{12}$$

depending on the parameter k. The wave function $\Psi^{(k)}$ is an eigenfunction of the Hermitean operator $\hat{p} - At^2/2$, which is conserved, with eigenvalue $\hbar k$. The constant factor $1/\sqrt{2\pi}$ is a normalization factor.

Following the same steps as in the preceding section, even though the potential depends on the time, it follows that also in this case the propagator must be given by Eq. (9) and making use of (12) we get

$$
\begin{aligned}
K(x_\mathrm{f}, t_\mathrm{f}; x_\mathrm{i}, t_\mathrm{i}) &= \frac{1}{2\pi} \exp\left\{ \frac{\mathrm{i}}{\hbar} \left[\frac{A}{2}({t_\mathrm{f}}^2 x_\mathrm{f} - {t_\mathrm{i}}^2 x_\mathrm{i}) - \frac{A^2}{40m}({t_\mathrm{f}}^5 - {t_\mathrm{i}}^5) \right] \right\} \\
&\quad \times \int_{-\infty}^{\infty} \exp\left[-\mathrm{i}\frac{\hbar(t_\mathrm{f} - t_\mathrm{i})}{2m} k^2 + \mathrm{i}\left(x_\mathrm{f} - x_\mathrm{i} - \frac{A}{6m}({t_\mathrm{f}}^3 - {t_\mathrm{i}}^3) \right) k \right] \mathrm{d}k \\
&= \sqrt{\frac{m}{2\pi \mathrm{i}\hbar(t_\mathrm{f} - t_\mathrm{i})}} \exp\left\{ \frac{\mathrm{i}m}{2\hbar} \left[\frac{(x_\mathrm{f} - x_\mathrm{i})^2}{t_\mathrm{f} - t_\mathrm{i}} + \frac{A}{3m}\Big({t_\mathrm{f}}^2(2x_\mathrm{f} + x_\mathrm{i}) - t_\mathrm{f} t_\mathrm{i}(x_\mathrm{f} - x_\mathrm{i}) - {t_\mathrm{i}}^2(2x_i + x_\mathrm{f}) \Big) \right.\right. \\
&\quad \left.\left. - \frac{A^2}{180m^2}(t_\mathrm{f} - t_\mathrm{i})\big(4{t_\mathrm{f}}^4 - {t_\mathrm{f}}^3 t_\mathrm{i} - 6{t_\mathrm{f}}^2{t_\mathrm{i}}^2 - t_\mathrm{f}{t_\mathrm{i}}^3 + 4{t_\mathrm{i}}^4 \big) \right] \right\}. \qquad (13)
\end{aligned}
$$

Apart from the normalization factor, the wave function (12) is of the form $\exp(\mathrm{i}S/\hbar)$, where S is a real-valued function that satisfies the Hamilton–Jacobi equation for the classical Hamiltonian $H = p^2/2m - Atx$ and is also R-separable [6, 7].

4. Concluding remarks

As shown in Sec. 2, even in cases where the Schrödinger equation admits separable solutions, the existence of R-separable solutions may be advantageous. As in the case of the usual separability, the R-separable solutions are related to the existence of a conserved operator.

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1. N.N. Lebedev, *Special Functions and Their Applications* (Prentice-Hall, Englewood Cliffs, N.J., 1965) (Dover, New York, reprinted 1972). Sec. 8.10.
2. P. Moon and D.E. Spencer, *Field Theory Handbook, Including Coordinate Systems, Differential Equations and Their Solutions*, 2nd ed. (Springer-Verlag, Berlin, 1988). Sec. IV. https://doi.org/10.1007/978-3-642-83243-7
3. G.F. Torres del Castillo, *An Introduction to Hamiltonian Mechanics* (Springer, Cham, 2018), Sec. 6.1.3. https://doi.org/10.1007/978-3-319-95225-3
4. L.D. Landau and E.M. Lifshitz, *Quantum Mechanics (Non-relativistic Theory)*, 3rd ed. (Pergamon, Oxford, 1991). §24.
5. H.J.W. Müller-Kirsten, *Introduction to Quantum Mechanics: Schrödinger Equation and Path Integral*, 2nd ed. (World Scientific, Singapore, 2012). Sec. 13.3. https://doi.org/10.1142/8428
6. G.F. Torres del Castillo and C. Sosa Sánchez, Solutions of the Schrödinger equation given by solutions of the Hamilton–Jacobi equation, *Rev. Mex. Fís.* **62** (2016) 534.
7. G.F. Torres del Castillo, *op. cit.*, Example 6.15.
8. F. Soto-Eguibar and H.M. Moya-Cessa, Solution of the Schrödinger Equation for a Linear Potential using the Extended Baker–Campbell–Hausdorff Formula, *Appl. Math. Inf. Sci.* **9** (2015) 175. https://dx.doi.org/10.12785/amis/090123
9. D.C. Khandekar and S.V. Lawande, Feynman path integrals: Some exact results and applications, *Phys. Rep.* **137** (1986) 115. https://doi.org/10.1016/0370-1573(86)90029-3