Phase-space representations via quasiprobability distributions

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Energy is a cornerstone concept in physics, embodying both conservation principles and the dynamics of systems. In quantum mechanics, energy manifests as the expectation value of the Hamiltonian operator, yet its intuitive understanding often remains elusive to students. This paper adopts a pedagogical approach to demystify quantum energy by employing three key phase-space representations: the Glauber-Sudarshan P-function, the Husimi Q-function, and the Wigner function. We show that, despite their distinct mathematical frameworks and interpretations, all three representations yield equivalent expressions for the mean energy of thermal states. This result provides an engaging platform for introducing concepts such as coherent states, operator ordering, and quasiprobability distributions. By bridging classical and quantum perspectives in an accessible format, we present a teaching model that not only conveys technical skills but also cultivates deeper conceptual insights, making it feasible to implement in a single lecture. This approach facilitates a richer understanding of the interplay between quantum mechanics and statistical physics, preparing students for advanced topics in quantum theory.

Keywords: Phase-space representation; quasiprobability distributions; coherent states; thermal mean energy.

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1. Introduction

The concept of energy stands as one of the most fundamental and universally acknowledged principles in the realm of physics, often regarded as the backbone of both classical and quantum theories. From the early formulations in classical mechanics, where it serves as a central component in the Lagrangian and Hamiltonian frameworks, to its intricate role in quantum mechanics, where it is associated with the Hamiltonian operator and its eigenvalues, energy embodies the essence of dynamical behavior across diverse physical systems.

As emphasized by Richard Feynman, the law of energy conservation governs all natural phenomena, reinforcing its importance in fields ranging from thermodynamics to quantum statistics and thermodynamic equilibrium. Yet, despite its foundational significance, the educational treatment of energy-especially in quantum mechanics-often focuses narrowly on operator algebra and eigenvalue problems, limiting students' intuitive grasp of its implications in thermal and semiclassical contexts.

Phase-space methods, particularly those incorporating quasiprobability distributions such as the Glauber-Sudarshan P-function, the Husimi Q-function, and the Wigner function, provide a compelling alternative to traditional pedagogical approaches. These representations not only elucidate the quantum-to-classical transition but also facilitate a more comprehensive understanding of energy as a thermal observable. By employing these frameworks, students can explore the connections between operator ordering, coherent states, and thermodynamic quantities in a manner that enhances conceptual clarity and intuition.

This article advocates for the integration of phase-space representations into the standard quantum mechanics curriculum by focusing on the mean energy of quantum systems-particularly using the thermal state of the quantum harmonic oscillator as a case study. By demonstrating the equivalence of the mean energy derived from different quasiprobability distributions, we aim to present an innovative approach that enriches the educational experience, bridging classical and quantum descriptions while providing a straightforward yet profound insight into the nature of energy in quantum mechanics. Through this treatment, we seek to cultivate a deeper understanding and appreciation of the interconnections between quantum theory and statistical mechanics, ultimately empowering students as they advance in their exploration of modern physics.

2. Glauber coherent states: bridging quantum and classical descriptions

Prior to delving into phase-space representations, it is beneficial to revisit the conventional computation of mean energy for the quantum harmonic oscillator (QHO), which serves as a fundamental model within quantum mechanics [4, 7]. The Hamiltonian for a one-dimensional QHO is expressed as:

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \tag{1}$$

where $\hbar$ represents the reduced Planck constant, ω signifies the oscillator's angular frequency, $\hat{a}^\dagger$ is the creation operator, and $\hat{a}$ is the annihilation operator. The number operator $\hat{n}$ is defined by $\hat{n} = \hat{a}^\dagger\hat{a}$, with eigenvalues $n = 0, 1, 2, \ldots$, corresponding to the number of quanta of excitation [4,7,29].

The eigenstates of the number operator $\hat{n}$, denoted by $|n\rangle$, form the so-called *Fock basis* [4,7,29]. They satisfy

$$\hat{n}|n\rangle = n|n\rangle, \qquad n = 0, 1, 2, \ldots, \tag{2}$$

and constitute a complete orthonormal basis of the Hilbert space with $\langle m|n\rangle = \delta_{mn}$ and

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbb{I}. \tag{3}$$

This basis underlies the evaluation of traces and partition functions in the canonical formalism.

These operators obey the canonical commutation relations [4,7,29]:

$$[\hat{a}, \hat{a}^\dagger] = \mathbb{I}, \tag{4a}$$

$$[\hat{n}, \hat{a}] = -\hat{a}, \tag{4b}$$

$$[\hat{n}, \hat{a}^\dagger] = \hat{a}^\dagger. \tag{4c}$$

The ladder relations act on the Fock states as [4,7,29]

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle, \qquad \hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle, \tag{5}$$

with the convention

$$\hat{a}\,|0\rangle = 0. \tag{6}$$

Together with the normalization of the vacuum and the commutation relation $[\hat{a}, \hat{a}^\dagger] = \mathbb{I}$, these formulas immediately imply

$$\hat{n}|n\rangle = \hat{a}^\dagger\hat{a}\,|n\rangle = n\,|n\rangle. \tag{7}$$

The energy eigenvalues of the QHO are expressed as:

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \tag{8}$$

where the quantum number n indicates the state of the system. For a given quantum system described by a density operator $\hat{\rho}$, the mean energy, or expectation value of the Hamiltonian, can be computed using the trace:

$$\langle\hat{H}\rangle = \mathrm{Tr}(\hat{\rho}\hat{H}). \tag{9}$$

In thermal equilibrium at temperature T, the state of the system is represented by the canonical ensemble density operator:

$$\hat{\rho} = \frac{e^{-\beta\hat{H}}}{\mathrm{Tr}(e^{-\beta\hat{H}})}, \tag{10}$$

where $\beta = 1/k_B T$ and k_B is the Boltzmann constant [4,7].

To compute the mean energy for a thermal state, we begin by evaluating the partition function $Z = \mathrm{Tr}(e^{-\beta\hat{H}})$:

$$Z = \sum_{n=0}^{\infty}\langle n|e^{-\beta\hat{H}}|n\rangle = \sum_{n=0}^{\infty} e^{-\beta E_n}$$
$$= \sum_{n=0}^{\infty} e^{-\beta\hbar\omega\left(n+\frac{1}{2}\right)} = e^{-\frac{\beta\hbar\omega}{2}}\sum_{n=0}^{\infty}\left(e^{-\beta\hbar\omega}\right)^n.$$

This series is a geometric series with a common ratio $r = e^{-\beta\hbar\omega}$. Since $0 < r < 1$ for positive temperatures, the sum converges to $1/(1-r)$:

$$Z = \frac{e^{-\frac{\beta\hbar\omega}{2}}}{1 - e^{-\beta\hbar\omega}}. \tag{11}$$

The mean energy can now be derived from the thermodynamic relation:

$$\langle\hat{H}\rangle = -\frac{\partial \ln Z}{\partial\beta},$$

and we calculate:

$$\ln Z = \ln\left(\frac{e^{-\frac{\beta\hbar\omega}{2}}}{1 - e^{-\beta\hbar\omega}}\right) = -\frac{\beta\hbar\omega}{2} - \ln\left(1 - e^{-\beta\hbar\omega}\right).$$

Differentiating with respect to β gives us:

$$\frac{\partial}{\partial\beta}\ln Z = -\frac{\hbar\omega}{2} - \frac{\hbar\omega e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}}.$$

Therefore, the mean energy can be expressed as:

$$\langle\hat{H}\rangle = \frac{\hbar\omega}{2} + \frac{\hbar\omega e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}}.$$

The second term can be simplified, leading to:

$$\frac{\hbar\omega e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} = \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}.$$

Thus, the final expression for the mean energy of the quantum harmonic oscillator becomes:

$$\langle\hat{H}\rangle = \hbar\omega\left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega} - 1}\right). \tag{12}$$

This well-established result for the mean energy of a harmonic oscillator in thermal equilibrium can also be represented using the hyperbolic cotangent function:

$$\langle\hat{H}\rangle = \frac{\hbar\omega}{2}\coth\left(\frac{\beta\hbar\omega}{2}\right). \tag{13}$$

This standard derivation serves as a crucial reference point for subsequent analyses utilizing phase-space methods, enabling students to assess the consistency across various formulations.

Prior to delving into mean energy analysis via phase-space methodologies, it is essential to introduce the concept of **coherent states**. These states act as both a conceptual and mathematical conduit between quantum mechanics and classical descriptions of the harmonic oscillator [9,12]. The harmonic oscillator stands as a pivotal model not only within quantum mechanics but also in statistical physics and quantum optics. The preceding calculation-anchored in the canonical ensemble and the operator formalism-provides a succinct yet potent framework for determining thermal expectation values of observables. Nevertheless, alternative formulations exist that recast quantum statistical mechanics in the lexicon of phase-space distributions, thus bridging quantum theory with its classical analog and enhancing our understanding of classical limits [16, 30, 31].

These formulations prove invaluable in cultivating an intuition regarding quantum states and their classical analogs. Specifically, quasiprobability distributions such as the Glauber-Sudarshan P-function, the Husimi Q-function, and the Wigner function enable the expression of quantum expectation values as weighted averages across complex phase space. Each framework is predicated on distinct operator orderings-namely normal, anti-normal, or symmetric-and possesses unique mathematical characteristics. Yet, they all arrive at identical physical outcomes. This congruence accentuates the internal consistency of quantum statistical mechanics and highlights the pedagogical significance of the phase-space approach [8, 9, 32, 33].

To grasp the construction of these phase-space representations, let us first revisit the definition and fundamental properties of coherent states. A *coherent state* $|\alpha\rangle$ is characterized as the right eigenstate of the annihilation operator:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \quad \alpha \in \mathbb{C}, \tag{14}$$

and can be formulated by applying the displacement operator $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ to the vacuum state:

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle. \tag{15}$$

The states $|\alpha\rangle$ are normalized, satisfying $\langle\alpha|\alpha\rangle = 1$, and comply with the completeness relation

$$\int \frac{\mathrm{d}^2\alpha}{\pi}\, |\alpha\rangle\langle\alpha| = \mathbb{I}, \tag{16}$$

thus offering a resolution of the identity [9], where the integration element is $d^2\alpha \equiv d(\mathrm{Re}\,\alpha)\, d(\mathrm{Im}\,\alpha)$. In polar coordinates $\alpha = re^{i\theta}$ one has $d^2\alpha = r\, dr\, d\theta$, so that

$$\int \frac{d^2\alpha}{\pi}\, e^{-|\alpha|^2} = \frac{1}{\pi}\int_0^{2\pi} d\theta \int_0^\infty r\, dr\, e^{-r^2} = 1.$$

This defines the natural integration measure on the complex plane for phase-space representations. Equation (16) also makes explicit that the set of coherent states is *overcomplete*: although non-orthogonal, they nonetheless span the Hilbert space redundantly. This redundancy is precisely what allows one to represent quantum states by smooth functions on phase space. Therefore, this forms an **overcomplete and non-orthogonal** basis within the Hilbert space.

Coherent states manifest several traits that render them akin to classical states: they minimize the Heisenberg uncertainty relation, their temporal evolution under the harmonic Hamiltonian follows classical trajectories in phase space, and quantum fluctuations exhibit a balanced distribution between position and momentum.

Owing to their classical-like characteristics and favorable mathematical properties, coherent states occupy a pivotal position in both quantum optics and statistical mechanics. They lay the groundwork for the aforementioned quasiprobability distributions, which will be examined in subsequent sections as tools for calculating thermodynamic quantities and elucidating the quantum-to-classical transition.

3. A Unified approach: Mean energy in P, Q, and Wigner representations

One of the most instructive exercises in quantum optics involves computing the mean energy $\langle\hat{H}\rangle$ of a thermal state using various quasiprobability representations. Despite their foundations in different operator orderings and distinct mathematical properties, the P, Q, and Wigner functions yield the same expectation value for the Hamiltonian. This consistency not only underscores the robustness of quantum expectation values across diverse phase-space formulations but also serves as an effective pedagogical tool for introducing these concepts and the formalism of second quantization.

It is worth recalling the connection between quasiprobability distributions and operator ordering. Each quasiprobability distribution corresponds to a specific operator-ordering prescription:

- **Normal ordering**: all creation operators $\hat{a}^\dagger$ are placed to the left of annihilation operators $\hat{a}$; this corresponds to the P-function.
- **Antinormal ordering**: all annihilation operators are placed to the left of creation operators; this corresponds to the Q-function.
- **Symmetric ordering**: creation and annihilation operators are symmetrized; this corresponds to the Wigner function.

These orderings explain why different quasiprobabilities arise, yet all reproduce the same mean values for suitably ordered operators.

3.1. Mean energy via the glauber-sudarshan P-function

The most general quantum state can be expressed as a superposition of projection operators onto coherent states, known as the Glauber–Sudarshan P-representation [8]:

$$\hat{\rho} = \int \frac{\mathrm{d}^2\alpha}{\pi} P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha|. \tag{17}$$

Here, $P(\alpha,\alpha^*)$ serves as a quasiprobability distribution in complex phase space, with coherent states $|\alpha\rangle$ providing the basis for this expansion. Such states are particularly pertinent in contextually relevant systems like the quantum harmonic oscillator and the quantized electromagnetic field, where they characterize maximally coherent states that exhibit classical-like behavior.

As previously mentioned, the function $P(\alpha,\alpha^*)$ is termed a *quasi*-probability distribution due to its potential to take on negative values or become highly singular, especially when representing nonclassical states characterized by sub-Poissonian statistics (see [33] and references therein). However, in scenarios where $P(\alpha,\alpha^*)$ is smooth and varies slowly across phase space, the nonorthogonality of coherent states becomes negligible, allowing the function to be interpreted as a conventional probability distribution [9].

Normalization of the density operator necessitates:

$$\mathrm{Tr}(\hat{\rho}) = \int \frac{\mathrm{d}^2\alpha}{\pi} P(\alpha,\alpha^*) = 1. \tag{18}$$

Thus, the expectation value of an observable $\hat{A}$ can be expressed as [33]:

$$\langle\hat{A}\rangle = \mathrm{Tr}(\hat{\rho}\hat{A}) = \int \frac{\mathrm{d}^2\alpha}{\pi} P(\alpha,\alpha^*)\langle\alpha|\hat{A}|\alpha\rangle. \tag{19}$$

Notably, the expression for the **average particle number** simplifies considerably:

$$\langle\hat{n}\rangle = \mathrm{Tr}(\hat{\rho}\hat{a}^\dagger\hat{a}) = \int \frac{\mathrm{d}^2\alpha}{\pi} P(\alpha,\alpha^*)|\alpha|^2 \equiv \langle|\alpha|^2\rangle_P. \tag{20}$$

Here, $\langle\cdots\rangle_P$ denotes the average over phase space weighted by the function $P(\alpha,\alpha^*)$.

For a thermal state within the canonical ensemble, the density operator is represented as

$$\hat{\rho} = (1 - e^{-\beta\hbar\omega})\, e^{-\beta\hbar\omega\hat{a}^\dagger\hat{a}}. \tag{21}$$

Inserting the relation (3) into above equation, the thermal density operator reads

$$\hat{\rho} = (1 - e^{-\beta\hbar\omega})\sum_{n=0}^{\infty} e^{-\beta\hbar\omega n}|n\rangle\langle n|, \tag{22}$$

The Glauber–Sudarshan P-function is defined by Eq. (17). Using the diagonal representation of thermal states in coherent states and the generating function

$$\langle\alpha|n\rangle = \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2},$$

one finds that the thermal state admits the Gaussian P-function given by

$$P(\alpha,\alpha^*) = \frac{1}{\langle\hat{n}\rangle}\exp\left(-\frac{|\alpha|^2}{\langle\hat{n}\rangle}\right), \tag{23}$$

with

$$\langle\hat{n}\rangle = \langle|\alpha|^2\rangle_P = \frac{1}{e^{\beta\hbar\omega} - 1}. \tag{24}$$

More generally, the expectation value of a normally ordered operator product is given by

$$\langle(\hat{a}^\dagger)^r\hat{a}^s\rangle_P = \int \frac{\mathrm{d}^2\alpha}{\pi} P(\alpha,\alpha^*)(\alpha^*)^r\alpha^s. \tag{25}$$

Specifically, using Eq. (24), the mean energy of the quantum harmonic oscillator can be expressed as

$$\begin{aligned}\langle\hat{H}\rangle &= \hbar\omega\left(\langle\hat{n}\rangle + \frac{1}{2}\right) = \hbar\omega\left(\frac{1}{e^{\beta\hbar\omega} - 1} + \frac{1}{2}\right)\\ &= \frac{\hbar\omega}{2}\coth\left(\frac{\beta\hbar\omega}{2}\right),\end{aligned} \tag{26}$$

which aligns with the canonical result previously obtained in Eq. (13).

Taking $r = s = 1$ in Eq. (25), the mean of $|\alpha|^2$ with respect to the P-distribution is

$$\langle|\alpha|^2\rangle_P = \int \frac{d^2\alpha}{\pi}\, P(\alpha,\alpha^*)\,|\alpha|^2. \tag{27}$$

It is also important to note that the statistical average over the complex plane in this derivation can be reformulated in polar coordinates. Writing $\alpha = re^{i\theta}$, we have $d^2\alpha = r\,dr\,d\theta$. Substituting (23) gives

$$\begin{aligned}\langle|\alpha|^2\rangle_P &= \frac{1}{\pi\langle n\rangle}\int_0^{2\pi} d\theta \int_0^{\infty} dr\, r\, e^{-r^2/\langle n\rangle}\, r^2\\ &= \frac{2}{\langle n\rangle}\int_0^{\infty} dr\, r^3 e^{-r^2/\langle n\rangle} = \frac{2}{\langle n\rangle}\frac{\langle n\rangle^2}{2} = \langle n\rangle,\end{aligned} \tag{28}$$

as expected.

3.2. Mean energy via the Husimi Q-function

A closely related phase-space distribution is obtained by considering the diagonal matrix element of the density operator $\hat{\rho}$:

$$Q(\alpha,\alpha^*) = \langle\alpha|\hat{\rho}|\alpha\rangle, \tag{29}$$

and is manifestly positive and normalized. It thus resembles a probability density. However, it is important to emphasize that the Q-function does not satisfy all the requirements of a classical probability distribution: in particular, its smoothing implies that it cannot reproduce expectation values of all operator orderings. We will highlight this distinction from the outset to avoid confusion [13, 14]. For a thermal state, the Q-function assumes a Gaussian form:

$$Q(\alpha,\alpha^*) = \frac{1}{1 + \langle\hat{n}\rangle}\exp\left(-\frac{|\alpha|^2}{1 + \langle\hat{n}\rangle}\right). \tag{30}$$

The Q-representation relates to operator averages in *anti-normal* order. In this context, the expectation value of an operator product is expressed as [13]

$$\langle\hat{a}^s(\hat{a}^\dagger)^r\rangle_Q = \int \frac{\mathrm{d}^2\alpha}{\pi} Q(\alpha,\alpha^*)(\alpha^*)^r\alpha^s, \tag{31}$$

where $\langle\cdots\rangle_Q$ symbolizes the statistical average with regard to $Q(\alpha,\alpha^*)$.

For the case where $s = r = 1$, we find

$$\langle \hat{a}\hat{a}^\dagger \rangle_Q = \int \frac{\mathrm{d}^2\alpha}{\pi} Q(\alpha,\alpha^*)|\alpha|^2 = \langle |\alpha|^2 \rangle_Q, \tag{32}$$

which implies

$$\langle \hat{n} \rangle = \langle |\alpha|^2 \rangle_Q - 1. \tag{33}$$

Thus, the mean energy in terms of the Q-function is given by

$$\begin{aligned} \langle \hat{H} \rangle &= \hbar\omega \left(\langle \hat{n} \rangle + \frac{1}{2} \right) = \hbar\omega \left(\langle |\alpha|^2 \rangle_Q - \frac{1}{2} \right) \\ &= \hbar\omega \left(\frac{1}{1 - e^{-\beta\hbar\omega}} - \frac{1}{2} \right) = \hbar\omega \left(\frac{1 + e^{\beta\hbar\omega}}{2(e^{\beta\hbar\omega} - 1)} \right) \\ &= \frac{\hbar\omega}{2} \coth \left(\frac{\beta\hbar\omega}{2} \right), \end{aligned} \tag{34}$$

confirming the equivalence with the P-representation and the result obtained via the standard trace formula in Eq. (13).

The statistical average can be evaluated following the same procedure used in the calculation of Eq. (28), leading to

$$\langle |\alpha|^2 \rangle_Q = \int \frac{d^2\alpha}{\pi} |\alpha|^2 Q(\alpha,\alpha^*) = \frac{1}{1 - e^{-\beta\hbar\omega}}, \tag{35}$$

where $Q(\alpha, \alpha^*)$ is given by Eq. (30).

3.3. Mean energy via the Wigner function

The Wigner function is related to the P-function through convolution [12]:

$$W(\alpha,\alpha^*) = 2 \int \frac{\mathrm{d}^2 z}{\pi} P(z,z^*) \exp(-2|\alpha - z|^2). \tag{36}$$

For a thermal state, this leads to the Gaussian form:

$$W(\alpha,\alpha^*) = \frac{1}{\langle \hat{n} \rangle + 1/2} \exp\left(-\frac{|\alpha|^2}{\langle \hat{n} \rangle + 1/2} \right), \tag{37}$$

where the inverse width is given by $1/(\langle \hat{n} \rangle + 1/2) = 2\tanh(\beta\hbar\omega/2)$.

The Wigner function facilitates the computation of expectation values in symmetric (Weyl) ordering:

$$\langle (\hat{a}^{\dagger r} \hat{a}^s)_S \rangle_W = \int \frac{\mathrm{d}^2\alpha}{\pi} W(\alpha,\alpha^*) (\alpha^*)^r \alpha^s, \tag{38}$$

where $(\cdots)_S$ denotes symmetric ordering [13, 14], and $\langle \cdots \rangle_W$ indicates the average over $W(\alpha, \alpha^*)$.

For the specific case where $r = s = 1$, we obtain:

$$\begin{aligned} \langle (\hat{a}^\dagger \hat{a})_S \rangle_W &= \int \frac{\mathrm{d}^2\alpha}{\pi} W(\alpha,\alpha^*)|\alpha|^2 \\ &= \frac{1}{2} \left(\langle \hat{a}\hat{a}^\dagger \rangle + \langle \hat{a}^\dagger \hat{a} \rangle \right), \end{aligned} \tag{39}$$

from which it follows that

$$\langle \hat{n} \rangle = \langle |\alpha|^2 \rangle_W - \frac{1}{2}. \tag{40}$$

Consequently, the mean energy in the Wigner representation becomes

$$\begin{aligned} \langle \hat{H} \rangle &= \hbar\omega \left(\langle \hat{n} \rangle + \frac{1}{2} \right) = \hbar\omega \langle |\alpha|^2 \rangle_W \\ &= \frac{\hbar\omega}{2} \coth \left(\frac{\beta\hbar\omega}{2} \right), \end{aligned} \tag{41}$$

with

$$\begin{aligned} \langle |\alpha|^2 \rangle_W &= \int \frac{\mathrm{d}^2\alpha}{\pi} |\alpha|^2 W(\alpha,\alpha^*) \\ &= \frac{1}{2} \coth \left(\frac{\beta\hbar\omega}{2} \right). \end{aligned} \tag{42}$$

Remarkably, the mean energy of a thermal state remains consistent whether computed via the P-function, the Q-function, or the Wigner function. This universal result reaffirms the coherence of different quasiprobability representations and aligns with the direct evaluation of $\mathrm{Tr}(\hat{\rho}\hat{H})$ as given in Eq. (13).

Remark: the phase-space averages of $|\alpha|^2$ are representation-dependent:

$$\langle |\alpha|^2 \rangle_P = \langle \hat{n} \rangle, \quad \langle |\alpha|^2 \rangle_Q = \langle \hat{n} \rangle + 1,$$
$$\langle |\alpha|^2 \rangle_W = \langle \hat{n} \rangle + \tfrac{1}{2},$$

where $\langle \hat{n} \rangle$ is given by Eq. (24). This is so because of different operator-ordering prescriptions (normal, antinormal and symmetric) underlying P, Q and W. Crucially, when forming the physical mean energy

$$\langle \hat{H} \rangle = \hbar\omega\big(\langle \hat{n} \rangle + \tfrac{1}{2} \big),$$

the representation-dependent constants cancel, so all three quasiprobability representations yield the same value of $\langle \hat{H} \rangle$ (cf. Eqs. (26)–(41)).

3.4. Pedagogical remark

The congruence of all three representations in yielding the same thermal expectation value for the energy,

$$\langle \hat{H} \rangle = \frac{\hbar\omega}{2} \coth \left(\frac{\beta\hbar\omega}{2} \right),$$

serves as a compelling demonstration of the internal consistency of quantum statistical mechanics. This equivalence is not merely a mathematical curiosity; rather, it offers a valuable pedagogical tool. It bridges distinct formulations of quantum theory, enhances understanding of operator ordering, and reinforces the interpretation of energy as a physically meaningful expectation value derived from phase space distributions.

4. Pedagogical advantages of the approach

Evaluating the mean energy $\langle \hat{H} \rangle$ using the P, Q, and Wigner quasiprobability distributions transcends computational exercises, providing a meaningful avenue to link abstract quantum formalism with intuitive constructs from phase space. The following outlines the educational benefits of adopting this approach in undergraduate or early graduate quantum mechanics courses.

4.1. Energy as a bridge between classical and quantum frameworks

Energy is a concept that students readily recognize and understand in physical contexts. Utilizing it as a unifying theme across various quantum representations enables students to connect formal mathematical tools-such as coherent states, quasiprobability distributions, and operator ordering-to a concept that holds a well-established physical interpretation. This strategy promotes a smoother conceptual transition from classical to quantum mechanics, reinforcing the role of quantum operators as generators of observables.

4.2. Early exposure to modern tools

While phase-space techniques are typically reserved for advanced studies in quantum optics or statistical mechanics, the approach presented here provides an accessible, pedagogically effective introduction to these topics at an earlier stage. Through this example, students are introduced to:

1. **Coherent states** and their classical-like dynamical properties,

2. The implications of **operator ordering** (normal, antinormal, and symmetric),

3. The interpretive roles of different **quasiprobability distributions**.

These concepts lay the groundwork for more advanced topics such as quantum field theory, quantum information, and open quantum systems.

4.3. Unification across representations

Demonstrating that a single physical observable-the mean energy-can be derived consistently from three distinct representational frameworks illustrates the internal coherence and robustness of quantum theory. This reinforces students' confidence in navigating multiple formalisms and counters the misconception that alternative approaches yield conflicting results. Instead, it highlights the adaptability of quantum mechanics and the invariance of physical predictions across representations.

4.4. Visualization and interpretation

Each quasiprobability distribution offers unique structural and interpretive features: (i) the P**-function** may exhibit singularities or negative values for nonclassical states; (ii) the Q**-function** is always smooth and positive, albeit lacking resolution; and (iii) the **Wigner function** can attain negative values, reflecting quantum interference and coherence.

By examining these attributes, students gain insights into the nuances of phase-space representations and their connection to classical intuition. Importantly, the fact that all distributions yield the same energy expectation value reinforces the reliability and consistency of quantum theory.

4.5. Feasibility within a single lecture

A significant pedagogical advantage of this approach lies in its practicality. As illustrated, the derivations of $\langle \hat{H} \rangle$ from all three representations are concise, conceptually transparent, and mathematically tractable. This makes it feasible to encompass the entire discussion within a single, focused lecture, enhancing the standard curriculum without necessitating significant restructuring or additional prerequisites.

4.6. Relationships and applications of the three quasiprobability distributions

The P, Q, and Wigner functions are related by Gaussian convolutions: the Q-function is a smoothed version of the Wigner function, which itself is a smoothed version of the generally more singular P-function. Since thermal states are Gaussian, this convolutional relationship ensures that all three quasiprobability distributions yield the same expression for the mean energy.

From a pedagogical viewpoint, each distribution also illustrates different physical aspects:

- The P-function is central in quantum optics, especially for identifying nonclassical states (where P becomes highly singular or negative).

- The Q-function is smooth, positive, and experimentally accessible in state tomography, providing an intuitive visualization of quantum states.

- The Wigner function directly generalizes the classical phase-space distribution and reveals quantum interference through its negative regions.

Presenting the three side by side allows students to appreciate both their common predictive power and their distinct interpretive uses.

4.7. Pedagogical takeaways

- **Unified viewpoint:** Despite their different definitions, the P, Q, and Wigner functions yield the same mean energy for thermal states. This illustrates how diverse formalisms can converge on identical physical predictions.

- **Operator ordering:** The three quasiprobability distributions correspond to normal, antinormal, and symmetric orderings, respectively. Introducing these concepts provides students with an early encounter with operator ordering in a concrete setting.

- **Overcompleteness as a bridge:** The redundant spanning property of coherent states makes phase-space representations possible, serving as a natural bridge between classical intuition and quantum mechanics.

- **Classical versus quantum intuition:** The Q-function looks like a probability density but is not a true probability distribution, highlighting subtle departures from classical reasoning.

4.8. Applications

Each representation has distinct practical uses:

- P**-function**: generates all normally ordered correlators; central in photodetection theory and in identifying classical versus nonclassical light (diagnosing nonclassicality) [34].

- Q**-function**: positive and smooth; experimental accessibility, useful for visualization and robust, coarse-grained diagnostics [35].

- **Wigner function**: quasi-distribution with possible negatividades; closest analogue of a classical phase-space distribution, revealing quantum interference; widely used to visualize quantum interference and to study dynamics in quantum optics, optomechanics, and quantum transport [36].

Together, these viewpoints strengthen conceptual understanding while ensuring that physical predictions (e.g., the thermal mean energy) remain invariant under changes of representation.

4.9. Final reflection

Integrating this unified treatment of mean energy through phase-space representations enriches quantum mechanics instruction by blending *quantitative reasoning*, interpretation, and conceptual synthesis. It nurtures both technical skills and a deeper understanding, offering students an early glimpse into the elegant structure of quantum theory-one that is both accessible and intellectually rewarding.

5. Conclusions

This study has effectively illustrated how the concept of energy –fundamental to physical theory –can be employed as a pedagogical bridge for introducing advanced quantum mechanical tools in a manner that is both engaging and accessible. By calculating the mean energy $\langle \hat{H} \rangle$ of a thermal state using the Glauber–Sudarshan P-function, the Husimi Q-function, and the Wigner function, we demonstrated that diverse phase-space representations, each tied to specific operator orderings, converge to the same physically meaningful results.

This fundamental equivalence imparts a significant educational insight: despite the varied mathematical representations in quantum mechanics, they are united by consistent physical principles. Such coherence across different formulations reassures students that the abstract structures of quantum theory are deeply grounded in physical reality. Acknowledging this unity aids in demystifying quantum mechanics, providing students with a conceptual anchor as they explore its more complex aspects.

By leveraging energy as a unifying theme, educators can seamlessly introduce essential quantum concepts–such as coherent states, quasiprobability distributions, and operator orderings– without necessitating extensive prior knowledge of quantum field theory or functional analysis. This approach fosters intuitive understanding by relating these abstract tools to thermodynamic observables, highlighting the profound connection between statistical mechanics and quantum optics. Consequently, it cultivates a deeper appreciation for the operational significance of quantum states and clarifies the measurement process in quantum theory.

Furthermore, the pedagogical value of this framework lies in its adaptability and modularity. It can be incorporated as a self-contained lecture or concise module within upper-level undergraduate or introductory graduate courses. In a limited timeframe, students encounter second quantization, phase-space methods, and operator algebra, all framed through a physically motivated computation. This creates an enriching environment conducive to developing both conceptual insight and mathematical proficiency.

Looking to the future, this strategy paves the way for broader educational explorations. Similar calculations involving other observables-such as position variance, entropy, or purity-could also be performed within this formalism, introducing students to themes like nonclassicality, decoherence, and quantum information. Thus, this approach offers a flexible platform for extending the curriculum to modern topics such as quantum thermodynamics, quantum control, and the classical-quantum transition.

In summary, the intentional incorporation of energy-based computations within phase-space quantum mechanics presents a robust pedagogical approach-one that highlights conceptual clarity, mathematical coherence, and interdisciplinary integration. We believe that employing this strategy can significantly enhance quantum mechanics education by bridging the gap between formalism and intuition, thereby equipping students for a more profound engagement with contemporary quantum theory.

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