Teaching strategy for introducing beginners to coherent states

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Handling coherent states by undergraduates students may be a hard task, as they have to deal with Glauber's series $e^{-(|\alpha|^2/2)}\sum\limits_{n=0}^{\infty}(\alpha^n/\sqrt{n!})\,\phi_n(x)$. We show here that the task can be greatly simplified by introduction of a novel compact formula for Glauber coherent states employed in by Ferrary *et al*., This expression is obtained by solving the basic differential equation associated to coherent states $a|\alpha\rangle = \alpha|\alpha\rangle$.

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1. Introduction

Coherent states (CS) constitute a well-developed and useful thematic structure that is applied in many physics areas. The literature is immense. An extremely short, illustrative but by no means exhaustive, is that of Refs. [1–6] and references therein. A coherent state is that quantum state that shows a dynamic closely resembling that of a classical harmonic oscillator (HO). Coherent states emerge in the quantum theory of a variegated range of systems [2].

However, the beginner usually faces difficulties in handling CS as defined by Glauber's series[i]. A somewhat more pedagogical treatment should be welcome. This is our present goal. We propose to start teaching by introducing the creation and destruction operators for the harmonic oscillator, then just solve the defining equation $a|\alpha\rangle = \alpha|\alpha\rangle$, which is easily accomplished.

In order to provide the reader with an alleviated learning job, we will use a compact closed expression for CS, recently used in [3], that can be easily handled by beginners and non-experts in Glauber's theory.

With this purpose we proceed as follows:

1) From the coherent states' defining equation $a|\alpha\rangle = \alpha|\alpha\rangle$ we obtain a compact form for the solution of the associated differential equation.

2) This solution is the compact CS expression that constitutes our main result.

3) We then show that this compact expression leads to the Glauber's series procedure.

4) Our formulation is used, as an example, to easily derive uncertainty relations.

2. Compact Coherent states

Let us briefly remind the reader about the standard coherent states of the harmonic oscillator (HO) $|\alpha\rangle$ [4–6]. A coherent state $|\alpha\rangle$ is a specific kind of quantum state of minimum uncertainty, the one that most resembles a classical state. It is applicable to the quantum harmonic oscillator, the electromagnetic field, etc., and describes a maximal kind of coherence and a classical kind of behavior.

It is very well known the annihilation operator for the one-dimensional harmonic oscillator is given by [4–6]

$$\hat{a} = \frac{mw\hat{x} + i\hat{p}}{\sqrt{2\hbar mw}}. \tag{1}$$

In the x-representation of Quantum Mechanics, this operator is expressed via

$$\hat{a}(x) = \frac{1}{\sqrt{2\hbar mw}}\left(mwx + \hbar\frac{d}{dx}\right). \tag{2}$$

Thus, a coherent state is defined as the eigenfunction

$$\hat{a}(x)\psi_\alpha(x) = \frac{1}{\sqrt{2\hbar mw}}\left(mwx\psi_\alpha(x) + \hbar\frac{d\psi_\alpha(x)}{dx}\right) = \alpha\psi_\alpha(x), \tag{3}$$

or, equivalently,

$$\frac{d\psi_\alpha(x)}{dx} = \left(\sqrt{\frac{2mw}{\hbar}}\alpha - \frac{mwx}{\hbar}\right)\psi_\alpha(x). \tag{4}$$

This is a simple differential equation that can be quite easily solved right now. Proceeding to do it constitutes the novel didactic tool of our paper. The solution of Eq. (4) reads

$$\psi_\alpha(x) = Ce^{-(mwx^2/2\hbar)}e^{\sqrt{(2mw/\hbar)}\alpha x}. \tag{5}$$

The constant C can be evaluated using the normalization condition

$$\int_{-\infty}^{\infty} |\psi_\alpha(x)|^2 dx = |C|^2 \int_{-\infty}^{\infty} e^{-(mw/\hbar)x^2} \times e^{\sqrt{2}(\alpha+\alpha^*)x} dx = 1. \quad (6)$$

Now, we effect the change of variables $\sqrt{(mw/\hbar)}x = y$. Accordingly,

$$\int_{-\infty}^{\infty} |\psi_\alpha(x)|^2 dx = |C|^2 \sqrt{\frac{\hbar}{mw}} e^{((\alpha+\alpha^*)^2/2)} \times \int_{-\infty}^{\infty} e^{-(y-((\alpha+\alpha^*)/\sqrt{2}))^2} dy = 1. \quad (7)$$

By recourse to the result given in the well known Table [8] in Ref. we then obtain

$$\int_{-\infty}^{\infty} e^{-(y-((\alpha+\alpha^*)/\sqrt{2}))^2} dy = \sqrt{\pi}. \quad (8)$$

As a consequence,

$$C = \left(\frac{mw}{\pi\hbar}\right)^{(1/4)} e^{-((\alpha+\alpha^*)^2/4)}. \quad (9)$$

Thus, we have for $\psi_\alpha(x)$ the expression

$$\psi_\alpha(x) = \left(\frac{mw}{\pi\hbar}\right)^{(1/4)} e^{-((\alpha+\alpha^*)^2/4)} \times e^{-(mwx^2/2\hbar)} e^{\sqrt{(2mw/\hbar)}\alpha x}, \quad (10)$$

or, equivalently,

$$\psi_\alpha(x) = \left(\frac{mw}{\pi\hbar}\right)^{(1/4)} e^{i\alpha_R\alpha_I} e^{-(\alpha^2/2)} e^{-(|\alpha|^2/2)} \times e^{-(mwx^2/2\hbar)} e^{\sqrt{(2mw/\hbar)}\alpha x}, \quad (11)$$

where $\alpha = \alpha_R + i\alpha_I$. As $e^{i\alpha_R\alpha_I}$ is an imaginary phase, it can be eliminated from (11) to finally obtain

$$\psi_\alpha(x) = \left(\frac{mw}{\pi\hbar}\right)^{(1/4)} e^{-(\alpha^2/2)} e^{-(|\alpha|^2/2)} \times e^{-(mwx^2/2\hbar)} e^{\sqrt{(2mw/\hbar)}\alpha x}. \quad (12)$$

Thus,

$$|\alpha\rangle = \left(\frac{mw}{\pi\hbar}\right)^{1/4} e^{-(\alpha^2/2)} e^{-(|\alpha|^2/2)} \times \int e^{-(mwx^2/2\hbar)} e^{\sqrt{(2mw/\hbar)}\alpha x} |x\rangle dx. \quad (13)$$

We have here achieved our goal: having at our disposal a compact, exact expression (12) for dealing with the coherent state $|\alpha\rangle$, that can be easily handled for any application one may have in mind. The issue is to guarantee that the above expression, that can be found in [3], is consistent with the standard, text-book treatment of coherent states. This we will tackle next.

3. Comparison with the standard Glauber theory

Let us briefly remind the reader of the standard Glauber treatment for the coherent states of the harmonic oscillator (HO) $|\alpha\rangle$ [4–6]. The states $|\alpha\rangle$ are normalized, *i.e.*, $\langle\alpha|\alpha\rangle = 1$, and they provide us with a resolution of the identity operator [4–6]

$$\int \frac{\mathrm{d}^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = 1, \quad (14)$$

which is a completeness relation for the coherent states [6]. The standard coherent states $|\alpha\rangle$ for the harmonic oscillator are eigenstates of the annihilation operator $\hat{a}$, with complex eigenvalues

$$\alpha = \frac{mwx + ip}{\sqrt{2\hbar mw}}, \quad (15)$$

which satisfy $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ [6]. These brief lines constitute the hard-core of the Glauber-approach.

We prove below that the coherent states (13) coincide with the above Glauber-ones.

The $n-$th HO eigenfunction is

$$\phi_n(x) = \left(\frac{m\omega}{\hbar}\right)^{(1/4)} \mathcal{H}_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right), \quad (16)$$

where $\mathcal{H}_n$ is Hermite's $n-$th order generalized function

$$\mathcal{H}_n(x) = \left(\pi^{1/2}2^n n!\right)^{-(1/2)} e^{-(x^2/2)} H_n(x), \quad (17)$$

while H_n is the concomitant Hermite polynomial. The gist of our demonstration is to start with Eq. (12), expand it in a Hermite series and verify that one arrives to the celebrated Glauber expansion for a coherent state.

In the x-representation, if we expand it in an appropriate basis, the coherent state (12) reads

$$\psi_\alpha(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{(1/4)} e^{-(\alpha^2/2)} e^{-(|\alpha|^2/2)} \times e^{-(mwx^2/2\hbar)} e^{\sqrt{(2mw/\hbar)}\alpha x} = \sum_{n=0}^{\infty} a_n\phi_n(x), \quad (18)$$

so that

$$a_n = \int \psi_\alpha(x)\phi_n(x) dx, \quad (19)$$

or

$$a_n = \left(\frac{mw}{\pi\hbar}\right)^{(1/4)} e^{-(\alpha^2/2)} e^{-(|\alpha|^2/2)} \times \int e^{-(mwx^2/2\hbar)} e^{\sqrt{(2mw/\hbar)}\alpha x} \phi_n(x) dx. \quad (20)$$

A Hermite formulation is thus the following

$$a_n = \left(\frac{mw}{\hbar}\right)^{(1/2)} \pi^{-1/4} e^{-(\alpha^2/2)} e^{-(|\alpha|^2/2)} \times \int_{-\infty}^{\infty} e^{-(mwx^2/2\hbar)} e^{\sqrt{(2mw/\hbar)}\alpha x} \mathcal{H}_n \times \left(\sqrt{\frac{mw}{\hbar}} x\right) dx. \quad (21)$$

We use now Eq. (17) with argument $\sqrt{(mw/\hbar)}x$ and have

$$a_n = \left(\frac{mw}{\hbar}\right)^{1/2} \pi^{-1/4} \left(\pi^{\frac{1}{2}} 2^n n!\right)^{-1/2} e^{-(\alpha^2/2)} e^{-(|\alpha|^2/2)} \times \int_{-\infty}^{\infty} e^{-(mwx^2/2\hbar)} e^{\sqrt{(2mw/\hbar)}\alpha x} H_n \times \left(\sqrt{\left(\frac{mw}{\hbar}\right)} x\right) dx. \quad (22)$$

We effect here the change of variables $\sqrt{(mw/\hbar)}x = y$ and find

$$a_n = \pi^{-1/4} \left(\pi^{1/2} 2^n n!\right)^{-1/2} e^{-(\alpha^2/2)} e^{-(|\alpha|^2/2)} \times \int_{-\infty}^{\infty} e^{-(y^2/2\hbar)} e^{\sqrt{2}\alpha y} H_n(y)\, dy, \quad (23)$$

or

$$a_n = \frac{\pi^{-1/4} e^{-(|\alpha|^2/2)}}{\left(n! 2^n \pi^{1/2}\right)^{1/2}} \int_{-\infty}^{\infty} e^{-\left(y-(\alpha/\sqrt{2})\right)^2} H_n(y)\, dy. \quad (24)$$

We appeal now to an Integral-Table result (see [7]) to obtain

$$a_n = \frac{\pi^{-1/4} e^{-(|\alpha|^2/2)}}{\left(n! 2^n \pi^{1/2}\right)^{1/2}} \pi^{1/2} 2^{n/2} \alpha^n, \quad (25)$$

and

$$a_n = \frac{\alpha^n}{\sqrt{n!}} e^{-(|\alpha|^2/2)}. \quad (26)$$

With the above expression, beginning with Eq. (12) for $\psi_\alpha(x)$, we reach Glauber's well-known and celebrated result.

$$\psi_\alpha(x) = e^{-(|\alpha|^2/2)} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \phi_n(x). \quad (27)$$

Thus, the states $\psi_\alpha(x)$ given by Eq. (12) have been shown to be Glauber's coherent states.

4. Uncertainties

As an application-example we give here some well known results that are needed to determine uncertainty relations. Instead of working with

$$\psi_\alpha(x) = e^{-(|\alpha|^2/2)} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \phi_n(x),$$

we just use the simple, and compact expression in Eq. (12).

For simplicity we take $mw/\hbar = 1$. For an ordinary coherent state $|\alpha >$ we have to compute four mean values.

- $\langle x^2 \rangle$

$$\langle x^2 \rangle = \pi^{-1/2} \int_{-\infty}^{\infty} e^{-1/2(x^2 - 2\sqrt{2}\alpha^* x + \alpha^{*2} + |\alpha|^2)} x^2 \times e^{-(1/2)(x^2 - 2\sqrt{2}\alpha x + \alpha^2 + |\alpha|^2)} dx \quad (28)$$

With the use of the Integral-Table result [8] we then find

$$\langle x^2 \rangle = (2i)^{-2} H_2 \left[\frac{i(\alpha^* + \alpha)}{\sqrt{2}}\right] \quad (29)$$

and thus

$$\langle x^2 \rangle = \frac{1}{2} + \frac{(\alpha + \alpha^*)^2}{2} \quad (30)$$

- $\langle x \rangle$

For $\langle x \rangle$ the situation is quite similar.

$$\langle x \rangle = \pi^{-1/2} \int_{-\infty}^{\infty} e^{-(1/2)(x^2 - 2\sqrt{2}\alpha^* x + \alpha^{*2} + |\alpha|^2)} x \times e^{-\frac{1}{2}(x^2 - 2\sqrt{2}\alpha x + \alpha^2 + |\alpha|^2)} dx \quad (31)$$

Using the Integral-Table result [8] again we obtain

$$\langle x \rangle = (2i)^{-1} H_1 \left[\frac{i(\alpha^* + \alpha)}{\sqrt{2}}\right], \quad (32)$$

and thus

$$\langle x \rangle = \frac{\alpha + \alpha^*}{\sqrt{2}}. \quad (33)$$

- $\langle p^2 \rangle$

For $\langle p^2 \rangle$, the integral is somewhat more complicated.

$$\langle p^2 \rangle = -\pi^{-1/2} \int_{-\infty}^{\infty} e^{-(1/2)(x^2 - 2\sqrt{2}\alpha^* x + \alpha^{*2} + |\alpha|^2)} \times \frac{\partial^2}{\partial x^2} e^{-(1/2)(x^2 - 2\sqrt{2}\alpha x + \alpha^2 + |\alpha|^2)} dx, \quad (34)$$

or

$$\langle p^2\rangle = \pi^{-1/2}\int\limits_{-\infty}^{\infty} e^{-(1/2)(x^2-2\sqrt{2}\alpha^* x+\alpha^{*2}+|\alpha|^2)} \times [1-(x-\sqrt{2}\alpha)^2] \times e^{-(1/2)(x^2-2\sqrt{2}\alpha x+\alpha^2+|\alpha|^2)}dx. \quad (35)$$

Now, by recourse to the Integral-Table result [8] we obtain

$$\langle p^2\rangle = 1-2\alpha^2 - i\sqrt{2}\alpha H_1\left[\frac{i(\alpha^*+\alpha)}{\sqrt{2}}\right] + \frac{1}{4}H_2\left[\frac{i(\alpha^*+\alpha)}{\sqrt{2}}\right], \quad (36)$$

or

$$\langle p^2\rangle = \frac{1}{2} - \frac{(\alpha-\alpha^*)^2}{2}. \quad (37)$$

- $\langle p\rangle$

For dealing with $\langle p\rangle$ one starts with

$$\langle p\rangle = -i\pi^{-1/2}\int\limits_{-\infty}^{\infty} e^{-(1/2)(x^2-2\sqrt{2}\alpha^* x+\alpha^{*2}+|\alpha|^2)} \times \frac{\partial}{\partial x}e^{-(1/2)(x^2-2\sqrt{2}\alpha x+\alpha^2+|\alpha|^2)}dx, \quad (38)$$

or

$$\langle p\rangle = i\pi^{-1/2}\int\limits_{-\infty}^{\infty} e^{-(1/2)(x^2-2\sqrt{2}\alpha^* x+\alpha^{*2}+|\alpha|^2)} \times (x-\sqrt{2}\alpha)e^{-(1/2)(x^2-2\sqrt{2}\alpha x+\alpha^2+|\alpha|^2)}dx, \quad (39)$$

and, finally,

$$\langle p\rangle = \frac{\alpha-\alpha^*}{i\sqrt{2}}. \quad (40)$$

4.1. Uncertainty relation

Accordingly, the well-known uncertainty relation for a coherent state that we were looking for becomes, in terms of the two variances Δx and Δp

$$\Delta x\Delta p = \frac{1}{2}, \quad (41)$$

i.e., minimal uncertainty, the main feature of coherent states.

5. Conclusions

In this brief discourse concerning coherent states (CS) we have shown that a beginner's task of mastering the subjected can be greatly facilitated by the use of Eq. (12), a compact closed expression for dealing with a given coherent state, instead of dealing with the conventional expression

$$\psi_\alpha(x) = e^{-(|\alpha|^2/2)}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\phi_n(x).$$

All CS-manipulations become greatly simplified by appeal to such Eq. (12).

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i. Of the form $\psi_\alpha(x) = e^{-(|\alpha|^2/2)}\sum_{n=0}^{\infty}(\alpha^n/\sqrt{n!})\phi_n(x)$, with details explained below in this text.

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